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Deuteron Stripping Reaction

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(Received May 3, 1953)

On the same assumption as Butler, deuteron stripping cross sections are obtained in terms of the nuclear widths in the single level formula. By using this expression the reduced widths are computed from the experimental data.

§ 1. Introduction

Studies of the angular distribution of the deuteron stripping reaction became an effective method to investigate the properties of nuclear levels by the recent theoretical progresses¹⁾⁻⁶⁾. But the absolute cross section was not yet estimated except Butler's calculation¹⁾ which is obtained on the assumption that the captured nucleon moves in a square well potential. Is there any method to express the absolute cross section by the well-known parameters without any assumption for the nuclear potential? The purpose of this paper is to show that the absolute cross section can be expressed only by a level width for one channel which is familiar in nuclear reactions. If we use the formula which is expressed by reduced widths, we can inversely estimate reduced widths from the experimental cross sections. Then this reduced width will be useful in studying the properties of low energy levels which can not be measured by the low energy neutron reaction and in examining the validity of our assumption for the calculation if we apply the formula for the nuclear levels which is measured by the low energy neutron reaction and compare the widths which are estimated from two different experiments.

Previous works have generally been based on the following assumptions for (d, p) reaction; (1) the interaction between the proton and the target nucleus can be neglected, (2) the interaction between the neutron and the proton is taken into account only in the initial state...the adiabatic approximation, (3) Coulomb interaction can be neglected, (4) D -state in the deuteron wave function can be neglected and (5) the radiative width can also be neglected. The effect of Coulomb interaction was estimated by Yoccoz⁸⁾ and the effect of D -state was shown to be small by Dalitz⁷⁾. We also assume as above, although it may not be so accurate to calculate the absolute cross section upon these assumptions. We first remark about the general relations between the reduced width and the logarithmic derivative of the wave function at the nuclear surface in the single level formalism in § 2. In § 3, we describe the total wave function in such a way that its logarithmic derivative for a fixed proton momentum is equal to the one which is given by the reduced width

in § 2, and calculate the differential cross sections. The computations of the reduced widths from the experimental cross section are given in the last section.

§ 2. One level formula in nuclear reaction

In carrying out our calculation we must calculate the logarithmic derivative and the wave function value at the nuclear surface, which are replaced by the usual ones of nuclear reaction. For this purpose we use Thomas's one level formalism⁹⁾ and its extension to the closed channel. We now briefly write the formula by the same notations as his ones.

The configuration space of nuclear system is divided into the internal region and the external region by the surface S which is defined through the relative distance of the pair of particles of each channel $r_c = a_c$, and the external region is further sub-divided into the channels c , which are defined by the pair of particles u , channel spin s , relative angular momentum l . For the entrance channel we use the notation e , and for the open channel e^+ and for the closed channel e^- . The wave function of the channel c is expressed by the product of the radial wave function of the relative motion and the other part ψ_c . ψ_c satisfies following orthonormal condition :

$$\int_S \psi_c \psi_{c'} dS = \delta_{cc'}. \quad (2.1)$$

The radial wave function of the relative motion are

$$E_{e^+}(x) = (M_e/\hbar k_e)^{1/2} (G + iF) \exp[i(l\pi/2 - \sigma)] \quad (2.2a)$$

$$E_{e^+}^*(x) = \text{comp. conj. of } E_{e^+}(x), \quad (2.2b)$$

they represent the outgoing and incoming wave of unit flux respectively in open channel, and satisfy the following condition

$$(\hbar/2iM_e)[E_{e^+}^*(x)E_{e^+}'(x) - E_{e^+}(x)E_{e^+}^{'*}(x)] = 1, \quad (2.3)$$

where the prime shows the derivative with respect to r_c . For closed channel we put

$$E_{e^-}(x) = (M_e/\hbar k_e)^{1/2} W_{-\eta, l+1/2}(2x), \quad (2.4)$$

where $M_{-\eta, l+1/2}(2x)$ is Whittaker function and for neutral particle

$$W_{0, l+1/2}(2x) = e^{-x} \sum_{r=0}^l \frac{(l+r)!}{r!(l-r)!(2x)^r}. \quad (2.5)$$

We now write the wave function and its normal derivative at S as

$$\Psi = \sum_c \psi_c V_c \quad (2.6a)$$

$$\text{grad } \Psi = \sum_c \psi_c (D_c - a_c^{-1} V_c), \quad (2.6b)$$

and the logarithmic derivative as

$$f_c = D_c/V_c. \quad (2.7)$$

Hereafter the wave function should take the value at surface $r_c = a_c$ when its argument is not written, and we put

$$g_c = E'_c / E_c. \quad (2.8)$$

For the external region of the open channel we consider the particular solution which has only one incoming wave in the channel e ,

$$\Psi = \phi_{e^+}(E_{e^+}^* - U_{e^+e^+} E_{e^+}) - \sum_{c \neq e^+} \phi_c U_{e^+c} E_c \quad (2.9)$$

and we get

$$f_{e^+} = f_{e^+}^{\text{Re}} + i f_{e^+}^{\text{Im}} = (E_{e^+}^* - U_{e^+e^+} E_{e^+}) (E_{e^+}^* - U_{e^+e^+} E_{e^+})^{-1}. \quad (2.10)$$

Next we define the reduced width in the following expression

$$\gamma_c^2 = \left(\frac{\hbar^2}{2M_c} \right) |V_c|^2 \int_{\tau} |\Psi|^2 d\tau, \quad (2.11)$$

where $\int_{\tau} d\tau$ is the integral over the internal region. For two solutions $\Psi_1(E_1)$, $\Psi_2(E_2)$ of the equation $H\Psi = E\Psi$, we apply Green's theorem and using the following definition

$$\gamma_{12,c}^2 = \left(\frac{\hbar^2}{2M_c} \right) V_{1c} V_{2c}^* \int_{\tau} \Psi_1 \Psi_2^* d\tau, \quad (2.12)$$

we get

$$E_2 - E_1 = \sum_c \gamma_{12,c}^2 (f_{1c} - f_{2c}^*) \quad (1.13)$$

and from this equation

$$g_{e^+}^{\text{Im}} - f_{e^+}^{\text{Im}} = \Gamma / 2\gamma_{e^+}^2, \quad (2.14)$$

where

$$\Gamma = \sum_c \Gamma_c, \quad \Gamma_c = 2\gamma_c^2 g_c^{\text{Im}}. \quad (2.15)$$

If we define the resonance energy by

$$f_e(E_r) = g_e(E_r), \quad (2.16)$$

we get the following relation from the real part of (2.13)

$$f_{e^+}^{\text{Re}} - g_{e^+}^{\text{Re}} = \gamma_{e^+}^{-2} (E_r + \Delta - E), \quad (2.17)$$

where

$$\Delta = \sum_c \Delta_c = - \sum_c \gamma_c^2 [g_c^{\text{Re}}(E) - g_c^{\text{Re}}(E_r)]. \quad (2.18)$$

Putting the equations (2.14) and (2.17) into (2.10), we get

$$U_{e^+e^+} = e^{2i\delta} \left[1 - i \frac{\Gamma - \Gamma_{e^+}}{(E_r + \Delta - E) + \frac{i}{2}\Gamma} \right], \quad (2.19)$$

where

$$e^{2i\delta} = E_e^+{}^*/E_e^+ \quad (2.20)$$

From the equations (2.10), (2.11) and (2.19), we get

$$|V_e^+|^2 = 2M_e/\hbar^2 \cdot \gamma_e^2 \Gamma_e^+ [(E_r + \mathcal{A} - E)^2 + I^2/4]^{-1}. \quad (2.21)$$

So far we are concerned with the open channel, but from now on we treat the closed channel neglecting the radiative width. In equation (2.13) we put $E_2 = E_r$ and $E_1 \rightarrow E_2$, and we get

$$[\dot{g}_e^- - \dot{f}_e^-]_{E=E_r} = \gamma_e^{-2} [1 + \sum_{e^-} \gamma_e^2 \dot{g}_e^-]_{E=E_r}, \quad (2.22)$$

where the dot shows the derivative with respect E . If we normalize the wave function as

$$\int |\Psi|^2 d\tau = 1, \quad (2.23)$$

where the integration is performed over the initial and all external regions, we get from (2.11)

$$\gamma_e^2 = (\hbar^2/2M_e^-) |V_e^-|^2 [1 - \sum_{e^-} |V_e^-|^2 J_e^-]^{-1}. \quad (2.24)$$

In this equation J_e^- is given as

$$J_e^- = |E_e^-|^{-2} \int_{a_e}^{\infty} |E_e^-(k_e r_e)|^2 dr_e. \quad (2.25)$$

Applying Green's theorem to E_e^- in the external region, we have

$$|E_e^-|^{-2} \int_{a_e}^{\infty} |E_e^-(k_e r_e)|^2 dr_e = \frac{\hbar^2}{2M_e} \frac{E_e' - E_e^- - E_e' \dot{E}_e^-}{E_e^2},$$

from which we get easily

$$J_e^- = \frac{\hbar^2}{2M_e^-} \dot{g}_e^-. \quad (2.26)$$

Putting (2.26) into (2.24) and solving it with respect to $|V_e^-|^2$, we get

$$|V_e^-|^2 = 2M_e^-/\hbar^2 \cdot \gamma_e^2 [1 + \sum_{e^-} \gamma_e^2 \dot{g}_e^-]^{-1}. \quad (2.27)$$

§ 3. The derivation of the cross section

We begin with the decomposition of the deuteron plane wave into partial waves which have s -components of proton and neutron spin m_{s_p} and m_{s_n} , neutron angular momentum and its s -component l_n and m_n respectively. In the following expression we define the spin-angle function of our system except proton

$$\mathcal{Y}_{Jlsm}^M = \sum_{m_l} \sum_{m_s} (l s m_l m_s | l s J M) Y_{lm_l}(\theta\phi) \chi_{sm_s j}, \quad (3.1)$$

where J and M are total angular momentum and its s -component, s and m_s are channel spin and its s -component, $\chi_{sm_s j}$ is the channel spin wave function, which is given by

$$\chi_{s_m s_j} = \sum_{m_j} \sum_{m_{s_n}} (j s_n m_j m_{s_n} | j s_n s m_s) \chi_{j m_j} \chi_{s_n m_{s_n}}. \quad (3.2)$$

In this equation $\chi_{j m_j}$ and $\chi_{s_n m_{s_n}}$ are spin wave function of the target nucleus and the neutron respectively. If we use Hulthén type as the deuteron wave function

$$\varphi(r) = N_d (e^{-\alpha r} - e^{-\beta r}) / r \quad (3.3)$$

and let r_p be the relative coordinate between the proton and the final nucleus, r_n the one between the neutron and the initial nucleus, r_a the one between the emitted or bound particle and the residual nucleus, r'_p the one between proton and initial nucleus, so that $r_p = r'_p - \frac{M_n}{M_i + M_n} r_n$, and let r be $r_n - r_p$, k_d the wave number of incident deuteron in c.m. system, then

$$e^{ik_d \frac{r_p' + r_n}{2}} \varphi(r) \chi_{s_d m_{s_d}} \chi_{j m} = \sum_{m s_p' m_n J M} K_{J l_n m_n s_d s_p' m_{s_p}}^M \chi_{s_p' m_{s_p}} \mathcal{Y}_{J l_n s_d}^M \\ \times \int e^{ik_p r_p} \frac{1}{r_n} \Phi_{l_n m_n}(k_p, r_n) d\mathbf{k}_p. \quad (3.4)$$

In the above equation $K_{J l_n m_n s_d s_p' m_{s_p}}^M$ is the geometrical factor

$$K_{J l_n m_n s_d s_p' m_{s_p}}^M = \sum_{m_{s_n} m_s} (s_n s_p m_{s_n} m_{s_p} | s_n s_p s_d m_{s_d}) \\ \times (j s_n m_j m_{s_n} | j s_n s m_s) (l_n s m_l m_s | l_n s J M) \quad (3.5)$$

and $\frac{1}{r_n} \Phi_{l_n m_n}(k_p, r_n)$ is given by

$$\frac{1}{r_n} \Phi_{l_n m_n}(k_p, r_n) = \frac{N_d i^{l_n} V_{l_n m_n}(Z')}{\pi} \frac{1}{Z' r_n} j_{l_n}(Z' r_n) \left[\alpha^2 + K'^2 - \frac{1}{\beta^2 + K'^2} \right], \quad (3.6)$$

where

$$j_l(x) = \sqrt{\frac{\pi x}{2}} J_{l+1/2}(x), \quad (3.7)$$

$$Z' = k_p - \frac{M_i}{M_i + M_n} k_p, \quad (3.8a)$$

$$K' = 1/2 \cdot k_d - k_p. \quad (3.8b)$$

By using these notations the wave function which represents the deuteron stripping reaction and has only an entrance channel $J M m_p l_n m_n$ are written as :
for internal region

$$\chi_{s_p m_{s_p}} \int B_{J M m s_p' m_n}(\mathbf{k}_p) \frac{1}{r_n} \Psi_{J M}(\mathbf{k}_p, i_n, r_n) e^{ik_p r_p} d\mathbf{k}_p, \quad (3.9a)$$

at surface

$$\Psi_{J M}(\mathbf{k}_p, i_n, a_n) = \mathcal{Y}_{J n s_j}^M u_j V_{J M l_n s_j} + \sum_{l' s' j' a'} \mathcal{Y}_{J l' s' j' a'}^M u_{j' a'} V_{J M l' s' j' a'} \quad (3.9b)$$

and for external region

$$\begin{aligned}
 & \chi_{s_p m_{s_p}} \mathcal{J}_{J l_n s j}^M \int e^{i k_p r_p} \frac{1}{r_n} \Phi_{l_n m_n}(\mathbf{k}_p, r_n) d\mathbf{k}_p \\
 & + \chi_{s_p m_{s_p}} \mathcal{J}_{J l_n s j}^M \mathcal{U}_j \int A_{J M l_n s j \alpha}(\mathbf{k}_p) \frac{1}{r_n} E_{l_n}(\mathbf{k}_p r_n) e^{i k_p r_p} d\mathbf{k}_p \\
 & + \sum_{l' s' j' \alpha'} \chi_{s_p m_{s_p}} \mathcal{J}_{J l' s' j' \alpha'}^M \mathcal{U}_{j' \alpha'} \int C_{J M l_n s j \alpha; l' s' j' \alpha'} \frac{1}{r_{\alpha'}} E_{l_{\alpha'}}(\mathbf{k}_{\alpha'} r_{\alpha'}) e^{i k_p r_p} d\mathbf{k}_{\alpha'}.
 \end{aligned} \quad (3.9c)$$

In the equation (3.9c) the first term represents incident deuteron wave, the second term elastically scattered or captured neutron and the third term represents other channel particles. For closed channel it includes also the neutron channel in which the initial nucleus is in an excited state, and for open channel it represents inelastically scattered neutron and other emitted particle. $\mathcal{U}_{j\alpha}$ is the spacial wave function of the target nucleus, $A_{J M l_n s j \alpha}$, $B_{J M m s p' l_n m_n}$ and $C_{J M l_n s j \alpha; l' s' j' \alpha'}$ must be determined by the logarithmic derivative and wave function value at the surface. For convenience, index e is used for entrance channel $J M l_n s j$ and c for $J M l_{\alpha'} s' j' \alpha'$ briefly. From (3.9c) we obtain the logarithmic derivative

$$f_e = [\Phi'_{l_n m_n}(\mathbf{k}_p) + A_e(\mathbf{k}_p) E'_{l_n}] [\Phi_{l_n m_n}(\mathbf{k}_p) + A_e(\mathbf{k}_p) E_{l_n}]^{-1}, \quad (3.10)$$

from which we get the expression for $A_e(\mathbf{k}_p)$

$$A_e(\mathbf{k}_p) = -[\Phi'_{l_n m_n}(\mathbf{k}_p) - f_e \Phi_{l_n m_n}(\mathbf{k}_p)] [E'_{l_n} - f_e E_{l_n}]^{-1}. \quad (3.11)$$

By comparing the left side and the right side of the equation (3.9) we obtain

$$B_e(\mathbf{k}_p) V_e(\mathbf{k}_p) = \Phi_{l_n m_n}(\mathbf{k}_p) + A_e(\mathbf{k}_p) E_{l_n}, \quad (3.12)$$

$$B_e(\mathbf{k}_p) V_e(\mathbf{k}_p) = C_{ec}(\mathbf{k}_p) E_{l_{\alpha'}}, \quad (3.13)$$

and from these equations we can determine $B_e(\mathbf{k}_p)$ and $C_{ec}(\mathbf{k}_p)$, and we calculate them separately for two cases.

We consider first the case in which the neutron is captured in negative energy state. When the proton energy E_p is near the resonance energy which corresponds to the neutron energy level, we calculate the denominator of (3.11) using the following relations

$$\begin{aligned}
 f_e(E) & \cong g_e(E_r) + (E_p - E_r) \dot{f}(E_r), \\
 g_e(E) & \cong g_e(E_r) + (E_p - E_r) \dot{g}(E_r).
 \end{aligned} \quad (3.14)$$

Here we neglected the imaginary part of the logarithmic derivative, which is supposed to be small for negative energy level. Putting the relation (3.14) into (3.11), (3.12) and (3.13) we get the following expressions;

$$\begin{aligned}
 A_e(\mathbf{k}_p) & \cong -(E_{l_n} \Phi'_{l_n m_n} - E'_{l_n} \Phi_{l_n m_n}) [E_{l_n} (\dot{g} - \dot{f})_{E=E_r} (E_p - E_r)]^{-1} E_{l_n}^{-1}, \\
 B_e(\mathbf{k}_p) & \cong -V_e^{-1} (E_{l_n} \Phi'_{l_n m_n} - E'_{l_n} \Phi_{l_n m_n}) [E_{l_n} (\dot{g} - \dot{f})_{E=E_r} (E_p - E_r)]^{-1},
 \end{aligned} \quad (3.15)$$

$$C_{ee}(\mathbf{k}_p) \cong -V_e V_e^{-1} (E_{l_n} \Phi'_{l_n m_n} - E'_{l_n} \Phi_{l_n m_n}) [E_{l_n} (\dot{g} - \dot{f})_{E=K_r} (E_p - E_r)]^{-1} E_{l_n}^{-1}.$$

By taking this into the equation (3.9) and summing over all entrance channels with the weight K_e we get the general wave function, which we integrate over \mathbf{k}_p so that the proton wave may be outgoing wave only for the large value of r_p , square the coefficient of the outgoing proton wave, integrate over all space except proton, take average over initial states, and sum over final states, we obtain the cross section. Since these procedures are written in detail in other authors' papers, we write only the results here:

$$\frac{d\sigma}{d\Omega} = q \sum_{l_n} |\theta_{l_n}|^2 \frac{\hbar^2}{2M} \gamma_e^2 (1 + \sum_0 \gamma_e^2 g_e)^{-1}, \quad (3.16)$$

where

$$\theta_{l_n} = \frac{1}{Z} \left(\frac{1}{\alpha^2 + K^2} - \frac{1}{\beta^2 + K^2} \right) (j'_{l_n} - g_{l_n} j_{l_n}), \quad (3.17a)$$

$$q = \frac{2J+1}{2j+1} \frac{k_p}{k_d} \frac{M_f}{M_f + M_n} \frac{M_i}{M_i + M_d} \pi N_d^2, \quad (3.17b)$$

$$Z^2 = k_d^2 + \left(\frac{M_i}{M_i + M_n} \right)^2 k_p^2 - \frac{2M_i}{M_i + M_n} k_p k_d \cos \theta_p, \quad (3.18a)$$

$$K^2 = \frac{1}{4} k_d^2 + k_p^2 - k_d k_p \cos \theta_p. \quad (3.18b)$$

For the second case in which the neutron is captured in virtual state we can calculate the cross section in the analogous way as for the first case. Thus we write here briefly the procedure and the results. If we define U in the following equation

$$f_e = (E_{l_n}^{*'} - U E_{l_n}') (E_{l_n}^* - U E_{l_n})^{-1}, \quad (3.19)$$

this U is considered to be same as U_{e2} in (2.19) by comparing the equation (2.9) with (3.19), because we assumed that the logarithmic derivative is same as for deuteron and neutron reaction. Putting (3.19) into (3.11) we get

$$A_e(\mathbf{k}_p) = \frac{\hbar(M_i + M_n)}{2M_i M_n} [(\Phi'_{l_n m_n} E_{l_n}^* - \Phi_{l_n m_n} E_{l_n}^{*'}) - U(\Phi'_{l_n m_n} E_{l_n} - \Phi_{l_n m_n} E_{l_n}')]. \quad (3.20)$$

From (3.6) and the following definition

$$e^{i\theta} |j'_{l_n} E_{l_n} - j_{l_n} E_{l_n}'| = j'_{l_n} E_{l_n} - j_{l_n} E_{l_n}', \quad (3.21)$$

(3.20) becomes

$$A_e(\mathbf{k}_p) = \frac{\hbar(M_i + M_n)}{2M_i M_n} \frac{N_d}{\pi} i_{l_n} Y_{l_n m_n}(Z') \theta_{l_n} E_{l_n} e^{-2i\theta} (1 - e^{-2i\theta} U_{ee}). \quad (3.22)$$

If we take (2.19) into (3.22),

$$A_e(\mathbf{k}_p) = \frac{\hbar(M_i + M_n)}{2M_i M_n} \frac{N_d}{\pi} i_{l_n} Y_{l_n m_n}(Z') \theta_{l_n} E_{l_n} e^{-3i\theta - 2i\delta} \left[p'_{sc} + \frac{(\Gamma - \Gamma_e)}{(E_r + \Delta - E_p) + (i\Gamma/2)} \right], \quad (3.23)$$

where

$$p'_{sc} = e^{i(\delta+\theta)} \sin(\delta+\theta). \quad (3.24)$$

This term corresponds to the potential scattering in neutron elastic scattering, and was neglected by Huby⁶⁾ as spurious. The reason was that when the neutron attached to the surface the deuteron disintegrated as the consequence of vanishing of the interaction between the neutron and the proton within the surface. But we are now calculating the (d, p) reaction, so we consider the state that the proton is far from the nucleus. Then the neutron will undergo the potential scattering as the usual neutron, by which the neutron is scattered not through the compound nucleus. Considering that the bond of the neutron and proton is cut as the consequence of this, we had better not to neglect it. In fact, the reason that Huby neglected is also true in certain amount, the term must be correctly treated by taking into account the n - p interaction and the formation of deuteron after stripping process. As we believe that it is more correct for us not to neglect it, we here include it in our formula: Putting (3.20) into (3.12) we obtain

$$B_e(\mathbf{k}_p) V_e(k_p) = - \frac{\hbar(M_i + M_n)}{2iM_i M_n} (E_{i_n}^* - U E_{i_n}) (\Phi'_{i_n m_n} E_{i_n} - \Phi_{i_n m_n} E'_{i_n})$$

and noting $V_e(k_p) = E_{i_n}^* - U_{ee} E_{i_n}$, we have

$$B_e(\mathbf{k}_p) = - \frac{\hbar(M_i + M_n)}{2iM_i M_n} (\Phi'_{i_n m_n} E_{i_n} - \Phi_{i_n m_n} E'_{i_n}). \quad (3.25)$$

Putting this equation into (3.13) we get

$$C_{ec}(\mathbf{k}_n) = - \frac{V_e}{E_{i_{\alpha'}}} \frac{\hbar(M_i + M_n)}{2iM_i M_n} (\Phi'_{i_n m_n} E_{i_n} - \Phi_{i_n m_n} E'_{i_n}) \quad (3.26)$$

and we replace V_e in above equation for (2.23), and putting (3.23) and (3.26) into (3.9c) we get particular solution of our problem. If we sum this solution over all entrance channels with the weight K_e we obtain the general solution, and further obtain differential cross section, which are given

$$\frac{d\sigma_{el}}{d\mathbf{k}_p} = q' \sum_{i_n s} |\theta_{i_n}|^2 |E_{i_n}|^2 |p'_{sc}|^2 + \frac{(\Gamma - \Gamma_e)/2}{(E_r + \Delta - E_p) + i\Gamma/2}^2, \quad (3.27a)$$

$$\frac{d\sigma_{inel}}{d\mathbf{k}_p} = q' \sum_{i_n s} |\theta_{i_n}|^2 |E_{i_n}|^2 \frac{2M_\alpha M_n \gamma_c^2}{\hbar(M_\alpha + M_n) |E_{i_{\alpha'}}|^2} \frac{\Gamma_e}{(E_r + \Delta - E_p)^2 + \Gamma^2/4}, \quad (3.27b)$$

where

$$q' = \frac{2J+1}{2j+1} \frac{M_i M_d}{k_d(M_i + M_d)} \frac{\hbar(M_i + M_n)^2}{(2M_i M_n)^2} N_d^2. \quad (3.28)$$

§ 4. Discussions

The results of our calculation are given by (3.16), (3.17), (3.18), (3.27), and (3.28). In these equations $|\theta_{i_n}|^2$ gives angular dependence of the emitted proton, and

is identical with Butler's result. Through this term the orbital angular momentum of the captured neutron can be obtained from the experimental angular distribution of the proton, from which we can infer the spin of the final nucleus. As such an analysis has been made by many authors, we do not repeat it here. Next, the term which determines the absolute cross section in the case of capture in negative energy state is $\hbar^2/2M \times \gamma_e^2(1 + \sum_c \gamma_c^2 j_c)^{-1}$, and it is approximately proportional to the reduced width of the entrance channel, and the term $\sum_c \gamma_c^2 j_c$ in the denominator is the correction term, which is generally small compared to 1. The reduced width of the entrance channel is obtained from experiment. We here give the formulae for j_l and \dot{j}_l which are necessary for these calculations. They are all for neutral particles and $x = \kappa a = ika$,

$$\begin{aligned} aj_0(x) &= -x, \\ aj_1(x) &= -\frac{1+x+x^2}{1+x}, \\ ag_2(x) &= -\frac{6+6x+3x^2+x^3}{3+3x+x^2}, \\ j_l &= \frac{dx_n}{dE_n} a \frac{dg_l}{dx}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} a \frac{dj_0}{dx} &= -1, \\ a \frac{dg_1}{dx} &= -\frac{2x+x^2}{(1+x)^2}, \\ a \frac{dg_2}{dx} &= -\frac{6x+12x^2+6x^3+x^4}{(3+3x+x^2)^2}. \end{aligned} \quad (4.2)$$

Table 1. Reduced width calculated from the experimental data of (d, p) reaction

Reaction	Final state excitation energy(MeV)	Captured neutron angular momentum l_n	Channel radius $a(10^{-13}\text{cm})$	Final state spin	Reduced width (10^{13}cm^{-1} MeV)	Reduced width in sum rule unit (%)
$\text{Be}^9(d, p)\text{Be}^{10} \text{ }^{10)}$	0	1	4.5	0	0.16	24
	3.37	1	4.5	2	0.21	31
$\text{O}^{16}(d, p)\text{O}^{17} \text{ }^{11)}$	0	2	5	$\frac{5}{2}$	0.073	12
	0.875	0	6	$\frac{1}{2}$	0.43	71
$\text{F}^{19}(d, p)\text{F}^{20} \text{ }^{12)}$	0	0	5.32	1	0.0009	0.16
	0	2	5.32	1	0.024	4.3
	0.65	2	5.32	1	0.26	46
				2	0.14	26
				3	0.10	18
	3.5	0	5.35	0	0.33	59
				1	0.074	13

Although the angular distribution of stripping reaction have been measured for many nuclei and at various energy levels, the absolute value is done for fewer nuclei. We try here to calculate the reduced widths from experimental cross sections. Angular distribution of the proton does not agree perfectly with the theoretical one, particularly at the minimum of distribution curve, so we used the experimental data at the first maximum or near forward direction, because our assumptions of calculation will be more correct at forward direction. The results are shown in Table 1. The channel radius a in 4th column is chosen so that the angular distribution may agree better with the experiment. Of course the reduced width varies with the choice of the value a . According to Teichmann and Wigner¹³⁾, the sum rule limit of the reduced width is given by

$$\sum_c \gamma_c^2 = 3\hbar^2/2Ma. \quad (4.3)$$

The reduced widths in the unit of this sum rule limit are given in 7th column. In carrying these computations the correction from the other channels than the entrance one is neglected. We see from Table 1 that the reduced width in sum rule unit is considerably large, showing that the single particle model of nucleus is a good approximation. When the target nucleus is excited by the neutron absorption, the entrance channel differs from the channels which are important in final nucleus. Then the reduced width of the entrance channel is considerably small¹⁴⁾. $F^{19}(\alpha, p)F^{20}$ g.s. reaction will be an example of this.

It is interesting to calculate the level width in the case of the capture into the virtual state, because the results can be compared with the one which is obtained from neutron scattering, but we do not discuss it here, for there is no appropriate experimental datum. It is noted that we are based on the assumption in § 1. The error which comes from our assumption may not be too small, particularly for the absolute value, and the estimate of the error is important. Therefore we might not be able to trust our reduced width and the sum rule limit as well as the neutron scattering.

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Statistical Mechanics of General Brownian Motions Underlying Irreversible Processes*

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A statistical mechanical theory of general Brownian motions hidden at the bottom of irreversible processes, especially of "linear dissipative processes" is outlined. Postulating that macroscopic observations are gross, i.e., that the macroscopic behavior of a large system is specified with a very small number of macro-observables and that their observed values are given by suitable time averages, it is concluded that macro-observables are approximately one-valued integrals of motion in M. S. Green's sense (Sec. 2) and that they have finite mutual correlations (Sec. 4). On the basis of these results we attempt to reformulate the equation of motion in classical mechanics (Liouville's equation) in terms of the observed values of macro-observables. In order to do this, a macroscopic counterpart of the phase space and that of a distribution function in it are introduced (Sec. 3). Next the secular motion of the macroscopic distribution function is calculated. In this way the general Brownian motions underlying irreversible processes are disclosed and their equation of motion is obtained: a Fokker-Planck equation (Sec. 4, 5). This equation is seen to agree exactly with that recently given by M. S. Green, who studied the present problem from the microscopic point of view as well, but in his case, because of some serious assumptions employed, it was no longer necessary to refer to the equation of motion in mechanics. Finally some discussions are given on the origin of the "Markoffian character" of the macroscopic motion and on the relation of the present theory to Hashitsume's semi-phenomenological one (Sec. 6).

§ 1. Introduction

Recent developments accomplished in the field of "thermodynamics of irreversible processes"¹⁾ now suggest us to investigate this phenomenological formalism from the molecular theoretical point of view, that is, to build up general statistical mechanics of irreversible processes. As is well known, this new thermodynamics cannot include the whole variety of non-equilibrium states, but can only describe the so-called "linear dissipative processes" which usually make fairly slow temporal progresses. This fact tells us that it is only an apparent mode of motions, i.e., its secular motion, that can be observed macroscopically; the rest, troublesome and rapidly changing mode being practically smoothed out by insensitive measuring apparatus. As a result in spite of a great number of degrees of freedom the motion of a large system is usually described by various empirical equations essentially simpler than those of mechanics. That is, various types of densities such as mass density, internal energy density or momentum density, or thermodynamic variables

* The main part of the present paper was already published in Japanese (Busseiron Kenkyu No. 57, 66 (1952)).

such as local pressure or local temperature take here the place of variables used in mechanics that are proper to the individual degree of freedom, in terms of such variables a special empirical law having been found each in every case. The above thermodynamics put in order these empirical laws in a unified theory on the basis of both thermostatics and Onsager's reciprocal theorem.²⁾

Lately N. Hashitsume³⁾ has made a step forward along a way from thermodynamics to statistical mechanics. Starting from "thermodynamics of irreversible processes" and making use of the fluctuation theory given by Einstein, Smoluchowski and others, he developed a semi-phenomenological theory in the form of the theory of general Brownian motions.⁴ Especially he derived the following generalized Fokker-Planck equation :

$$\frac{\partial W(a, t)}{\partial t} = \sum_{k, l} \frac{\partial}{\partial a_k} \left[D_{kl} \left\{ \frac{\partial W(a, t)}{\partial a_l} - \frac{1}{k} \frac{\partial S(a)}{\partial a_l} W(a, t) \right\} \right], \quad (1)$$

where $W(a, t)$ and $S(a)$ stand for the probability distribution function and the entropy of a macroscopic state, $a \equiv \{a_k\}$, at time t respectively, and D_{kl} denotes the generalized diffusion constant. With the intention of finding a mark of forthcoming general statistical-mechanical attacks, he avoided there to fall back on any special mechanical model. Hence (1) can be considered as giving rather fundamental features of dissipative mechanisms in irreversible processes. In other words, it seems now obvious that the true character of any dissipative process could be ascertained by disclosing the general Brownian motions hidden at its bottom.

Having no relation to Hashitsume, but almost at the same time, M. S. Green⁵⁾ has done his useful work on time-dependent phenomena from the statistical mechanical side. He has surely succeeded in determining the explicit form of a Fokker-Planck equation in terms of microscopic variables, which is supposed to reproduce various irreversible processes and which is directly compared with Hashitsume's. Hence it can be said that thermodynamics of irreversible processes has been interrelated practically to the molecular structure of matters. Now Green's theory starts from the following two assumptions: First the macroscopic behavior of a large system is described by his so-called "gross variables" that are approximately one-valued integrals of motion. Secondly its equation of motion is expressed in the form of the Markoffian stochastic processes in respect of those gross-variables. As to the first assumption, it can be supposed to be natural, especially when one examines each special example (see examples cited by Green⁵⁾). From the theoretical point of view, however, it would be necessary to find the reason why gross variables can be considered as approximate one-valued integrals of motion, or why one looks only at such variables and not at the others. Moreover, one cannot be satisfied also with the second assumption because its position seems inconsistent with the deterministic nature of mechanics. Namely there still remains an important problem in statistical mechanics unsolved: one has to answer the question how and when the deterministic law turns stochastic. One sees therefore a serious gap existing between Green's theory and the molecular mechanics, or one may say that Green's theory cannot be regarded, in the true sense of the word, as statistical-mechanical but rather statistical.

In order to fill up the above gap we shall develop a formalism in the present article. It will start from the first principle of classical mechanics and lead to the same results as Green's without his assumptions, thus the general Brownian motions that underlie the irreversible processes being seen through from the molecular theoretical point of view. The quantum mechanical treatment will be discussed separately in a subsequent article.

For the sake of convenience the notations used in this paper are much the same as Green's.

§ 2. Macro-observables

As stated in the preceding section, there exist apparently distinct two types of equation of motion in the case of systems with great many degrees of freedom. One of them is mechanical, therefore is fundamental. In this case some means should be necessary to be found in measuring coordinates and momenta of all the particles composing the system in question. Practically, however, one has never had any such experimental means. We have, thus, been forced to appeal to more or less gross observations, which have led to the other type of motion, as is easily imagined. Here the adjective "gross" should be interpreted in *two* ways: First, the number of physical quantities measured phenomenologically is far smaller than that of the degrees of freedom of the system, so that its microscopic state can be only incompletely specified in this case. Next, any macroscopic observation can be performed in no case as accurately as in mechanics, i.e., it is gross in its usual sense. In any case without any such observational method, we would lose sight of the whole picture of the matter. From these considerations we can naturally conclude that the essential difference between these two types of equation of motion is due solely to a choice that is made from different observational styles on both sides. Henceforth we may name the observables in mechanics and in the phenomenological theories as *micro-* and *macro-observables* respectively.

Macro-observables are in general classified into two classes. One of them is defined mechanically, so is considered as a sort of phase function, and the other can be defined only thermodynamically. In order to give a statistical mechanical foundation to the phenomenological theories, first one should be exclusively concerned with the observables of mechanical nature, then those of thermodynamic nature could be introduced through ergodic-theoretical considerations. In the present paper we are mainly interested in the description of motion of a large system in terms of the macro-observables of mechanical nature which will from now on be called briefly macro-observables. The introduction of thermodynamics will be postponed to a future occasion.

The instantaneous macroscopic state of any dynamical system can precisely be specified by the position $X = (X_1, \dots, X_N)$ of its representative point in a suitable phase space. On the other hand the macroscopic state is, as stated above, only grossly specified by a far smaller number of macro-observables than N , which can be expressed as functions of the phase X , say $A_i(X)$, ($i=1, \dots, h$). Now it has to be emphasized that any instantaneous macro-observation must be regarded as extending over some time interval τ which is supposed to be proper to the measuring apparatus employed and to be long from the

microscopic point of view.⁽¹⁾ Hence we may naturally consider that the complete operation of measurement of a macro-observable $A_i(X)$ yields its time average over τ , i.e., that the observed value corresponds to

$$\bar{A}_i(X) = \frac{1}{\tau} \int_0^\tau ds A_i(X_s), \quad (2)$$

where X_s denotes the image point of the phase X after the lapse of s .

After all, collecting the above ideas, we postulate that the objects of macroscopic observations are solely confined to the phase functions $A_1(X)$, $A_2(X)$, ..., $A_h(X)$, ($h \ll N$), the observed values of which are given by (2), and that in terms of these time averages the macroscopic behavior of a large system can be specified. In the author's opinion this postulate is the most fundamental one among those which support the present theory, while the others, being more or less of secondary importance, will be introduced occasionally in later stages.

In spite of the simple and primitive form of the postulate it is possible to draw from it immediately a number of important consequences.

The first of them is concerned with existence of two time scales. One is most exact and is always employed in the micro-mechanics. The other is more gross, on which one can safely neglect time intervals of the order of magnitude τ . It is the latter scale by which one follows the secular motions of a large system. We shall again return to this point in the next section.

Secondly, the nature of the macro-observables reveals itself. In order that $\bar{A}_i(X)$ may be qualified for giving an instantaneous and definite value on the macroscopic time scale, $A_i(X)$ cannot be an arbitrary phase function but must satisfy the following two conditions. In the first place, $\bar{A}_i(X)$ has to be almost independent of τ , or to be dependent only on the order of magnitude of τ . Indeed the duration τ of a macro-observation may not be precisely fixed on the microscopic time scale, but its order of magnitude can at most be adjusted at our will. Consequently, although $A_i(X)$ would generally fluctuate very quickly with frequencies $\tau_0 \ll \tau$, the time-smoothed function $\bar{A}_i(X_s)$ must be almost linear to an extent of τ on the microscopic time scale (Fig. 1). In the second place, the difference between $\bar{A}_i(X)$ and $\bar{A}_i(X_\tau)$ must be at most of the same order of magnitude as that of errors in macro-

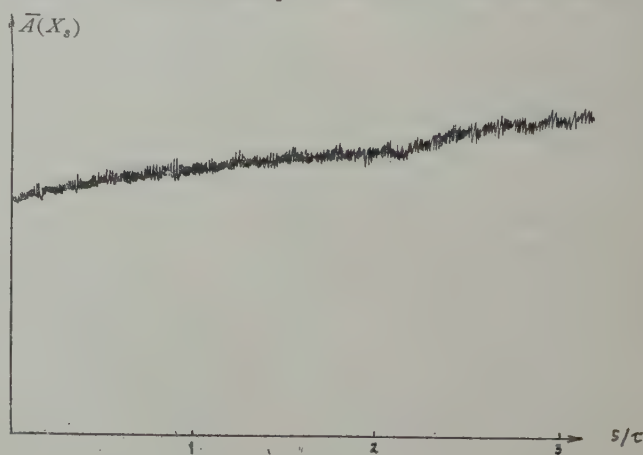


Fig. 1

observations. Otherwise, since τ should be neglected on the macroscopic time scale, $\bar{A}_i(X)$ could not be supposed to have a unique instantaneous value on that time scale. In other words, it is because the time average from 0 to τ as well as that from $-\tau/2$ to $+\tau/2$ must be assigned to one and the same instant on the macroscopic time scale. After all, the graph of $\bar{A}_i(X_s)$ must be not only linear but nearly horizontal as well, or, in short, almost constant to an extent of τ on the microscopic time scale (Fig. 1).

Finally one can infer another nature of $A_i(X)$. In order that $\bar{A}_i(X)$ could play a role as the macroscopic counterpart of $A_i(X)$ to describe the macroscopic behavior of our system, the deviation of $A_i(X)$ from $\bar{A}_i(X)$ at almost all the instants must be considerably small compared with $A_i(X)$ itself, otherwise the essential part of the motion of our system would go far out of our observation.

The above statements concerning the behavior of macro-observables correspond to, and allow one to gain an insight into, Green's another assumption that *macro-observables are approximately one-valued integrals of motion*.*

Having obtained these important consequences from our postulate, in the following considerations we need no more special property of macro-observables except one which will appear in Sec. 4 as a conclusion from our postulate as well. Adding a few words by the way, it will be one of the fundamental problems in statistical mechanics to prove the existence of phase functions with the aforesaid nature. For example, Khinchin⁵⁾ has proved that in the case of ideal gases "sum-functions" have very small fluctuations on an energy-surface. If one could generalize this theorem to take account of interactions between component particles and further to replace "sum of one-body functions" by "sum of two-body functions", the present theory would then stand on its more solid foundation.**

§ 3. The macroscopic counterpart of a distribution function in the phase space

The value of $\bar{A}_i(X)$, that is the observed value of $A_i(X)$, is denoted by a_i , ($i=1, \dots, h$). The set $a=\{a_i\}$ specifies the macroscopic state of our system, so that one can construct a h -dimensional Euclidean space as the macroscopic transcription of the original N -dimensional phase space. For the sake of brevity, we shall call the former *macrophase space*, or simply, *a-space*.

Now divide the macrophase space into an infinite number of small cubic cells, the side length Δ_i of each of which is of the same order of magnitude as that of errors in macro-observation. Since the volume $\Delta = \prod_{i=1}^h \Delta_i$ of each cell is very small, summation over all the cells whenever it occurs can be quite naturally replaced by a suitable integration. By the following inequalities:

$$a_i - \frac{1}{2} \Delta_i \leq A_i(X) < a_i + \frac{1}{2} \Delta_i, \quad (i=1, \dots, h) \quad (3)$$

where a is the central point of a given cell, we define a point set, the characteristic func-

* This idea was originally proposed by J. v. Neumann,⁷⁾ to whom the term "macro-observable" is due.

** Mass density and momentum density are written in the form of "sum of one-body function" and energy density has the form "sum of two-body function."

tion and the volume of which are denoted by $E_a[A(X)]$ and $w(a)\Delta$ respectively. There $A(X)$ stands for the set of the functions $\{A_i(X)\}$. Just in the same way one can introduce another characteristic function $E_a[\bar{A}(X)]$.

The following relation will be frequently used in the subsequent calculations:

$$\int dX \cdots = \sum_{a'} \int dX E_{a'}[A(X)] \cdots = \sum_{a'} w(a') \Delta \langle \cdots \rangle_{a'} = \int da' w(a') \langle \cdots \rangle_{a'}, \quad (4)$$

where

$$\langle \cdots \rangle_a = \frac{1}{w(a)\Delta} \int dX E_a[A(X)] \cdots. \quad (5)$$

Now consider a statistical ensemble specified by a distribution function $F(X, t)$ in the phase space. As the transcription of it into a -space the following distribution function might be proposed:

$$f(a, t) = \frac{1}{\Delta} \int dX F(X, t) E_a[A(X)]. \quad (6)$$

Due to our postulate, however, it is more reasonable to prefer the next one to (6):

$$f^*(a, t) = \frac{1}{\Delta} \int dX F(X, t) E_a[\bar{A}(X)]. \quad (7)$$

For, the procedure transcribing a distribution function cannot be considered as a mathematical one but a purely physical one associated with observational operations. Indeed, whilst $F(X, t)$ is determined when one examines microscopically each member composing a certain ensemble, its macroscopic counterpart is obtained when one has another look at every member of the ensemble by means of macroscopic observations. Therefore it is not the value of $A(X)$ itself but its observed value $\bar{A}(X)$ that comes into question.

Here it is worth-while to note that $f^*(a, t)$ is already a smooth function of time, so that one can infer the macroscopic motion of a given ensemble directly from that of $f^*(a, t)$. The proof is: According to the discussions given in the preceding section, $\bar{A}(X)$ is nearly constant in an interval of length τ , so that for $\tau \gtrsim s$

$$\begin{aligned} f^*(a, t+s) &= \frac{1}{\Delta} \int dX F(X, t+s) E_a[\bar{A}(X)] = \frac{1}{\Delta} \int dX F(X_{-s}, t) E_a[\bar{A}(X)] \\ &= \frac{1}{\Delta} \int dX F(X, t) E_a[\bar{A}(X_s)] \approx \frac{1}{\Delta} \int dX F(X, t) E_a[\bar{A}(X)] = f^*(a, t), \end{aligned} \quad (8)$$

where use has been made of the well-known theorems of mechanics that $F(X, t+s) = F(X_{-s}, t)$ and $dX = dX_{-s}$. Because of the discontinuous character of the characteristic functions, the fourth equality sign is not strictly correct, and it is the calculation of this small discrepancy that will become problem in the following sections.

Next we introduce another distribution function $\bar{f}(a, t)$:

$$\bar{f}(a, t) = \frac{1}{\tau} \int_0^\tau ds f(a, t+s). \quad (9)$$

Substituting (6) into (9), one obtains

$$\bar{f}(a, t) = \frac{1}{\Delta} \int dX F(X, t) \overline{E_a[A(X)]}, \quad (10)$$

where

$$\overline{E_a[A(X)]} = \frac{1}{\tau} \int_0^\tau ds E_a[A(X_s)], \quad (11)$$

Because of small fluctuation in $A(X)$, as was already discussed in the last section, one easily has

$$\bar{f}(a, t) = f^*(a, t). \quad (12)$$

For, neglecting infinitesimal quantities of the second order in respect of the above fluctuation the following relations can be readily verified:

$$E_a[\bar{A}(X)] = E_a[A(X)] + \frac{1}{\tau} \int_0^\tau ds \int_0^s ds' \sum_{k=1}^h V_k(X_{s'}) \frac{\partial E_a[A(X)]}{\partial A_k(X)} = \overline{E_a[A(X)]}, \quad (13)$$

where

$$V_k(X) = dA_k(X)/dt. \quad (14)$$

§ 4. The equation of motion of $f^*(a, t)$: Green's equation

From (12) and (9) one obtains

$$\partial f^*(a, t) / \partial t = \partial \bar{f}(a, t) / \partial t = \{f(a, t + \tau) - f(a, t)\} / \tau. \quad (15)$$

Before proceeding further on, a special mention must be made here of an important consequence that can be easily derived from (15). Since $\bar{A}(X)$ is almost independent of τ , $f^*(a, t)$, and therefore $\bar{f}(a, t)$, are hardly dependent on τ . Accordingly, so are also the first and second expressions in (15). On the other hand, as $f(a, t)$ has no bearing on τ , it will be natural to expect that the third expression in (15) would depend on τ , unless any particular condition concerning macro-observables happens to be satisfied. Indeed, (15) which leads us to the equation of motion of $f^*(a, t)$, as will be soon shown, reveals at the same time another fundamental feature of macro-observables. At any rate $\{f(a, t + \tau) - f(a, t)\} / \tau \equiv \partial f(a, t) / \partial t$ is only a difference quotient on the microscopic time scale, its differential quotient on that time scale being given by $\partial f(a, t) / \partial t$. But if $\partial f(a, t) / \tau$ would have a plateau value independent of τ , it may be considered as a differential quotient on the macroscopic time scale and as giving the secular variation of $f(a, t)$, and therefore may be called *macroscopic time derivative* of $f(a, t)$. Thus one can say that the time appearing in $f^*(a, t)$ or in $\bar{f}(a, t)$ is substantially macroscopic in contrast with that appearing in $f(a, t)$ which still retains its original microscopic character.

Substituting (6) into (15), one has

$$\frac{\delta f(a, t)}{\tau} = \frac{1}{\tau \Delta} \int_0^\tau ds \int dX \frac{\partial F(X, t+s)}{\partial s} E_a[A(X)]. \quad (16)$$

On the other hand the equation of motion of classical mechanics can be written in the form:⁵⁾

$$\frac{\partial F(X, t)}{\partial t} = - \sum_{r=1}^N \frac{\partial}{\partial X_r} \{ U_r(X) F(X, t) \}, \quad (17)$$

where

$$U_r(X) = dX_r/dt. \quad (18)$$

On the basis of (16) and with the use of (17), one has, after carrying out partial integration once, come to

$$\frac{\delta f(a, t)}{\tau} = \frac{1}{\tau \Delta} \int_0^\tau ds \int dX \sum_{r=1}^N U_r(X) F(X, t+s) \sum_{k=1}^h \frac{\partial E_a[A(X)]}{\partial A_k(X)} \frac{\partial A_k(X)}{\partial X_r}. \quad (19)$$

From (14) and (18) one obtains

$$\sum_{r=1}^N U_r(X) \frac{\partial A_k(X)}{\partial X_r} = \sum \frac{dX_r}{dt} \cdot \frac{\partial A_k(X)}{\partial X_r} = V_k(X). \quad (20)$$

Making use of (20), (19) may be rewritten in the form:

$$\frac{\delta f(a, t)}{\tau} = \frac{1}{\tau \Delta} \int_0^\tau ds \int dX F(X, t+s) \sum_k V_k(X) \frac{\partial E_a[A(X)]}{\partial A_k(X)}. \quad (21)$$

Through the procedure similar to those employed in (8) one can bring (21) into

$$\frac{\delta f(a, t)}{\tau} = \frac{1}{\tau \Delta} \int_0^\tau ds \int dX F(X, t) \sum_k V_k(X_s) \frac{\partial E_a[A(X_s)]}{\partial A_k(X_s)}. \quad (22)$$

As was fully discussed in Sec. 2, for $s \lesssim \tau$,

$$A_k(X_s) - A_k(X) = \int_0^s V_k(X_{s'}) ds' \ll A_k(X). \quad (23)$$

Because of this relation one can expand the derivatives of the characteristic function occurring in the integrand of (23) in respect of $A_k(X_s) - A_k(X)$, and neglect quantities of the second order with respect to them. Using (4) furthermore, the result is:

$$\begin{aligned} \frac{\delta f(a, t)}{\tau} &= \frac{1}{\tau} \int_0^\tau ds \int d\alpha' w(\alpha') \sum_k \langle F(X, t) V_k(X_s) \rangle_{\alpha'} \frac{\partial \delta(a - \alpha')}{\partial \alpha'_k} \\ &+ \frac{1}{\tau} \int_0^\tau ds \int_0^s ds' \int d\alpha' w(\alpha') \sum_k \sum_l \langle F(X, t) V_k(X_s) V_l(X_{s'}) \rangle_{\alpha'} \frac{\partial^2 \delta(a - \alpha')}{\partial \alpha'_k \partial \alpha'_l}. \end{aligned} \quad (24)$$

where $\delta(a-a') = \prod_{i=1}^n \delta(a_i - a'_i)$ and $\delta(a_i - a'_i)$ is Dirac's δ -function. After carrying out partial integrations (24) can be brought into

$$\begin{aligned} \frac{\partial f(a, t)}{\tau} = & \sum_k \frac{\partial}{\partial a_k} \left[-\tau w(a) \frac{1}{\tau} \int_0^\tau ds \langle F(X, t) V_k(X_s) \rangle_a \right. \\ & \left. + \sum_l \frac{\partial}{\partial a_l} \left\{ \tau w(a) \frac{1}{\tau} \int_0^\tau ds \int_0^s ds' \langle F(X, t) V_k(X_s) V_l(X_{s'}) \rangle_a \right\} \right]. \end{aligned} \quad (25)$$

Here we introduce the following assumptions:

$$\langle F(X, t) V_k(X_s) \rangle_a = \langle F(X, t) \rangle_a \langle V_k(X_s) \rangle_a, \quad (26)$$

$$\langle F(X, t) V_k(X_s) V_l(X_{s'}) \rangle_a = \langle F(X, t) \rangle_a \langle V_k(X_s) V_l(X_{s'}) \rangle_a.$$

Although these assumptions might seem too drastic and to need cautious analyses, we will not discuss here further on, but a brief comment will be given later (Sec. 6).

By straightforward calculations it can easily be shown that

$$\begin{aligned} \langle V_k(X_s) \rangle_a = & \frac{1}{w(a)A} \int dX V_k(X) E_a[A(X)] \\ & - \frac{1}{w(a)A} \int dX V_k(X) \sum_l \int_0^s ds' V_l(X_{s'}) \frac{\partial E_a[A(X)]}{\partial A_l(X)} \\ = & \langle V_k(X) \rangle_a + \frac{1}{w(a)} \sum_l \frac{\partial}{\partial a_l} \left\{ \int_0^s ds' \langle V_k(X) V_l(X_{s'}) \rangle_a w(a) \right\}. \end{aligned} \quad (27)$$

Furthermore, to the approximation hitherto made

$$\int_0^s ds' \langle V_k(X_s) V_l(X_{s'}) \rangle_a = \int_0^s ds' \langle V_k(X_s) V_l(X_{s-s'}) \rangle_a = \int_0^s ds' \langle V_k(X) V_l(X_{-s'}) \rangle_a. \quad (28)$$

Making use of relations (26), (27) and (28) one obtains for (25)

$$\begin{aligned} \frac{\partial f(a, t)}{\tau} = & \sum_k \frac{\partial}{\partial a_k} \left[- \left\{ \langle V_k \rangle_a + \frac{1}{w(a)} \sum_l \frac{\partial}{\partial a_l} [\xi_{kl}(a, \tau) w(a)] \right\} f(a, t) \right. \\ & \left. + \sum_l \frac{\partial}{\partial a_l} \{ \xi_{kl}(a, \tau) f(a, t) \} \right], \end{aligned} \quad (29)$$

where

$$\xi_{kl}(a, \tau) = \frac{1}{\tau} \int_0^\tau ds \int_0^s ds' \langle V_k(X) V_l(X_{-s'}) \rangle_a. \quad (30)$$

As was already discussed at the beginning of this section, because of our postulate we must require that the right-hand side of (29) is independent of τ . This requirement now

reveals us another, previously announced, feature of macro-observables: $\xi_{kl}(a, \tau)$ must have a plateau value, say $\bar{\xi}_{kl}(a)$, which is independent of τ . Under this condition the left-hand side of (29) gives the secular variation of $f(a, t)$, therefore equals to $\partial f^*(a, t)/\partial t$. That is,

$$\frac{\partial f^*(a, t)}{\partial t} = \sum_k \frac{\partial}{\partial a_k} \left[- \left\{ \langle V_k \rangle_a + \frac{1}{w(a)} \sum_l \frac{\partial}{\partial a_l} [\xi_{kl}(a) w(a)] \right\} f(a, t) + \sum_l \frac{\partial}{\partial a_l} \{ \xi_{kl}(a) f(a, t) \} \right]. \quad (31)$$

Now we take the time average of the both sides of (31) from 0 to τ . Then, while the left-hand side remains unchanged, on the right-hand side $f(a, t)$ is replaced by $\bar{f}(a, t)$, therefore by $f^*(a, t)$. The final result is:*

$$\frac{\partial f^*(a, t)}{\partial t} = \sum_k \frac{\partial}{\partial a_k} \left[- \left\{ \langle V_k \rangle_a + \frac{1}{w(a)} \sum_l \frac{\partial}{\partial a_l} [\xi_{kl}(a) w(a)] \right\} f^*(a, t) + \sum_l \frac{\partial}{\partial a_l} \{ \bar{\xi}_{kl}(a) f^*(a, t) \} \right], \quad (32)$$

where

$$\bar{\xi}_{kl}(a) = \{ \xi_{kl}(a) + \xi_{lk}(a) \} / 2. \quad (33)$$

It is seen that the motion of $f^*(a, t)$ is described by the Fokker-Planck equation already obtained by M. S. Green.⁽⁷⁾ If one compares (32) with Hashitsume's equation (1), one can say that the general Brownian motions underlying irreversible processes have now been disclosed from the classical mechanical point of view (see Sec. 6).

§ 5. The phenomenological equation of motion

As is well known, one can easily derive from (32) the average motion, that is, the phenomenological equation of motion. Here, however, another more direct method will be given.

Consider an ensemble uniformly distributed in a point set, say $E_a[A(X)]$, in the phase space at time t . The part of the ensemble that will be found in a point set $E_{a'}[A(X)]$ at a later time $t+\tau$ will be given

$$(1/w(a) \Delta) \cdot E_a[A(X)] E_{a'}[A(X_\tau)]. \quad (34)$$

Therefore the ensemble average of a macro-observable $A_k(X)$ at $t+\tau$ equals to

$$\langle\langle a_k(t+\tau) \rangle\rangle = \int da' a'_k \int dX E_a[A(X)] E_{a'}[A(X_\tau)] \frac{1}{w(a) \Delta}. \quad (35)$$

Using the expansion technique frequently used hitherto, (35) can be transformed into

* It can be directly verified that the difference between both the right-hand sides of (31) and (32) is negligible to the approximation hitherto made.

$$\langle\langle a_k(t+\tau) \rangle\rangle = \dot{a}_k + \int_0^\tau ds \langle V_k(X_s) \rangle_a. \quad (36)$$

Here one can put $\dot{a}_k = \langle\langle a_k(t) \rangle\rangle$. Then, with the help of (27) one obtains

$$\frac{d\langle\langle a_k(t) \rangle\rangle}{dt} = \left[\langle V_k \rangle_a + \frac{1}{w(a)} \sum_l \frac{\partial}{\partial a_l} \{ \xi_{kl}(a) w(a) \} \right]_{a=\langle a \rangle}, \quad (37)$$

where d/dt means the macroscopic time derivative introduced in the preceding section. These are Green's phenomenological equations of motion.⁵⁾

§ 6. Discussions

Throughout the above calculations we have neglected quantities of the second order in respect of fluctuation in $A(X)$. This approximation just corresponds to neglecting the third order moments with respect to $V(X)$'s, as easily understood. Due to the discussion given in Sec. 2 we are assured of the validity of these approximate calculations.

Next a few words must be said about our postulate. We have inserted in it a condition: the number of macro-observables is far smaller than that of degrees of freedom. This condition however has never been used in the present calculations. Moreover one of the conclusions that $\hat{\xi}_{kl}(a, \tau)$ has a plateau value $\hat{\xi}_{kl}(a)$ would presumably involve the above condition. Therefore one can still more take off it from our postulate.

As to the assumption (26) we will give here only the following note. Consider an ensemble average $\langle F(X, t) G(X) \rangle_a$, where $G(X)$ is an arbitrary phase function. In order that it equals to $\langle F(X, t) \rangle_a \langle G(X) \rangle_a$, it is necessary and sufficient that $F(X, t)$ has the form:

$$\bar{\bar{F}}(X, t) = \int da' f(a', t) \frac{E_a[A(X)]}{w(a') \Delta}. \quad (38)$$

This new distribution function is seen *uniform* within each point set $E_a[A(X)]$ in the phase space and seems to express an equilibrium state in an extended sense: local equilibrium. Of course the replacement of $F(X, t)$ by $\bar{\bar{F}}(X, t)$ in (25) is by no means self-evident. Since, however, it would be very difficult to justify it, we will only give an outline of the trouble into which we have now got. Take the time average of both sides of (25) from 0 to τ . Then, since the left-hand side is to give the secular variation of $f(a, t)$, it should remain unchanged, whereas on the right-hand side $F(X, t)$ is replaced by its time average. Thus the problem to be solved can be formulated in the following way: For an arbitrary distribution function $F(X, t)$

$$\frac{1}{\tau} \int_0^\tau ds F(X, t+s) = \bar{\bar{F}}(X, t). \quad (39)$$

This relation resembles to, but is more specialized than, the well-known *ergodic theorem* in the equilibrium theory.⁸⁾

According to the above theorem, if $A_i(X)$'s were exact and controllable one-valued integrals of motion, then

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T ds F(X, t+s) = \bar{F}(X, t). \quad (40)$$

Furthermore, suppose the left-hand side of (40) converges so rapidly that

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T ds F(X, t+s) = \frac{1}{\tau} \int_0^\tau ds F(X, t+s). \quad (41)$$

Then, at least in order to prove (39), all the $A_i(X)$'s would not be necessarily exact integrals of motion but need to be only approximate ones in the sense that they are constant within a time interval of the length τ . This condition is practically satisfied in the case of our real $A_i(X)$'s. Hence (39) substantially reduces to (41). In the meantime another feature of macro-observables can be inferred from various examples of irreversible processes: If the whole system is imagined to be suitably divided into a large number of small sub-systems, the one-valued integrals of motion of every sub-system which is now considered as independent of the others are just given by our macro-observables.⁵⁾ After all it is now to be desired to prove (41) in the case of such a sub-system, i.e., of a macroscopically small but microscopically still large system.

When one compares the present theory with Hashitsume's, the above-mentioned problem is seen to take its another appearance. As was already pointed out by Hashitsume,³⁾ in a special case when $\langle V_k \rangle = 0$ and $\hat{\xi}_{kl} = \text{constant}$, (32) reduces to (1), if one puts

$$S(a) = k \log w(a), \quad (42)$$

and $D_{kl} = \hat{\xi}_{kl}$. Here one sees that Boltzmann's principle again appears to play the same role as it once did in relating thermostatics to mechanics. This fact suggests that the states considered in the present theory are not so far from the true equilibrium but those in local equilibrium, and that (42) will naturally result from (39). Thus concluding the discussion of this section, it is the very relation (39) that gives our system its Markoffian character and that at the same time connects the present theory directly with thermodynamics of irreversible processes. On these circumstances we shall discuss separately on a future occasion.

§ 7. Summary and acknowledgments

We postulated that the macro-observations should be markedly distinguished from the micro-observations in respect of their observational styles, the observed values of the former being given by the time average (2), and that in terms of these time averages the macroscopic behavior of a large system can be described.

The results obtained were: First there exist two time scales, one is macroscopic and the other is microscopic. Secondly the macro-observables are approximately one-valued

integrals of motion in Green's sense. Next the correlations of macro-observables must have plateau values independent of τ . Finally under the assumption (26) the macroscopic motion of the system is described by the Fokker-Planck equation that was already found by M. S. Green.

Finally some discussions were given referring to the justification of the assumption (26) and the possibility of introducing thermodynamics into the present theory. In particular either of these problems was reduced to the generalized ergodic theorem (39), its proof however having been untouched.

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Ionization of Gas by Electrons

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A statistical analysis is made for the number of ions produced by an electron when it is absorbed in a hydrogen gas. The average number and the mean square fluctuation of the ion pairs as functions of the incident energy of electron satisfy inhomogeneous linear integral equations which yield Fano's results in the first approximation.

§ 1. Introduction

A charged particle in its passage through a gas produces a number of ion pairs along its path. The fact that the number is proportional to the energy lost in the gas by the incident particle is well known and is often used for the measurement of the energy of charged particles. Theoretical considerations on this subject have been tried by several authors^{(1),(2),(3),(4),(5)}, but the linearity mentioned above has not been deduced very clearly. Recently a few experiments^{(6),(7)} have been carried out which seem to indicate a slight deviation of the number of ion pairs from the linear relation to the energy lost by the incident particle producing these ions.

In the present paper the authors try to analyse the above problem by a statistical method. For the sake of simplicity we consider the case in which an electron is absorbed in a gas of atomic hydrogen. Our formalism, however, can be easily generalized for the case of more complex nature without much modification.

The probability distribution function for an electron with a given energy to produce a given number of ion pairs in the gas is defined which satisfies a set of nonlinear integral equations. Instead of solving these equations the linear integral equations for the average number of ion pairs and their mean square fluctuation are obtained, whose approximate solutions are estimated for high values of the incident energy.

§ 2. Fundamental equations

When an electron collides with a gas atom it will either ionize or excite the atom. The elastic collision can be ruled out in this case because it changes neither the energy

of the electron nor the number of ions. We neglect the electron exchange effect and the recoil of the gas atom hit by the electron as well as the higher order processes such as radiative collision and recombination of ion pairs etc. Thus we can define the following collision probabilities: $q_n(E)$ is the probability per collision that an electron with an incident energy E excites a gas atom to its n -th level with the excitation energy ϵ_n , and $d\epsilon q(E, \epsilon)$ is the probability per collision that the same electron ionizes the atom losing its energy by an amount between ϵ and $\epsilon + d\epsilon$. In the latter case the secondary electron has a kinetic energy $\epsilon - \epsilon_I$, where ϵ_I is the ionization potential of the gas atom.

The above defined $q_n(E)$ and $q(E, \epsilon)$ are normalized in the sense that

$$\sum_{n=2}^{\infty} q_n(E) + \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) = 1. \quad (1)$$

The actual forms for the q 's are obtained with the corresponding cross sections by

$$q_n(E) = \frac{\sigma_n(E)}{\sum_{n=2}^{\infty} \sigma_n(E) + \int_{\epsilon_I}^E d\epsilon \sigma(E, \epsilon)}, \quad (2)$$

$$q(E, \epsilon) = \frac{\sigma(E, \epsilon)}{\sum_{n=2}^{\infty} \sigma_n(E) + \int_{\epsilon_I}^E d\epsilon \sigma(E, \epsilon)}$$

In the following we assume that these σ 's and consequently q 's have been given by quantum-mechanical calculations.

Let $P(E, I)$ be the probability that an electron with an incident energy E produces exactly I pairs of ions in the gas in which it is entirely absorbed. $P(E, I)$ is normalized with respect to I :

$$\sum_{I=0}^{\infty} P(E, I) = 1. \quad (3)$$

Considering the change of $P(E, I)$ on one collision between the electron and a gas atom we obtain the following set of non-linear integral equations for $P(E, I)$;

$$P(E, I) = \sum_{n=2}^{\infty} q_n(E) P(E - \epsilon_n, I) + \sum_{i=0}^{I-1} \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) P(E - \epsilon, i) P(\epsilon - \epsilon_I, I - i - 1),$$

for $I = 1, 2, 3 \dots$ (4)

with

$$P(E, 0) = \sum_{n=2}^{\infty} q_n(E) P(E - \epsilon_n, 0), \quad \text{for } E > \epsilon_I,$$

and

$$P(E, I) = \delta_{I0} \quad \text{for } 0 < E < \epsilon_I.$$

Note that the above equations take into account all contributions from the secondary, tertiary etc., electrons as well as that of the primary.

We apply a Laplace transformation to $P(E, I)$ by

$$Q(E, \lambda) = e^{-\lambda} \sum_{I=0}^{\infty} e^{-\lambda I} P(E, I), \quad (5)$$

and obtain the Laplace transform of eqs. (4) as

$$\begin{aligned} Q(E, \lambda) &= \sum_{n=2}^{\infty} q_n(E) Q(E - \epsilon_n, \lambda) \\ &+ \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) Q(E - \epsilon, \lambda) Q(\epsilon - \epsilon_I, \lambda), \end{aligned} \quad (6)$$

which should be solved with the initial condition

$$Q(E, \lambda) = e^{-\lambda} \quad \text{for } 0 < E < \epsilon_I.$$

The $P(E, I)$ is given in terms of $Q(E, \lambda)$ by

$$P(E, I) = \frac{i}{2\pi} \int_0^{-2\pi i} Q(E, \lambda) e^{\lambda(I+1)} d\lambda.$$

§ 3. Average number and mean square fluctuation

Instead of solving eq. (6) or (4) directly we obtain the equations for the moments of the number of ion pairs by expanding eq. (6) into the power series of λ and regarding the equation as an identity with respect to λ . Thus the first and second moments

$$I(E) = \sum_{I=0}^{\infty} P(E, I) I, \quad \text{and} \quad I^2(E) = \sum_{I=0}^{\infty} P(E, I) I^2$$

satisfy the following equations,

$$\begin{aligned} I(E) &= q(E) + \sum_{n=2}^{\infty} q_n(E) I(E - \epsilon_n) \\ &+ \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) \{I(E - \epsilon) + I(\epsilon - \epsilon_I)\}, \end{aligned} \quad (7)$$

and

$$\begin{aligned} I^2(E) &= q(E) + 2 \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) \{I(E - \epsilon) + I(\epsilon - \epsilon_I) + I(E - \epsilon) I(\epsilon - \epsilon_I)\} \\ &+ \sum_{n=2}^{\infty} q_n(E) I^2(E - \epsilon_n) + \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) \{I^2(E - \epsilon) + I^2(\epsilon - \epsilon_I)\}. \end{aligned} \quad (8)$$

where

$$q(E) = \int_{\epsilon_I}^E d\epsilon q(E, \epsilon).$$

From eqs. (7) and (8) the equation for the mean square fluctuation

$$D(E) = \sum_{I=0}^{\infty} [I - I(E)]^2 P(E, I) = I^2(E) - [I(E)]^2 \quad (9)$$

is obtained as

$$D(E) = r(E) + \sum_{n=2}^{\infty} q_n(E) D(E - \epsilon_n) + \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) \{D(E - \epsilon) + D(\epsilon - \epsilon_I)\}, \quad (10)$$

where

$$r(E) = \sum_{n=2}^{\infty} q_n(E) \{I(E) - I(E - \epsilon_n)\}^2 + \int_{\epsilon_I}^E d\epsilon q(E, \epsilon) \{I(E - \epsilon) + I(\epsilon - \epsilon_I) + 1 - I(E)\}^2. \quad (11)$$

Both eqs. (7) and (10) are inhomogeneous linear integral equations of similar type which should be solved with the initial conditions

$$I(E) = 0, \text{ and } D(E) = 0 \quad \text{for } 0 < E < \epsilon_I. \quad (12)$$

We can easily prove from eq. (7) that

$$0 < I(E) < E/\epsilon_I,$$

which has an obvious physical interpretation.

§ 4. Formal and approximate solutions

Because of the initial condition (12) eqs. (7) and (10) can be expressed as sums of finite series. Let $K_n(E, \epsilon)$ be defined by

$$K_1(E, \epsilon) = \begin{cases} q(E, E - \epsilon) + q(E, \epsilon + \epsilon_I), & \epsilon_I < \epsilon < E - \epsilon_I \\ q(E, E - \epsilon), & E - \epsilon_I < \epsilon < E - \epsilon_2 \end{cases}$$

with

$$q(E, \epsilon) = \sum_{n=2}^{\infty} q_n(E) \delta(\epsilon - \epsilon_n) \quad \text{for } \epsilon_2 \leq \epsilon < \epsilon_I,$$

and

$$K_n(E, \epsilon) = \int_{\epsilon + (n-1)\epsilon_2}^{E - \epsilon_2} d\epsilon' K_{n-1}(E, \epsilon') K_1(\epsilon', \epsilon), \quad n = 2, 3, \dots$$

Then $I(E)$ and $D(E)$ are given by

$$I(E) = q(E) + \sum_{n=1}^N \int_{\epsilon_I}^{E - n\epsilon_2} d\epsilon K_n(E, \epsilon) q(\epsilon), \quad (13)$$

and

$$D(E) = r(E) + \sum_{n=1}^N \int_{\epsilon_I}^{E - n\epsilon_2} d\epsilon K_n(E, \epsilon) r(\epsilon), \quad (14)$$

where N is an integer which satisfies

$$\epsilon_I + N\epsilon_2 < E < \epsilon_I + (N+1)\epsilon_2.$$

The above solutions are exact, but the complex nature of the $K_n(E, \epsilon)$ makes it hard to estimate $I(E)$ or $D(E)$ for high values of E by the above formulas.

We know experimentally that the average number of the ion pairs $I(E)$ is roughly proportional to the energy E ,

$$I(E) = x_0 E, \quad x_0 \simeq \text{const.} \quad (15)$$

In order to determine the value of x_0 we insert (15) into eq. (7) and obtain

$$x_0 = \frac{q(E)}{\sum_{n=2}^{\infty} q_n(E) \epsilon_n + \epsilon_I q(E)} \quad (16)$$

which varies very slowly with E and becomes constant as E increases, thus confirming the assumption (15). The energy loss per ion pair is given by $1/x_0$ with (16), which is essentially the same expression as the one obtained by Fano¹⁾ by means of the consideration that the energy loss greater than ϵ_I contributes to x_0 only by an amount ϵ_I because the secondary electrons can also produce ion pairs along their paths. In our formalism eq. (16) is only the first approximation and the better result would be obtained by solving eq. (8) starting with the first approximation.

Likewise if we assume that

$$D(E) = \gamma_0 I(E), \quad \gamma_0 \simeq \text{const.} \quad (17)$$

we obtain from eqs. (10), (11) and (15),

$$\gamma_0 = \{x_0^2 \sum_{n=2}^{\infty} q_n(E) \epsilon_n^2 + (1 - x_0 \epsilon_I)^2 q(E)\} / q(E), \quad (18)$$

which is again essentially the Fano's expression¹⁾ obtained by his ingenious consideration.

§ 5. Numerical evaluation and comparison with experiment

Our q 's are evaluated using Bethe's⁸⁾ formulas for σ 's in (2). They are approximately,

$$\begin{aligned} q_n(E) &= \frac{1}{\phi_0 - (\epsilon_I/E)} \cdot \frac{2^8}{3} \cdot \frac{n^7 (n-1)^{2n-5}}{(n+1)^{2n+5}}, \\ q(E, \epsilon) &= q_1(E, \epsilon) + q_2(E, \epsilon), \\ q_1(E, \epsilon) &= \frac{1}{\phi_0 - (\epsilon_I/E)} \cdot \frac{2^7}{9e^4} \left\{ 4 \left(\frac{\epsilon_I}{\epsilon} \right)^4 - \left(\frac{\epsilon_I}{\epsilon} \right)^5 \right\} \frac{1}{\epsilon_I}, \\ q_2(E, \epsilon) &= \begin{cases} 0 & \epsilon \lesssim 5\epsilon_I, \\ \frac{1}{\phi_0 - (\epsilon_I/E)} \left(\frac{\epsilon_I}{\epsilon} \right)^2 \frac{1}{\epsilon_I} & \epsilon \gtrsim 5\epsilon_I. \end{cases} \end{aligned}$$

In the above expressions $q_1(E, \epsilon)$ and $q_2(E, \epsilon)$ refer respectively to the soft and violent

collisions of the electron with an atomic electron; Φ_0 comes from the normalization (1) and $\Phi_0 = 1.2$.

Using these expressions the values of x_0 and γ_0 are obtained by eqs. (16) and (18), which are shown in Fig. 1. For high values of E , x_0 and γ_0 are constants and

$$x_0 = 0.48/\epsilon_I, \quad \gamma_0 = 0.47.$$

The corresponding value of the average energy loss per ion pair is 28 eV. The discrepancy of our numerical value of x_0 from that of Fano's is probably due to the fact that he has neglected the effect of the violent collision, which contributes more to the numerator than to the denominator of eq. (16).

Instead of assuming (15) we can put

$$I(E) = x_1(E - E_0), \quad (15')$$

with x_1 and E_0 constants, then

$$x_1 = \frac{q(E)}{\sum_{n=2}^{\infty} q_n(E) \epsilon_n + (\epsilon_I + E_0) q(E)}, \quad (16')$$

and assuming (17) again (and writing γ_1 in the place of γ_0)

$$\gamma_1 = [x_1^2 \sum_{n=2}^{\infty} q_n(E) \epsilon_n^2 + \{1 - x_1(\epsilon_I + E_0)\}^2 q(E)] / q(E), \quad (18')$$

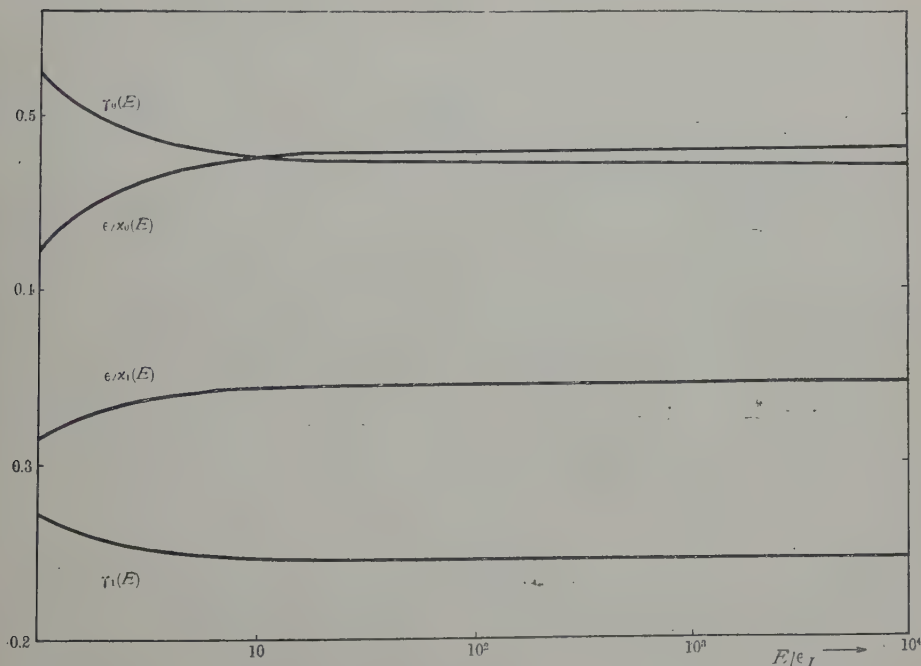


Fig. 1. The results of numerical evaluation of eqs. (16), (16'), (18) and (18') are shown. One will see that they are practically constant for $E > 10 \epsilon_I = 136$ eV.

It would be a tedious calculation to find the value of E_0 from eq. (7) but we can estimate it by comparing (16') with the experimental value* $1/x_1 \sim 39 \text{ eV}^{(9)}$ and get $E_0 \sim 11 \text{ eV}$. Using these values of x_1 and E_0 , γ_1 becomes 0.25. That the value of γ_1 is smaller than unity indicates that the distribution $P(E, I)$ is narrower than the Poisson distribution. E_0 introduced above is analogous to the ionization defect defined by Knipp and Ling⁽⁷⁾ in their calculation on the ionization yields of heavy charged particles. x_1 and γ_1 are also shown in Fig. 1.

The estimations made above are only very crude and the better approximations are now under consideration.

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Theory of the Interactions with Higher Derivatives and its Application to the Non-local Interaction

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Extending Yang-Feldman's method to the system with the interaction Lagrangian with higher derivatives, we obtain the Hamiltonian in the interaction representation. Secondly, we clarify that the Hamiltonian thus obtained is nothing else than one given by usual Heisenberg-Pauli's method using the canonical conjugates differed from the independent variables of the variation, that is, if we take such canonical variables, the system is reduced to the familiar case without higher time derivatives. We can apply this theory to the non-local interaction straightforwardly and then explain the differences between the non-local interaction and the non-localized action. Also we discuss the possibilities of the removals of the divergences by introducing the non-localities.

§ 1. Introduction

The formulations of the present quantum field theory have been pursued from the points of view of the Hamilton formalism or of the Lagrange formalism, and have become very excellent and beautiful. However, in the practical views, this theory has many difficulties as is well known. One of these difficulties is the well-known divergence problem. Although we could avoid it tentatively by the renormalization techniques, the problem has not yet been solved satisfactorily, and moreover we have arrived at the limitations of the renormalization techniques at present. The second is the discrepancies between the theoretical results and the experimental data and this tendency is becoming gradually more serious along with the development of the experimental techniques. Of course, this may be partly due to the incorrect treatments of the problems, that is, the incompleteness of the approximation method employed and then new approximation methods have been analysed exhaustively by many authors. But it will be true that there yet remain in their analysis the convictions, whether it is consciously or not, that the nature favours the renormalizable theories. Does there exist any philosophy indicated that the present renormalizability would become good foundation in our descriptions of the nature? This is a mythology in the present situations.

In order to get rid of such mythology and to advance further, it is yet necessary to construct the divergence-free field theory. Such attempts have appeared with the title "NON", i.e., the non-linear theory¹⁾, the non-local field theory²⁾, the non-localized action theory³⁾, the non-local interaction theory⁴⁾, etc. Of course, the aims of these theories are to solve not only this problem, but also the other interesting problems, such as the unified description in the non-linear theory, the explanation of the mass spectrum in the non-local

field theory and the removals of the mathematical ambiguities in the non-localized action, etc. However, for the present it seems hopeless to expect the further developments of the non-linear theory because of the difficulties of quantization and also the development of the non-localized action because of the existence of the negative energies. As far as we are concerned with the removal of the divergences, the non-local field theory and the non-local interaction theory do not differ from each other essentially. Although the non-local field theory will become the background of the latter in the further developments, we do not find any reason why we are obliged to prefer the non-local field up to now.

In these circumstances, we interest in the non-local interaction theory. The most important question in this theory is how it can remove the divergences by using the concept of the non-locality without introducing the negative energies.

In order to answer this question, it is necessary to restrict to the method how to treat the non-local interaction. Unfortunately, we have not yet the satisfactory method treating this theory but it seems to be accepted by many authors that even in this case Yang-Feldman's method⁽⁵⁾ can be applied logically.

Many authors have believed that in the non-locally interacting system, there does not exist the Hamilton formalism. This is only reason why many authors have employed Yang-Feldman's method, because in this theory we could obtain S -matrix even if the Hamilton formalism would not exist. It may be true that in the non-locally interacting system we can not describe the system step by step following the time variations and therefore we can not define the Hamiltonian as the time-displacement operator. But as it will be clear later, the circumstances are not so pessimistic if we distinguish the essential differences between the interaction representation and the Heisenberg's one and those between the canonical variables and the independent variables of the variation. In the interaction representation, we can define the canonical variables step by step and in this way can construct the Hamilton formalism. In fact, the essential point of Yang-Feldman's method is to define the interaction Hamiltonian firstly and then construct the S -matrix from it. If we do not pass through this procedure and construct the S -matrix by connecting the incoming waves with the outgoing waves, it will be very difficult to confirm the unitary character of the S -matrix.

The purpose of this paper is to construct the theory of the interaction Lagrangian with higher derivatives and then applied it to the non-local interaction as the infinite sum of such interactions. It seems to us an allowable way to approach to the non-local interaction.

§ 2. Extension of Yang-Feldman's method and Hamiltonian

When the interaction Lagrangian with higher derivatives has been given, we can construct the Hamiltonian in the interaction representation following the method which is the extension of Yang-Feldman's method. This procedure has been discussed generally by Takahashi and Umezawa⁽⁶⁾. In this chapter, we will try to take a little detailed discussions practically.

We start with the Lagrangian which describes the system of the fermion field interacting with the neutral boson field,

$$\bar{L}_{t_0 t} = \bar{L}_0 + \bar{L}_I,$$

$$\bar{L}_0 = - \int dx \bar{\psi}(x) M(\partial) \psi(x) + \frac{1}{2} \int dx \phi(x) A(\partial) \phi(x), \quad (2.1)$$

$$\bar{L}_I = g \int dx \bar{O}(\partial) \bar{\psi}(x) \cdot O(\partial) \psi(x) \cdot P(\partial) \phi(x),$$

where $\bar{\psi}(x)$, $\psi(x)$ and $\phi(x)$ are field quantities of fermion and boson in the Heisenberg representation respectively and $M(\partial)$, $A(\partial)$, $\bar{O}(\partial)$, $O(\partial)$ and $P(\partial)$ are the differential operators of which suffices are constructed suitably to bring the whole Lagrangian to be world scalar.

Using the variation method, we gain the field equations in the Heisenberg representation

$$\begin{aligned} M(\partial) \psi(x) &= \bar{O}(-\partial) \frac{\delta \bar{L}_I}{\delta \bar{O}(\partial) \bar{\psi}(x)}, \\ A(\partial) \phi(x) &= -P(-\partial) \frac{\delta \bar{L}_I}{\delta P(\partial) \phi(x)}. \end{aligned} \quad (2.2)$$

When we want to solve these equations, we are obliged to give the some initial conditions, and there we take the quantities satisfying the free equations

$$M(\partial) \psi^{in}(x) = 0 \quad \text{and} \quad A(\partial) \phi^{in}(x) = 0 \quad (2.3)$$

as these initial conditions. In the case of the interaction with higher derivatives, there does not generally exist the unitary transformation which connects them with the Heisenberg operators. Consequently, we have to define the quantities $\psi(x, \sigma)$, $\bar{\psi}(x, \sigma)$ and $\phi(x, \sigma)$ which are connected with the above quantities by the unitary transformation and then replace these quantities with the Heisenberg operators. And with the suitable boundary conditions, the relations between them are given as follows:

$$\begin{aligned} \psi(x) &= \psi(x, \sigma) - \int dx' \bar{O}(-\partial) N(\partial) D^\sigma(x-x') \frac{\delta \bar{L}_I}{\delta \bar{O}(\partial') \bar{\psi}(x')} + \int^\sigma dx' H_1(x, x'), \\ \phi(x) &= \phi(x, \sigma) + \int dx' P(-\partial) R(\partial) D^\sigma(x-x') \frac{\delta \bar{L}_I}{\delta P(\partial') \phi(x')} + \int^\sigma dx' H_2(x, x'). \end{aligned} \quad (2.4)$$

The last terms in the right-hand sides of the above equations have values only above the surface σ , and satisfy the equations

$$\int^\sigma dx' M(\partial) H_1(x, x') = 0 \quad \text{and} \quad \int^\sigma dx' A(\partial) H_2(x, x') = 0,$$

where

$$M(\partial) N(\partial) = (\square - \alpha^2) \quad \text{and} \quad A(\partial) R(\partial) = (\square - \mu^2)$$

hold. α and μ are masses of fermion and boson respectively.

According to Yang and Feldman, the transformation function which transforms the field quantities such as

$$\begin{aligned}\phi(x, \sigma) &= U^{-1}(\sigma, \sigma') \phi(x, \sigma') U(\sigma, \sigma') \\ &= U^{-1}(\sigma, -\infty) \phi^{\text{in}}(x) U(\sigma, -\infty)\end{aligned}\quad (3.5)$$

can be written by using the hermitian operator $\mathcal{H}_I(x/\sigma)$

$$U(\sigma, \sigma') = 1 - i \int_{\sigma'}^{\sigma} dx' \mathcal{H}_I(x'/\sigma) + \dots, \quad (2.5)$$

which satisfies

$$\begin{aligned}& \left[\phi(x, \sigma), \int_{\sigma'}^{\sigma} dx' \mathcal{H}_I(x'/\sigma) \right] \\ &= i \int_{\sigma'}^{\sigma} dx' \left[\bar{O}(-\partial) N(\partial) \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{O}(\partial') \bar{\phi}(x')} - H_1(x, x') \right], \text{ etc.}\end{aligned}$$

These equations give the definition of Hamiltonian in the interaction representation and these definitions can be modified to

$$\begin{aligned}-i [\phi(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= \bar{O}(-\partial) N(\partial) \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{O}(\partial') \bar{\phi}(x')} - H_1(x, x'), \\ i [\bar{\phi}(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= P(-\partial) R(\partial) D(x-x') \frac{\partial \bar{L}_I}{\partial P(\partial') \bar{\phi}(x')} + H_2(x, x')\end{aligned}\quad (2.7)$$

as the relations between the integrands of the above equations.

If we define the new quantities such as

$$\begin{aligned}\bar{O}(-\partial) O(\partial) N(\partial) &= F(\partial), & P(-\partial) P(\partial) R(\partial) &= E(\partial), \\ O(\partial) \phi &= \mathbf{A}, & P(\partial) \bar{\phi} &= \mathbf{B}, \\ O(\partial) \phi(x, \sigma) &= A(x, \sigma), & P(\partial) \bar{\phi}(x, \sigma) &= B(x, \sigma), \\ O(\partial) H_1(x, x') &= G_1(x, x'), & P(\partial) H_2(x, x') &= G_2(x, x'),\end{aligned}\quad (2.8)$$

the above equations (2.4) and (2.7) can be rewritten

$$\begin{aligned}\mathbf{A}(x) &= A(x, \sigma) - \int dx' F(\partial) \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{\mathbf{A}}(x')} + \int dx' G_1(x, x'), \\ \mathbf{B}(x) &= B(x, \sigma) + \int dx' E(\partial) D(x-x') \frac{\partial \bar{L}_I}{\partial \mathbf{B}(x')} + \int dx' G_2(x, x')\end{aligned}\quad (2.9)$$

and

$$\begin{aligned}-i [A(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= F(\partial) \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{\mathbf{A}}(x')} - G_1(x, x'), \\ i [B(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= E(\partial) D(x-x') \frac{\partial \bar{L}_I}{\partial \mathbf{B}(x')} + G_2(x, x')\end{aligned}\quad (2.10)$$

respectively.

The operator $\mathcal{H}_I(x/\sigma)$ obtained in this way can be called Hamiltonian, if the facts

that it is the time-displacement operator and it satisfies the integrability condition are confirmed. These proofs have been accomplished by Takahashi and Umezawa.⁽⁶⁾

In order to get this Hamiltonian practically, the first thing we must do is to express the Heisenberg operators in the right-hand sides of (2.9) by the quantities of x/σ in virtue of (2.9), and the next is to determine the suitable four-dimensional divergence terms G_i 's so as to be able to have the answer. Such quantities G_i 's do always exist and become the essential reasons why we can have the interaction Hamiltonian in this theory. We will try to get it practically.

From (2.9), we have

$$A(x) = A(x/\sigma) + \frac{1}{2}g \int dx' [F(\partial), \varepsilon(x-x')] A(x-x') B(x') + \int dx' G_1(x, x'),$$

and then they are also rewritten as

etc.

$$A = A - g [F(\partial)]_\kappa (AB) + \int dx' G_1(x, x'), \quad \text{etc.}$$

or

$$A = A - g [F(\partial)]_\kappa (AB) + O(g^2), \quad (2.11)$$

$$B = B + g [E(\partial)]_\mu (\bar{A}A) + O(g^2)$$

by using the definition of the operator (see Appendix)

$$\begin{aligned} [F(\partial)]_\kappa = & -\frac{1}{X_\kappa^+ X_\kappa^-} \left[F(\partial) - \frac{1}{X_\kappa^+ + X_\kappa^-} \{ X_\kappa^+ F(\partial - n X_\kappa^-) + X_\kappa^- F(\partial + n X_\kappa^+) \} \right], \\ X_\kappa^\pm = & (\Delta - \kappa^2)^{1/2} \pm (n\partial), \quad n_\mu = (0, 0, 0, i). \end{aligned} \quad (2.12)$$

Hereafter, we mean that

$$f(\sigma) A \equiv f(\sigma) A(x/\sigma) (= \lim_{x \rightarrow 0} f(\sigma) A(x, \sigma))$$

unless otherwise stated.

Using these equations of (2.11), (2.10) become

$$\begin{aligned} -i [A(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= F(\partial) A(x-x') \\ &+ [g A' B' - g^2 B' [F(\partial')]_\kappa (A' B') + g^2 A' [E(\partial')]_\mu (\bar{A}' A') + O(g^3)] \\ &\quad - G_1(x, x'), \\ i [B(x, \sigma), \mathcal{H}_I(x'/\sigma)] &= E(\partial) D(x-x') \\ &+ [g \bar{A}' A' - g^2 \bar{A}' [F(\partial')]_\kappa (A' B') - g^2 [\tilde{F}(-\partial')]_\kappa (\bar{A}' B') \cdot A' + O(g^3)] \\ &\quad + G_2(x, x'), \end{aligned}$$

where operator \tilde{F} in the last equation means the operator in which the spinor suffices are transposed from F . In order to derive the Hamiltonian from them, we may choose the additional terms G_i 's as follows :

$$\begin{aligned}
G_1(x, x') &= \frac{g^2}{2} [(F(\partial) A(x-x') A') \cdot [E(\partial')]_\mu (\bar{A}' A')] \\
&\quad - [E(-\partial')]_\mu (F(\partial) A(x-x') A') \cdot (\bar{A}' A')] \\
&\quad - \frac{g^2}{2} [(F(\partial) A(x-x') B') \cdot [F(\partial')]_\mu (A' B')] \\
&\quad - [\tilde{F}(-\partial')]_\mu (F(\partial) A(x-x') B') \cdot (A' B')] + O(g^3), \\
G_2(x, x') &= \frac{g^2}{2} [E(\partial) D(x-x') \bar{A}' \cdot [F(\partial')]_\mu (A' B')] \\
&\quad - [\tilde{F}(-\partial')]_\mu (E(\partial) D(x-x') \bar{A}') \cdot (A' B')] \\
&\quad + \frac{g^2}{2} [\tilde{F}(-\partial')]_\mu (\bar{A}' B') \cdot (E(\partial) D(x-x') A') \\
&\quad - (\bar{A}' B') \cdot [F(\partial')]_\mu (E(\partial) D(x-x') A')] + O(g^3).
\end{aligned} \tag{2.14}$$

After all we may have the interaction Hamiltonian

$$\begin{aligned}
\mathcal{H}_I &= -g \bar{A} A \cdot B - \frac{g^2}{2} (\bar{A} A) \cdot [E(\partial)]_\mu (\bar{A} A) + \frac{g^2}{2} (\bar{A} B) \cdot [F(\partial)]_\mu (A B) \\
&\quad + \frac{g^2}{2} [\tilde{F}(-\partial)]_\mu (\bar{A} B) \cdot (A B) + O(g^3),
\end{aligned} \tag{2.15}$$

but this is not the correct answer. The correct one is given from (2.15) by a slight technique by which the ranks of the time derivatives of the field quantities in (2.15) are reduced to zero for spinor's and one for boson's in virtue of the equations of motion

$$M(\partial)\phi=0 \quad \text{and} \quad A(\partial)\phi=0.$$

If we designate ones after these procedures by the rambling underlines, we have the last answer

$$\begin{aligned}
\mathcal{H}_I &= -g \underline{\bar{A} A B} - \frac{g^2}{2} (\underline{\bar{A} A}) \cdot [\underline{E(\partial)}]_\mu (\underline{\bar{A} A}) \\
&\quad + \frac{g^2}{2} (\underline{\bar{A} B}) [\underline{F(\partial)}]_\mu (\underline{A B}) + \frac{g^2}{2} [\underline{\tilde{F}(-\partial)}]_\mu (\underline{\bar{A} B}) \cdot (\underline{A B}) + O(g^3).
\end{aligned} \tag{2.16}$$

The Hamiltonian thus obtained is generally expressed by the infinite series of the coupling constant, that is, in the case of the interaction with higher derivatives, the Hamiltonian does not close with the finite terms concerning with the coupling constant.

The conditions whether the Hamiltonian closes or not, depend on the values

$$\begin{aligned}
&-\frac{1}{2} [F(\partial), \varepsilon(x-x')] A(x-x'), \\
&-\frac{1}{2} [E(\partial), \varepsilon(x-x')] D(x-x').
\end{aligned} \tag{2.17}$$

From (2.8), the differential operators are constructed of

$$F(\partial) = O(-\partial)O(\partial)N(\partial), \quad E(\partial) = P(-\partial)P(\partial)R(\partial)$$

and therefore if the highest derivatives entering into the interaction Lagrangian are assumed to be $\bar{O}(\partial) \sim \partial^{\bar{a}}$, $O(\partial) \sim \partial^a$ and $P(\partial) \sim \partial^b$, and the spins of the spinor and boson fields are s_a and s_b respectively, we have

$$\begin{aligned} N(\partial) \sim \partial^{2s_a} \quad \text{and} \quad R(\partial) \sim \partial^{2s_b} \quad \text{for} \quad x \neq 0, \quad \mu \neq 0, \\ N(\partial) \sim \partial_0 \quad \text{and} \quad R(\partial) \sim \partial^0 \quad \text{for} \quad x = 0, \quad \mu = 0 \end{aligned}$$

and then

$$\begin{aligned} F(\partial) \sim \partial^{2s_a + \bar{a} + a} \quad \text{or} \quad \partial^{\bar{a} + a + 1}, \\ F(\partial) \sim \partial^{2s_b + 2b} \quad \text{or} \quad \partial^{2b}. \end{aligned}$$

Then the sufficient conditions by which the Hamiltonian closes are $2s_b + 2b \leq 2$ and $2s_a + \bar{a} + a < 2$.

On the contrary, in the case of $2s_a + \bar{a} + a = 2$, it is usual that the Hamiltonian does not close irrespective of the ranks of $2s_b + 2b$. For the other cases, the Hamiltonian becomes generally the infinite series of the coupling constants, but there are some exceptions in which the highest derivatives drop automatically in virtue of the other formal requirements, such as the antisymmetry of the interaction and then their effective ranks become so low that the Hamiltonians do close.

The more concrete conclusions drawn from the above conditions are as follows:

A) In order to get the Hamiltonian with finite terms of the coupling constant, one must control the ranks of derivatives in the interaction Lagrangian at most to be i) one for spin 0 field ($s(\tau)$ and $ps(p\tau)$) and zero for spin 1 field (but there is one exception stated above, i.e., $v(t)$) with the linearly coupled fields, and ii) one for spin 0 field (meson current) and zero for spin 1/2 field with the bilinearly coupled fields, and iii) when the mass of the field is zero, the case becomes to be similar for spin 0 and spin 1/2 for boson and spinor fields respectively.

B) The Hamiltonians of the interaction of spin 3/2 (and more) particle and of the Konopinski-Uhlenbeck's interaction do not close.

As the examples, we indicate some results in which the Hamiltonians do not close for the sake of the later discussions:

Example 1. The case of $L_1 = -g \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} \phi$, where ϕ^* and ϕ are the quantities of the charged scalar field and ϕ is one of the neutral scalar field

$$\mathcal{H}_1 = g \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\mu} \phi + \frac{g^2 \phi^2}{1 + g\phi} n_\mu n_\nu \frac{\partial \phi^*}{\partial x_\mu} \frac{\partial \phi}{\partial x_\nu}. \quad (2.18)$$

Example 2. The case of $L_1 = g \bar{\psi} \phi f(\square) \phi$, where $\bar{\psi}$ and ψ are 1/2 spinor, ϕ is scalar

$$\mathcal{H}_1 = -g \bar{\psi} \phi f(\mu^2) \phi + \frac{g^2}{2} \underbrace{\bar{\psi} \phi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \phi}_{\sim O(g^3)}. \quad (2.19)$$

Example 3. The case of $L_I = g f(\square) \bar{\psi} f(\square) \psi \cdot \phi$, where $\bar{\psi}$ and ψ are $1/2$ spinor, ϕ is scalar

$$\begin{aligned} \mathcal{H}_I = & -g f^2(x^2) \bar{\psi} \psi \phi - \frac{g^2}{2} f^2(x^2) (\bar{\psi} \psi) \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma \partial - x) (\psi \phi) \\ & + \frac{g^2}{2} f^2(x^2) \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\tilde{\gamma} \partial + x) (\bar{\psi} \phi) \cdot (\psi \phi) + O(g^3). \end{aligned} \quad (2.20)$$

Example 4. The case of Konopinski-Uhlenbeck's interaction $L_I = g(\mathbf{B}_\mu \bar{\psi} \partial_\mu \psi - \mathbf{B}_\mu^* \partial_\mu \bar{\psi} \cdot \psi)$, where \mathbf{B}_μ and \mathbf{B}_μ^* have the bilinear forms of the heavy particle field

$$\begin{aligned} \mathcal{H}_I = & -g \bar{\psi} [1 - g(n\gamma)(nB)] [B_\nu + (n\gamma)(nB)\gamma_\nu] \partial_\nu^s \psi + c.c. \\ & + \frac{g^2}{2} \bar{\psi} [B_\nu \partial_\nu^s \{ (n\gamma)(nB^*) \psi \} + (n\gamma)(nB) \partial_\nu^s \{ B_\nu^* \psi \} + (nB) \gamma_\nu \partial_\nu^s \{ (nB^*) \psi \} \\ & - (n\gamma)(nB) (\partial_\nu^s B_\nu^*) \psi - (nB) \gamma_\nu (nB^*) \partial_\nu^s \psi] - c.c. \\ & - g \bar{\psi} [(n\gamma)(nB) + (n\gamma)(nB^*)] [1 - g \{ (n\gamma)(nB) + (n\gamma)(nB^*) \}] \psi + O(g^3), \end{aligned} \quad (2.21)$$

where $\partial_\nu^s = \partial_\nu + n_\nu(n\partial)$: space derivative.

§ 3. S-matrix

It is easily to construct the S -matrix from the interaction Hamiltonian thus obtained. As already stated, the Hamiltonian obtained in the previous section only involves at most one rank of the time derivatives and therefore we can define the Dyson's chronological operator uniquely. From (2.6), we have

$$\begin{aligned} S = U(\infty, -\infty) &= \cdots U(\sigma_0, \sigma_{-1}) U(\sigma_1, \sigma_0) U(\sigma_2, \sigma_1) \cdots \\ &= P \exp [-i \int dx \mathcal{H}_I(x)]. \end{aligned} \quad (3.1)$$

But the interaction Hamiltonian is generally expressed as the infinite series of the coupling constant

$$\mathcal{H}_I(x) = \sum_{r=1}^{\infty} \mathcal{H}_I^{(r)}(x, n) = -\underline{L}_I(x) + \sum_{r=2}^{\infty} \mathcal{H}_I^{(r)}(x, n), \quad (3.2)$$

in which the field quantities satisfy the free equations

$$M(\partial)\psi=0, \quad \tilde{M}(-\partial)\bar{\psi}=0 \quad \text{and} \quad A(\partial)\phi=0.$$

The first term in (3.2) is constructed by the operation by which the ranks of the time-derivatives in the given interaction Lagrangian are reduced to be at most one in virtue of the free equations, and the following terms are the additional terms which appear through the change of the representation from Heisenberg's to the interaction. In these terms, there are the normal-independent terms in addition to the ones depending on the surface normals. The latter only play a role by which the S -matrix does not depend the paths of integration and they can be also obtained from the integrability condition. While, the former play the essential roles and can not uniquely obtained only from the integrability condition aside from this theory.

In order to investigate the detailed characters of the interaction Hamiltonian, we take

$$L_I = g \bar{\psi} \psi f(\square) \phi$$

as the prototype example. In this case, there does not exist any normal independent term and the Hamiltonian is given from (2.19)

$$\mathcal{H}_I = -g \bar{\psi} \psi f(\mu^2) \phi + \frac{g^2}{2} \bar{\psi} \psi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \psi + O(g^3).$$

Then we have S -matrix

$$S = P \exp i \int dx \left[g \bar{\psi} \psi f(\mu^2) \phi - \frac{g^2}{2} \bar{\psi} \psi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \psi + O(g^3) \right]. \quad (3.3)$$

Here we decompose this into the normal constituents of the scalar field $\langle \phi \rangle$'s. This is easily done by using the operation

$$\phi \rightarrow \langle \phi \rangle + \frac{1}{2} \int dx' D_F(x-x') \frac{\delta}{\delta \langle \phi' \rangle}$$

and by Feynman's ordered operator calculus⁷⁾, and we get

$$S = P' \exp i \int dx \left[g \bar{\psi} \psi f(\mu^2) \langle \phi \rangle + \frac{g^2}{4} f^2(\mu^2) \bar{\psi} \psi \int dx' D_F(x-x') \bar{\psi}' \psi' - \frac{g^2}{2} \bar{\psi} \psi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \psi + O(g^3) \right]. \quad (3.4)$$

If we use the definition

$$\begin{aligned} D_F(x-x') &= D^{(1)}(x-x') - 2i \bar{D}(x-x') \\ &= D^{(1)}(x-x') + 2i \frac{1}{\square - \mu^2} \delta(x-x') \end{aligned}$$

in the second terms in (3.4), we have

$$\frac{i g^2}{4} \int dx' \bar{\psi} \psi f^2(\mu^2) D^{(1)}(x-x') \bar{\psi}' \psi' - \frac{g^2}{2} \bar{\psi} \psi \frac{f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \psi,$$

and after the cancellation of this last term with a part of the third term in S -matrix, we have

$$S = P' \exp i \int dx \left[g \bar{\psi} \psi f(\mu^2) \langle \phi \rangle + \frac{i g^2}{4} \bar{\psi} \psi \int dx' \{ f^2(\mu^2) D^{(1)}(x-x') - 2i f^2(\square) \bar{D}(x-x') \} \bar{\psi}' \psi' + O(g^3) \right]. \quad (3.5)$$

In this expression, we can understand the Green function D_F in the second term is replaced by

$$f^2(\mu^2) D_F(x-x') \rightarrow f^2(\mu^2) D^{(1)}(x-x') - 2i f^2(\square) \bar{D}(x-x') \quad (3.6)$$

due to the existence of $\mathcal{H}_I^{(2)}$ in the Hamiltonian. Similar to this circumstance, we can

understand that the existence of $\mathcal{H}_I^{(r)}$ take the operation by which some parts of the r -th order terms constructed from the superpositions of the lower-order terms are cancelled out to become the new r -th order terms and therefore we can also understand why the interaction Hamiltonian is generally expressed as the infinite series.

If we consider only such operational aspect and the equalities

$$f^2(\square)D_F \equiv f^2(\mu^2)D^{(1)} - 2if^2(\square)\bar{D}, \quad f(\square)\phi \equiv f(\mu^2)\phi, \quad (3.7)$$

and define the symbol P^* such as

$$P^* \exp [ig \int dx \bar{\psi} \psi f(\square)\phi] \equiv \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \int dx_1 \cdots \int dx_n P(\bar{\psi}\psi(1), \dots, \bar{\psi}\psi(n)) \\ \times f(\square_1) \cdots f(\square_n) P(\phi(1), \dots, \phi(n)), \quad (3.8)$$

we can express S -matrix formally for practical use

$$S = P^* \exp [ig \int dx \bar{\psi} \psi f(\square)\phi]. \quad (3.9)$$

The circumstances in the general case are similar to the above. That is, if we also understand

$$E(\partial)D_F \equiv \underline{E(\partial)}D^{(1)} - 2iE(\partial)\bar{D}, \quad P(\partial)\phi \equiv \underline{P(\partial)}\phi, \\ F(\partial)A_F \equiv \underline{F(\partial)}A^{(1)} - 2iF(\partial)\bar{A}, \quad O(\partial)\psi \equiv \underline{O(\partial)}\psi, \quad (3.10)$$

and define the symbol P^* such as

$$P^* \exp [ig \int dx \bar{O}(\partial)\bar{\psi} O(\partial)\psi P(\partial)\phi] \\ \equiv \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} \int dx_1 \cdots \int dx_n \bar{O}(\partial_1) O(\partial_1) \cdots \bar{O}(\partial_n) O(\partial_n) P(\bar{\psi}\psi(1), \dots, \bar{\psi}\psi(n)) \\ \times P(\partial_1) \cdots P(\partial_n) P(\phi(1), \dots, \phi(n)), \quad (3.11)$$

we can express S -matrix formally

$$S = P^* \exp [ig \int dx \bar{O}(\partial)\bar{\psi} O(\partial)\psi P(\partial)\phi]. \quad (3.12)$$

This is nothing else the S -matrix obtained usually by many authors according to Yang-Feldman's method by connecting the incoming waves with the outgoing waves.

From these results, we can see that so far as we are concerned with the problem how to obtain S -matrix, there may exist any direct way to find S -matrix without the knowledge of the interaction Hamiltonian, but it may be purely conventional method and it does not throw light into the essential characters of Yang-Feldman's method.

§ 4. The connections with the method of quantization

As indicated before, we can obtain the interaction Hamiltonian from the given interaction Lagrangian with higher derivatives by extending Yang-Feldman's method. In this theory, it will be the most essential point that we assume the free equations with the definite spins and masses as the initial conditions.

According to the current method of quantization, we take any set of the canonical conjugates (Q_r, P_r) when the Lagrangian is given, and then construct the total Hamiltonian as follows :

$$\mathcal{H}_{tot} = \sum_r P_r \dot{Q}_r - L.$$

And if we want to go through the quantum theory, we take the procedures of quantization

$$[P_r, Q_r]_{\pm} = \text{const}$$

for every sets. As there are many sets of the canonical conjugates in the Lagrangian with higher time derivatives, many elementary quanta appear and some of them have the negative energies so as to make the total energy to be indefinite (Pais-Uhlenbeck). The circumstances may be similar to the case of the interaction Lagrangian with higher derivatives and even if we assume the definite positive energy particles as the initial conditions, there will appear the other kinds of negative energy particles in the intermediate states. However, this consideration does not generally hold in our theory. The differences are due to the different stand points on the quantization method. We want to clarify these circumstances in this chapter.

Let us recall the procedure in the well-known Heisenberg-Pauli's method of quantization. In this theory, one starts with the free Lagrangian and define the canonical conjugates

$$Q^0(x), \quad P^0(x) = \frac{\delta L_0}{\delta \dot{Q}^0(x)}$$

and then Hamiltonian is deduced from the usual definition

$$\mathcal{H}_0 = P^0 \dot{Q}^0 - L_0.$$

This is the first stage in this procedure.

The problem in the next stage is how to treat the system of two or more fields interacting each other. In the usual case, the answer is very easily given. The method is that we also define the canonical conjugates in this system (i.e., in the Heisenberg representation)

$$Q(x), \quad P(x) = \frac{\delta L_{tot}}{\delta \dot{Q}(x)}$$

and then obtain the Hamiltonian analogously

$$\mathcal{H}_{tot} = P \dot{Q} - L_{tot}.$$

The quantization is adopted for the one of both sets

$$[P^0(x), Q^0(x')]_{t=t'} = -i \delta(\mathbf{r} - \mathbf{r}') \quad \text{or} \quad [P(x), Q(x')]_{t=t'} = -i \delta(\mathbf{r} - \mathbf{r}')$$

and both results are consistent with each other, because we can connect both sets by the unitary transformation.

However, when higher derivatives enter into the interaction Lagrangian, there does no further exist more such unitary transformation generally and the consistency of the two kinds of quantization does break down.

There are two ways to avoid this difficulty. One is the method by which we take many sets of the canonical conjugates (Q_r^0, P_r^0) in the free system so as to connect automatically with ones (Q_r, P_r) in the interacting system by the unitary transformation. According to the consideration in § 6, this method becomes to be equivalent to Pais-Uhlenbeck's method of quantization. Second one which we adopt is the method by which we reserve the original canonical conjugates (Q^0, P^0) and then take the quantities connecting with them by the unitary transformation

$$\begin{aligned} Q_t (\equiv Q(x/\sigma)) &= U^{-1}(\sigma, -\infty) Q^0(x) U(\sigma, -\infty)_{x|\sigma}, \\ P_t (\equiv P(x/\sigma)) &= U^{-1}(\sigma, -\infty) P^0(x) U(\sigma, -\infty)_{x|\sigma} \end{aligned} \quad (4.1)$$

as the set of the canonical conjugates in the interacting system. This is the stand point of the interaction representation.

In the first method, we are obliged to change the initial conditions because of the existence of the interaction with higher derivatives and can not avoid the difficulty of negative energy particles. On the contrary, in the second method, although we have the interaction with higher derivatives, we reserve the numbers of the canonical conjugates throughout and so there does not exist such difficulty. This is generally possible when we can define the quantities $Q(x/\sigma)$ and $P(x/\sigma)$ in the interaction representation.

Therefore, if we take

$$\begin{aligned} Q_t &= F(Q, \dot{Q}, \ddot{Q}, \dots) \equiv Q(x/\sigma) \longleftrightarrow Q^0(x) \quad (\text{numerically equal}), \\ P_t &= G(Q, \dot{Q}, \ddot{Q}, \dots) \equiv P(x/\sigma) \longleftrightarrow P^0(x) \quad (\text{numerically equal}) \end{aligned} \quad (4.2)$$

as the canonical conjugates in the interacting system and

$$\dot{Q}_t = \frac{d}{dt} F(Q, \dot{Q}, \ddot{Q}, \dots) \longleftrightarrow \dot{Q}^0(x) \quad (4.3)$$

we have

$$P_t = \frac{\delta L_{t_0 t}}{\delta \dot{Q}_t}, \quad \frac{\delta L_{t_0 t}}{\delta Q_t^{(r)}} = 0 \quad (r < 1). \quad (4.3')$$

Then the system is reduced to the familiar one and we can obtain the interaction Hamiltonian very easily as follows:

$$\begin{aligned} \mathcal{H}_I &= \mathcal{H}_{t_0 t}(Q_t, \dot{Q}_t) - \mathcal{H}_0(Q^0, P^0) \\ &= (P_t \dot{Q} - P^0 \dot{Q}^0) - L_{t_0 t}(Q_t, \dot{Q}_t) + L_0(Q^0, \dot{Q}^0). \end{aligned} \quad (4.4)$$

As the matter of course, the Hamiltonian thus obtained satisfies the usual requirements; i) There should exist no time-differentials for the spinor field, ii) for the boson field, at most one rank of time-differentials and iii) one of the canonical conjugates of the charged field must be the complex conjugate of the other.

In order to see the consistency of the theory, we indicate that the temporal changes of the canonical variables are induced by the Hamiltonian obtained. From (3.4), we have

$$\begin{aligned}\phi(x) = & \phi(x, \sigma) + \int dx' \left[\frac{1}{2} [\bar{O}(-\partial) N(\partial), \varepsilon(x-x')] \Delta(x-x') \frac{\partial L_I}{\partial \bar{O}(\partial') \bar{\psi}'} \right. \\ & + \frac{1}{2} [\varepsilon(x-x') - \varepsilon(\sigma, x')] O(-\partial) N(\partial) \Delta(x-x') \frac{\partial L_I}{\partial \bar{O}(\partial') \bar{\psi}'} \Big] \\ & + \int dx' \left[\frac{1 + \varepsilon(x-x')}{2} H_1(x-x') - \frac{1}{2} [\varepsilon(x-x') - \varepsilon(\sigma, x')] H_1(x, x') \right].\end{aligned}$$

and combining the definitions of the interaction Hamiltonian (2.9) it becomes

$$\begin{aligned}\phi(x) - \frac{1}{2} \int dx' [\bar{O}(-\partial) N(\partial), \varepsilon(x-x')] \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{O}(\partial') \bar{\psi}(x')} - \int_{\sigma}^x dx' H_1(x, x') \\ = \phi(x, \sigma) - i \int dx' \frac{1}{2} [\varepsilon(x-x') - \varepsilon(\sigma, x')] [\phi(x, \sigma), \mathcal{H}_I(x'/\sigma)].\end{aligned}\quad (4.5)$$

Then if we bring to x on the surface σ , the left-hand side of the above equation becomes to be equal to the canonical coordinate here adopted*

$$\begin{aligned}Q_i = \phi(x) - \frac{1}{2} \int dx' [\bar{O}(-\partial) N(\partial), \varepsilon(x-x')] \Delta(x-x') \frac{\partial \bar{L}_I}{\partial \bar{O}(\partial') \bar{\psi}'} \\ - \int_{\sigma(x)}^x dx' H_1(x, x') \equiv \phi(x/\sigma)\end{aligned}$$

and if we differentiate (4.5) with time and then bring to x on the surface σ , we have

$$\dot{Q}_i = \dot{\phi}(x/\sigma) - i [\phi(x/\sigma), \int d^3 x' \mathcal{H}_I(x'/\sigma)]_{t=t'}, \quad (4.6)$$

which is the well-known equation.

§ 5. Examples and discussions of the method of quantization

The general procedure of the method of quantization stated above and the concrete contents of the equations (4.2), (4.3) and especially the essential foundation of this method (4.3') can be confirmed by the several examples.

We begin with the ordinary examples such as $ps(pv)$ and $v(v)$ for indicating the consistency of this theory with usual one and then take more general cases by which we are interested.

Example 1. The case of the neutral pseudoscalar theory with pseudovector coupling

If we take

$$L_I = \bar{\psi}_\mu \frac{\partial \phi}{\partial x_\mu}, \quad \mathcal{H}_\mu = i f \bar{\psi} \gamma_5 \gamma_\mu \psi \quad (5.1)$$

* The last term in the left side of (4.5) has value only on the surface, then if we bring x on the surface σ , it becomes $-\int_{\sigma(x)}^x dx' H_1(x, x')$.

as the interaction Lagrangian, we have from (2.4) and (2.9)

$$\begin{aligned} j_\mu &= j_\mu, & (\psi &= \phi, \bar{\psi} = \bar{\phi}), \\ \phi &= \phi, & (n\partial)\phi &= (n\partial)\phi + (nj). \end{aligned} \quad (5.2)$$

Here we take as the sets of the canonical conjugates

$$\begin{aligned} q &= \phi \equiv \phi, & p &\equiv (n\partial)\phi, \\ Q &= \psi \equiv \psi, & P &\equiv \bar{\psi}(n\gamma) \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} \dot{q} &= (n\partial)\phi, & \dot{p} &= \frac{\delta L_{\text{tot}}}{\delta \dot{q}}, \\ \dot{Q} &= (n\partial)\psi, & \dot{P} &= \frac{\delta L_{\text{tot}}}{\delta \dot{Q}} \end{aligned} \quad (5.4)$$

and then obtain the interaction Hamiltonian from (4.4)

$$\mathcal{H}_I = -j_\mu \frac{\partial \phi}{\partial x_\mu} + \frac{1}{2} [n_\mu j_\mu]^2. \quad (5.5)$$

This is the same which reduced by the ordinary method.

Example 2. The case of the neutral vector field with vector coupling

If we take

$$L_I = j_\mu U_\mu, \quad j_\mu = i g \bar{\psi} \gamma_\mu \psi \quad (5.6)$$

as the interaction Lagrangian, we have from (2.4) and (2.9)

$$\begin{aligned} j_\mu &= j_\mu, & (\psi &= \phi, \bar{\psi} = \bar{\phi}), \\ U_\mu &= U_\mu - \frac{1}{\mu^2} n_\mu (nj), \\ (n\partial) U_\mu &= (n\partial) U_\mu + \frac{1}{\mu^2} \partial_\mu^s (nj). \end{aligned} \quad (5.7)$$

We take as the sets of the canonical conjugates

$$\begin{aligned} q_\mu &= U_\mu + \frac{1}{\mu^2} n_\mu (nj) \equiv U_\mu, & p_\mu &\equiv (n\partial) U_\mu - \partial_\mu (nU), \\ Q &= \psi \equiv \psi, & P &\equiv \bar{\psi}(n\gamma) \end{aligned} \quad (5.8)$$

and

$$\begin{aligned} \dot{q}_\mu &= (n\partial) (U_\mu + \frac{1}{\mu^2} n_\mu (nj)), & \dot{p}_\mu &= \frac{\delta L_{\text{tot}}}{\delta \dot{q}_\mu}, \\ \dot{Q} &= (n\partial)\psi, & \dot{P} &= \frac{\delta L_{\text{tot}}}{\delta \dot{Q}} \end{aligned} \quad (5.9)$$

and then obtain from (4.4) the well-known interaction Hamiltonian

$$\mathcal{H}_I = -j_\mu U_\mu + \frac{1}{2\mu^2} [n_\mu j_\mu]^2. \quad (5.10)$$

Example 3. The case of $L_I = -g \partial_\mu \psi^* \partial_\mu \psi \phi$ where ψ^*, ψ, ϕ are boson's

We have from (2.4) and (2.9)

$$\begin{aligned} \phi &= \phi, & (n\partial)\phi &= (n\partial)\phi, \\ \psi &= \psi, & (n\partial)\psi &= \frac{1}{1+g\phi} (n\partial)\psi, \end{aligned} \quad (5.11)$$

$$\psi^* = \psi^*, \quad (n\partial)\psi^* = \frac{1}{1+g\phi} (n\partial)\psi^*$$

and taking as the sets of the canonical conjugates

$$\begin{aligned} q &= \phi \equiv \phi, & p &\equiv (n\partial)\phi, \\ Q &= \psi \equiv \psi, & P &\equiv (n\partial)\psi^*, \\ Q^* &= \psi^* \equiv \psi^*, & P^* &\equiv (n\partial)\psi \end{aligned} \quad (5.12)$$

and their relations

$$\begin{aligned} \dot{q} &= (n\partial)\dot{\phi}, & \dot{p} &= \frac{\delta L_{tot}}{\delta \dot{q}}, \\ \dot{Q} &= (n\partial)\dot{\psi}, & \dot{P} &= \frac{\delta L_{tot}}{\delta \dot{Q}}, \\ \dot{Q}^* &= (n\partial)\dot{\psi}^*, & \dot{P}^* &= \frac{\delta L_{tot}}{\delta \dot{Q}^*}, \end{aligned} \quad (5.13)$$

we obtain the interaction Hamiltonian from (4.4)

$$\mathcal{H}_I = g \partial_\mu \psi^* \partial_\mu \psi \phi + \frac{g^2 \phi^2}{1+g\phi} n_\mu n_\nu \partial_\mu \psi^* \partial_\nu \psi, \quad (5.14)$$

which is the infinite series of the coupling constant and does agree with the previous result (2.18).

In the above three examples, there also exist the unitary transformations connected with the canonical conjugates in the free field with ones in the interacting system and therefore it is natural to agree with ones deduced by the usual method.

Next we take the examples in which such unitary transformations do not exist anyhow. As the Hamiltonians and also the sets of the canonical conjugates are generally expressed by infinite series of the coupling constants, we approximate them upto the second order throughout.

Example 4. The case of Konopinski-Uhlenbeck's interaction*

* This case is also treated by Umezawa and Takahashi (Prog. Theor. Phys. 9 (1953), 501), but the procedure is a slight different from ours and therefore we also treat it in this paper.

We take

$$L_I = g(\mathbf{B}_\mu \bar{\psi} \partial_\mu \psi - \mathbf{B}_\mu^* \partial_\mu \bar{\psi} \cdot \psi) \quad (5.15)$$

as the interaction Lagrangian. Upto the second order approximation, the quantities of the heavy particles do not play the essential roles and then we consider only the quantities of the light particles.

As the Lagrangian has generally ambiguities of four divergent quantities, we can modify it by taking them suitably so as to become as follows:

$$\begin{aligned} L_{tot} = & -\bar{\psi}(\gamma \partial + \kappa) \psi + g(\mathbf{B}_\mu \bar{\psi} \partial_\mu \psi - \mathbf{B}_\mu^* \partial_\mu \bar{\psi} \cdot \psi) + \frac{\partial \mathcal{V}_\mu}{\partial x_\mu} + L_{\text{heavy}} \\ = & -[\bar{\psi} + g \bar{\psi}(n\gamma)(n\mathbf{B}) - \frac{g^2}{2} \bar{\psi}(n\gamma)(n\mathbf{B})(n\gamma)(n\mathbf{B}^*) + O(g^3)] \\ & \times (\gamma \partial + \kappa) [\psi + g(n\gamma)(n\mathbf{B}^*)\psi - \frac{g^2}{2}(n\gamma)(n\mathbf{B})(n\gamma)(n\mathbf{B}^*)\psi + O(g^3)] \\ & + g \bar{\psi}(\mathbf{B}_\nu + (n\gamma)(n\mathbf{B})\gamma_\nu) \partial_\nu \psi - g \partial_\nu \bar{\psi}(\mathbf{B}_\nu^* + \gamma_\nu(n\gamma)(n\mathbf{B}^*)) \psi \quad (5.16) \\ & + g \kappa \bar{\psi}[(n\gamma)(n\mathbf{B}) + (n\gamma)(n\mathbf{B}^*)] \psi \\ & - \frac{g^2}{2} \bar{\psi}(n\gamma)(n\mathbf{B})[(n\gamma)(n\mathbf{B}^*)(\gamma \partial) \psi - (\gamma \partial) \{(n\gamma)(n\mathbf{B}^*)\} \psi] \\ & + \frac{g^2}{2} [\partial_\nu \bar{\psi} \gamma_\nu (n\gamma)(n\mathbf{B}) - \partial_\nu \{\bar{\psi}(n\gamma)(n\mathbf{B})\} \gamma_\nu] (n\gamma)(n\mathbf{B}^*) \psi + O(g^3) + L_{\text{heavy}}. \end{aligned}$$

while we have the canonical conjugates

$$\begin{aligned} Q = & \psi + g(n\gamma)(n\mathbf{B}^*)\psi - \frac{g^2}{2}(n\gamma)(n\mathbf{B})(n\gamma)(n\mathbf{B}^*)\psi + O(g^3) \equiv \psi, \\ -P(n\gamma) = & \bar{\psi} + g \bar{\psi}(n\gamma)(n\mathbf{B}) - \frac{g^2}{2} \bar{\psi}(n\gamma)(n\mathbf{B})(n\gamma)(n\mathbf{B}^*) + O(g^3) \equiv \bar{\psi} \end{aligned} \quad (5.17)$$

from (2.4) and (2.14). Then we can indicate

$$P = \frac{\delta L_{tot}}{\delta \dot{Q}}, \quad \frac{\delta L_{tot}}{\delta \ddot{Q}} = 0,$$

if we have

$$\dot{Q} = (n\partial)[\psi + g(n\gamma)(n\mathbf{B}^*)\psi - \frac{g^2}{2}(n\gamma)(n\mathbf{B})(n\gamma)(n\mathbf{B}^*)\psi + O(g^3)] \quad (5.18)$$

and consequently the interaction Hamiltonian becomes to one given in (2.21). The Hamiltonian obtained here has not the time derivatives of $\bar{\psi}$ and ψ from the beginning without the techniques used in § 2. This fact justifies the previous techniques dropping the higher time derivatives by using the free equations of motion.

Example 5. The case of the unified interaction of spinor field $L_I = g f(\square) \bar{\psi} f(\square) \psi$

We take this interaction Lagrangian as the most simply one which does not

introduce any normal dependent terms in the interaction Hamiltonian.

By using the above stated modifications, we can rewrite it as follows :

$$\begin{aligned}
 L_{tot} &= -\bar{\psi}(\gamma\partial + x)\psi + g f(\square) \bar{\psi} f(\square) \psi + \frac{\partial \tau v_\mu}{\partial x_\mu} \\
 &= -\left[\bar{\psi} + \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\partial \bar{\psi} \gamma + x \bar{\psi}) + O(g^2) \right] \\
 &\quad \times (\gamma\partial + \mu) \left[\psi - \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x)\psi + O(g^2) \right], \quad (5.19)
 \end{aligned}$$

where μ is the real root of the equation

$$g f^2(x^2) + x - \mu = 0 \quad (5.20)$$

and μ becomes x if the coupling constant g tends to zero.

While we have from (2.4)

$$\psi = \psi + g \frac{f(\square) - f(x^2)}{\square - x^2} (\gamma\partial - x) f(\square) \psi + O(g^2)$$

or

$$\begin{aligned}
 \psi &= \psi + \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \psi \\
 &\quad + \frac{1}{2} g (\square - x^2) \left[\frac{f(\square) - f(x^2)}{\square - x^2} \right]^2 (\gamma\partial - x) \psi + O(g^2),
 \end{aligned}$$

but we can modify it without changing the result as follows :

$$\psi = \psi + \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \psi + O(g^2). \quad (5.21)$$

Then if we take as the canonical conjugates

$$\begin{aligned}
 Q &= \psi - \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \psi + O(g^2) \equiv \psi, \\
 -P(n\gamma) &= \bar{\psi} + \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\partial \bar{\psi} \gamma + x \bar{\psi}) + O(g^2) \equiv \bar{\psi}, \quad (5.23)
 \end{aligned}$$

and as its time derivative

$$\dot{Q} = (n\partial) \left[\psi - \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \psi + O(g^2) \right], \quad (5.23)$$

we have

$$P = \frac{\partial L_{tot}}{\partial \dot{Q}}, \quad \frac{\partial L_{tot}}{\partial Q^{(r)}} = 0 \quad (r > 1),$$

and the interaction Hamiltonian becomes

$$\mathcal{H}_I = (\mu - x) \bar{\psi} \psi = [-g f^2(x^2) + \dots] \bar{\psi} \psi \quad (5.24)$$

in which there does not exist any time derivative of the field quantities.

Example 6. The case of $L_I = g \bar{\psi} \psi f(\square) \phi$

We modify the Lagrangian as before

$$\begin{aligned}
 I_{tot} &= -\bar{\psi}(\gamma\partial + \kappa)\psi - \frac{1}{2} [\partial_\lambda \phi \partial_\lambda \phi + \mu^2 \phi^2] + g \bar{\psi} \psi f(\square) \phi + \frac{\partial \omega_\mu}{\partial x_\mu} \\
 &= -[\bar{\psi} - g^2 \bar{I} + O(g^3)](\gamma\partial + \kappa)[\psi - g^2 I + O(g^3)] \\
 &\quad - \frac{1}{2} \left[\partial_\nu \left\{ \phi + g \frac{f(\square) - f(\mu^2)}{\square - \mu^2} (\bar{\psi} \psi) + O(g^3) \right\} \right]^2 \\
 &\quad - \frac{1}{2} \mu^2 \left\{ \phi + g \frac{f(\square) - f(\mu^2)}{\square - \mu^2} (\bar{\psi} \psi) + O(g^3) \right\}^2 \\
 &\quad + g \bar{\psi} \psi f(\mu^2) \left[\phi + g \frac{f(\square) - f(\mu^2)}{\square - \mu^2} \bar{\psi} \psi + O(g^3) \right] \\
 &\quad - \frac{1}{2} g^2 \bar{\psi} \psi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} \bar{\psi} \psi - g^2 \bar{\psi}(\gamma\partial + \kappa) I + g^2 (\partial \bar{I} \gamma - \kappa \bar{I}) \psi + O(g^2).
 \end{aligned} \tag{5.25}$$

While we obtain from (2.4) and (2.9)

$$\begin{aligned}
 \phi &= \phi - g \frac{f(\square) - f(\mu^2)}{\square - \mu^2} \bar{\psi} \psi + O(g^3), \\
 \psi &= \psi + g^2 I + O(g^3),
 \end{aligned} \tag{5.26}$$

where

$$\begin{aligned}
 I &= -\frac{1}{2} \int_{-x}^{x} dx' \left[S(x-x') \psi' \frac{f^2(\square') - f^2(\mu^2)}{\square' - \mu^2} (\bar{\psi}' \psi') \right. \\
 &\quad \left. - (\bar{\psi}' \psi') \frac{f^2(\square') - f^2(\mu^2)}{\square' - \mu^2} (S(x-x') \psi') \right]
 \end{aligned} \tag{5.27}$$

from (2.14). It is more convenient to represent this by the quantities of the interacting system. Then we can employ the following form:

$$\begin{aligned}
 I &= -\frac{1}{2} \int_{-x}^{x} dx' \left[S(x-x') \psi' \frac{f^2(\square') - f^2(\mu^2)}{\square' - \mu^2} \frac{1}{2} \{ (\bar{\psi}' + \bar{\psi}') \psi' \} \right. \\
 &\quad \left. - (\bar{\psi}' \psi') \frac{f^2(\square') - f^2(\mu^2)}{\square' - \mu^2} \frac{1}{2} \{ S(x-x') (\psi' + \psi') \} \right],
 \end{aligned} \tag{5.27'}$$

where the meanings of the rambling underlines are same as § 2, that is, we must replace the time derivatives operating on \underline{A} with the combinations of the special derivatives as if it satisfies the free equations of motion (for example, $(n\partial)\underline{\psi}$ is read as $-(n\gamma)(\gamma\partial + \kappa)\psi$. This procedure does not destroy the equivalence to (5.27) in the result. Further if we consider the last three terms in (5.25), they become

$$\begin{aligned}
 & -\frac{1}{2}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi) \\
 & +\frac{1}{4}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi)-\frac{1}{4}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi) \\
 & +\frac{1}{4}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi)-\frac{1}{4}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi) \\
 & =-\frac{1}{2}g^2(\bar{\psi}\psi)\frac{f^2(\square)-f^2(\mu^2)}{\square-\mu^2}(\bar{\psi}\psi)
 \end{aligned} \tag{5.28}$$

in which any time derivative of ψ and $\bar{\psi}$ does not more appear.

Consequently, if we take as the sets of the canonical conjugates

$$\begin{aligned}
 Q &= \psi - g^2 \mathbf{I} + O(g^3) \equiv \phi, \\
 -P(n) &= \bar{\psi} - g^2 \bar{\mathbf{I}} + O(g^3) \equiv \bar{\psi}, \\
 q &= \phi + g \frac{f(\square) - f(\mu^2)}{\square - \mu^2}(\bar{\psi}\psi) + O(g^3) \equiv \phi, \\
 p &\equiv (n\partial)\phi
 \end{aligned} \tag{5.29}$$

and their time derivatives

$$\begin{aligned}
 \dot{Q} &= (n\partial)[\phi - g^2 \mathbf{I} + O(g^3)], \\
 \dot{q} &= (n\partial)\left[\phi + g \frac{f(\square) - f(\mu^2)}{\square - \mu^2}(\bar{\psi}\psi) + O(g^3)\right],
 \end{aligned} \tag{5.30}$$

we have

$$P = \frac{\delta L_{tot}}{\delta \dot{Q}}, \quad p = \frac{\delta L_{tot}}{\delta \dot{q}}, \quad \frac{\delta L_{tot}}{\delta Q^{(r)}} = 0, \quad \frac{\delta L_{tot}}{\delta q^{(r)}} = 0 \quad (r > 1)$$

and the interaction Hamiltonian becomes from (4.4)

$$\mathcal{H}_I = -g^2 \bar{\psi}\psi f(\mu^2)\phi + \frac{1}{2}g^2 \bar{\psi}\psi \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2}(\bar{\psi}\psi) + O(g^2). \tag{5.13}$$

This is the same result as obtained in § 2.

From the above six examples, we can confirm that our procedures are equivalent with usual Heisenberg-Pauli's method of quantization using only one set of the canonical conjugates of each field.

There would arise some questions of course. The first will be whether we can describe the dynamical system correctly or not. And the second will be by what procedure we can give the differences from usual Ostrogradsky-like quantization.

From the stand point of the classical mechanics, the laws of the motion employed by us may not describe all of the given dynamical problems. It is usual to give the initial conditions as much as the numbers of time-derivatives involved in Lagrangian. On the

contrary, we give not such initial conditions, but the different ones. These conditions may involve the family of above stated initial conditions by which only one particular solution is given. With these initial conditions, we can only answer the way how this solution is obtained approximately in the course of the infinite past to the infinite future, that is, we can describe the time-variations of the canonical variables along the particular path in the sense of the canonical mechanics.

This description seems to be very peculiar at first sight, but may be natural from the stand point of the present field theory in which we can define the elementary particles independently from their interactions. In order to clarify this point more concretely, it is necessary to consider at the relations of the second problem, i.e., the differences from usual Ostrogradsky-like quantization.

We take the simplest case given in Example 5 of § 4. The given Lagrangian can be written as follows:

$$\begin{aligned} L_{tot} &= -\bar{\psi}(\gamma\partial + \alpha)\psi + g f(\square)\bar{\psi}f(\square)\psi + \partial w_\mu / \partial x_\mu \\ &= -\bar{\psi}(\gamma\partial + \mu) \left[1 - g \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} (\gamma\partial - \mu) \right] \psi + \frac{\partial w'_\mu}{\partial x_\mu}, \end{aligned} \quad (5.32)$$

where μ is the real root of the equation

$$g f^2(x^2) + x - \alpha = 0$$

and μ becomes α if $g \rightarrow 0$. This equation has other many roots μ_1, \dots, μ_{n-1} , if $f^2(x^2)$ is assumed to be n -th-order polynomial, and therefore using this fact the equation (5.32) becomes

$$L_{tot} = g \bar{\psi}(\gamma\partial + \mu)(\gamma\partial + \mu_1) \cdots (\gamma\partial + \mu_{n-1})\psi + \partial w'_\mu / \partial x_\mu, \quad (5.33)$$

where $\lim_{g \rightarrow 0} \mu = \alpha$ and $\lim_{g \rightarrow 0} g \mu_1 \cdots \mu_{n-1} = -1$.

If we follow with Ostrogradsky and Pais-Uhlenbeck's methods, we define the new variables

$$\psi_0 = [-F'(-\mu)]^{-1/2} \frac{F(\gamma\partial)}{(\gamma\partial + \mu)} \psi, \quad \psi_i = [-F'(-\mu)]^{-1/2} \frac{F(\gamma\partial)}{(\gamma\partial + \mu_i)} \psi \quad (5.34)$$

$$F(\gamma\partial) = g(\gamma\partial + \mu)(\gamma\partial + \mu_1) \cdots (\gamma\partial + \mu_{n-1}) \quad (i=1, \dots, n-1),$$

and the Lagrangian becomes

$$L_{tot} = -\bar{\psi}_0(\gamma\partial + \mu)\psi_0 - \sum_{i=1}^{n-1} \frac{F'(-\mu)}{F'(-\mu_i)} \bar{\psi}_i(\gamma\partial + \mu_i)\psi_i. \quad (5.35)$$

In this treatment, we are obliged to have $2n$ initial conditions in order to describe the system and then take quantization for these n sets of the canonical variables.

While, in our theory, we start with (5.32) not with (5.33) and modify it such as

$$\begin{aligned} L_{tot} &= - \left[1 + g \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} (\gamma\partial + \mu) \right]^{1/2} \bar{\psi}(\gamma\partial - \mu) \left[1 - g \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} (\gamma\partial - \mu) \right]^{1/2} \psi \\ &= - \left[\psi + \frac{1}{2} g \frac{f^2(\square) - f^2(\alpha^2)}{\square - \alpha^2} (\partial\bar{\psi}\gamma + \alpha\bar{\psi}) + O(g^2) \right] \end{aligned}$$

$$\times (\gamma\partial + \mu) \left[\phi - \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \phi + O(g^2) \right].$$

Then we take

$$\begin{aligned} \phi &= \phi - \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\gamma\partial - x) \phi + O(g^2), \\ \bar{\phi} &= \bar{\phi} + \frac{1}{2} g \frac{f^2(\square) - f^2(x^2)}{\square - x^2} (\partial\bar{\phi}\gamma + x\bar{\phi}) + O(g^2) \end{aligned} \quad (5.36)$$

as the independent canonical variables and go to the interaction representation.

Comparing both methods, we can understand the following remarkable differences:

- i) The field quantities which describe the elementary particle with mass μ are defined by different definitions (5.34) and (5.36). Therefore, our description is not equivalent with one taking only one set of the independent variables and dropping the other as in Ostrogradsky's method.
- ii) From the standpoint of quantization, we describe only the quantum effects of this set of the canonical variables which involve the classical effects of all variables as the particular solution, whereas Ostrogradsky's method describes quantum effects of all variables together with their classical effects. We consider only one set describes the elementary particle, whereas the other treats as if all variables correspond to different elementary particles.
- iii) However, Ostrogradsky's method can not have the picture of free field and its variations by the interaction, i.e., the interaction in this case, while it is possible to have such picture in our theory and if the interaction representation does exist, our treatment is unique answer.
- iv) Ostrogradsky's method can not avoid the so-called negative energy difficulty contrary to our's.

§ 6. The non-local interaction and the non-localized action

We have treated the cases of the interaction Lagrangian with higher derivatives, but the theory developed thus far does not limit the ranks and forms of derivatives in it. While the non-local interactions are considered formally as the infinite sum of such interactions and then can be also treated by this theory. However, as such infinite series are formal and are expressed essentially by the integral form, there are some questions concerning with this point, but if we can obtain the results not to be such infinite series, our treatment may be satisfactory.*

The Lagrangian of the non-local interaction is given generally as⁴⁾

$$\begin{aligned} \bar{L}_I &= g \int \int \int dx_1 dx_2 dx_3 F(x_1, x_2, x_3) \bar{\phi}(x_1) \phi(x_2) \Gamma\phi(x_3) \\ &= g \int \int \int dx dr_1 dr_2 F(r_1, r_2) \bar{\phi}\left(x + \frac{r_1}{2} + \frac{r_2}{2}\right) \phi\left(x - \frac{r_1}{2}\right) \Gamma\phi\left(x + \frac{r_1}{2} - \frac{r_2}{2}\right), \end{aligned} \quad (6.1)$$

* The similar treatment not using such infinite sum can be given. C. Hayashi, prepared paper.

where $x = \frac{1}{2} \left[\frac{x_1 + x_2}{2} + x_3 \right]$, $r_1 = \frac{x_1 + x_2}{2} - x_3$ and $r_2 = x_1 - x_3$. The function F is the so-called form factor and depends only on the relative coordinates r_1 and r_2 from the requirement of the displacement invariance.

* In order to bring the desired form, we use the Fourier transformation

$$F(r_1, r_2) = \frac{1}{(2\pi)^8} \iint dL dl G(L, l) e^{iLr_1 + ilr_2}$$

and have

$$\bar{L}_1 = g \int dx \{ dk_1 \{ dk_2 \{ dk_3 e^{-i(k_1 + k_2 + k_3)x} G\left(k_1 + k_3, \frac{1}{2}(k_1 - k_2)\right) \phi(-k_1) l \phi(k_2) \phi(k_3). \quad (6.2)$$

We understand that this form is generally equivalent with

$$L_1 = g \bar{O}(\partial) \bar{\phi} \Gamma O(\partial) \phi P(\partial) \phi. \quad (6.3)$$

As the special case, we take

$$F(r_1, r_2) = \delta(r_2) g(r_1) \quad \text{and} \quad g(r_1) = \frac{1}{(2\pi)^4} \int dq e^{iqr_1} f(-q^2)$$

and then (6.2) becomes ($\Gamma=1$)

$$L_1 = g \bar{\phi} \phi f(\square) \phi, \quad (6.4)$$

which is the familiar form treated before, whereas if we take

$$F(r_1, r_2) = g(r_2) \delta(r_1),$$

we have

$$L_1 = g O(\partial) \bar{\phi} O(\partial) \phi \cdot \phi. \quad (6.5)$$

We can discuss about the problem of the divergence difficulties from S -matrix. According to § 3, the Green functions of each field are modified such as

$$\begin{aligned} \bar{O}(-\partial) O(\partial) N(\partial) \underline{D_F} &\equiv \bar{O}(-\partial) O(\partial) N(\partial) \underline{D^{(1)}} - 2i \bar{O}(-\partial) O(\partial) N(\partial) \bar{D}, \\ P(-\partial) P(\partial) R(\partial) \underline{D_F} &\equiv P(-\partial) P(\partial) R(\partial) \underline{D^{(1)}} - 2i P(-\partial) P(\partial) R(\partial) \bar{D}, \end{aligned} \quad (6.6)$$

where the rambling underlines mean the procedure by which parts of the differential operators are reduced to constants in virtue of $\partial(\square - x^2)$ and $\partial(\square - \mu^2)$. \bar{D} and \bar{D} -functions are all modified in the non-local interaction, whereas $D^{(1)}$ and $D^{(1)}$ -functions are only partly modified and especially in the case of $P(\partial) = f(\square)$, we have

$$f^2(\square) R(\partial) D_F \equiv f^2(\mu^2) R'(\partial) D^{(1)} - 2i f^2(\square) R(\partial) \bar{D},$$

where $D^{(1)}$ -function is not entirely modified. Consequently, if we want to discuss the divergence problem, it is sufficient to consider how $D^{(1)}$ and $D^{(1)}$ -functions are modified in this theory. From this consideration, we can conclude i) any non-local interactions with the form factors depending only on one relative coordinate, i.e., two points of the coordinates, can not eliminate the divergence at all. For instance, the interaction (6.4)

does not modify $D^{(1)}$ -function and also (6.4) does not all boson kernels, ii) while, the suitable non-local interactions depending on three points have some possibilities for this aim, but the concrete analytic form has not been given upto now.*

Secondly we want to clarify the question why we can construct the Hamilton-formalism even if the non-local interaction exists. According to usual understanding, one can not describe step by step the non-locally interacting system and cannot construct such formalism. This understanding may be correct as far as we are concerned with Heisenberg representation. But if there does exist the interaction representation, the circumstances are not same in it as Heisenberg's. The canonical conjugates $Q(x/\sigma)$ and $P(x/\sigma)$ defined in this representation have not such non-localities, that is, though the quantities in Heisenberg representation would have such non-localities, the canonical variables $Q(x/\sigma)$ and $P(x/\sigma)$ are constructed by combing them so as to cancel out such non-localities. This is the reason why we can construct the Hamilton formalism in the non-locally interacting system. If it may be true, there appears new question whether the non-localities involving the system reflect themselves correctly into our description or not. It is very difficult to answer it, because we do not know what the theory reflecting correctly such non-localities. Of course, it is true that in our theory parts of non-localities (which are called usually "non-localities," because they associate higher time derivatives in Lagrangian) are thrown away. This fact appears in the modifications of Green functions in which parts of higher time derivatives are replaced by the constant or spacial derivatives using the properties of Green functions. But the other parts of non-localities which due to higher spacial derivatives do not affect the Hamilton formalism and are retained also in this theory.

One of the important conclusions from this Hamilton formalism is the establishment of differences between the non-local interaction and the non-localized action. Let us consider familiar simple example and take as Lagrangian

$$L_{tot} = \frac{1}{2} \phi(\square - \mu^2) \phi + g w f(\square) \phi, \quad (6.7)$$

where w is the source. Following to our theory, we take

$$\begin{aligned} q &= \phi + g \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} w + O(g^2), \\ p &= (n\partial)\phi + g(n\partial) \frac{f^2(\square) - f^2(\mu^2)}{\square - \mu^2} w + O(g^2), \quad \left(\dot{p} = \frac{\delta L_{tot}}{\delta \dot{q}} \right) \end{aligned} \quad (6.8)$$

as the set of the canonical conjugates and then construct the Hamiltonian in usual method. The Hamiltonian thus obtained does not more involve any time derivative, \ddot{q}, \dots and quantization is accomplished with this one set

$$[p, q] = -i.$$

* In their paper, Kristensen and Møller give only the cut-off character of the form-factor, but not its analytic form and Bloch does too. From usual deductions, the form factor obtained reduces divergence on one hand, while induces it on other hand, such as $e^{-\Pi^2}$.

While in the theory of the non-localized action, we replace the field quantity such as

$$B = f(\square)\phi \quad (6.9)$$

and then start with

$$L_{tot} = \frac{1}{2} B f^{-2}(\square) (\square - \mu^2) B + g' \mu B. \quad (6.10)$$

If $f^{-2}(\square) = \prod_{i=1}^n (\square - \mu_i^2)$, we can define the new quantities

$$\phi_0 = f^{-2}(\square) B (= f^{-1}(\square)\phi), \quad \phi_i = (\square - \mu_i^2) \frac{f^{-2}(\square)}{\square - \mu_i^2} B \left(= \frac{\square - \mu_i^2}{\square - \mu_i^2} f^{-1}(\square)\phi \right), \quad (i=1, \dots, n) \quad (6.11)$$

and Lagrangian becomes

$$L_{tot} = \frac{1}{2} \eta_i \phi_0 (\square - \mu_i^2) \phi_0 + \frac{1}{2} \sum_{i=1}^n \gamma_i \phi_i (\square - \mu_i^2) \phi_i + g' \mu (\gamma_0 \phi_0 + \sum_{i=1}^n \gamma_i \phi_i). \quad (6.12)$$

If we take

$$q_0 = \phi_0, \quad p_0 = \gamma_0 (n\partial) \phi_0; \quad q_i = \phi_i, \quad p_i = \gamma_i (n\partial) \phi_i, \quad (i=1, \dots, n) \quad (6.13)$$

as the sets of the canonical conjugates, we can construct the Hamiltonian. The Hamiltonian thus obtained does also no more involve any time derivative of the canonical variables and quantization is accomplished with these many sets

$$[p_0, q_0] = -i, \quad [p_i, q_i] = -i \quad (i=1, \dots, n).$$

Although both theories are similar at the point which we can define the interaction representations, there does exist the essential differences. In the former, we take only one set of the canonical variables, whereas the latter has many sets and if we want to bring into the quantum theory, the well-known negative energy difficulty appears in the latter contrary to the former.

In the former, we restrict ourselves with the same numbers of the sets involving in the free Lagrangian, whereas the latter take the possible numbers of the sets so as to connect the free field quantities with Heisenberg quantities by the unitary transformation and generally the numbers of the sets increase more than the former. Of course, as there does not exist any principles which determine the free parts uniquely, if we start with (6.12) at the beginning, we also arrive at the same results as the latter. But as far as the free Lagrangian being given, we can take uniquely the numbers of sets. In order to avoid the negative energy, we are obliged to take the free Lagrangian as (6.7) and have one set of the canonical conjugates.

The non-localized action without the negative energy does not increase the numbers of the canonical conjugates and therefore becomes to be equivalent to the non-local interaction, that is, in this case the separation between free parts and interaction parts is not accomplished uniquely and the definition of the free Lagrangian is ambiguous unless we introduce any other principles, such as the picture of harmonic oscillator proposed by Takahashi and Umezawa.

At the end of this chapter, we can conclude that we can treat the non-local interactions in this theory without fear for the negative energy.

§ 7. Conclusions

By using the extensions of Yang-Feldman's method to the system with the interaction Lagrangian with higher derivatives, we can obtain the Hamiltonian in the interaction representation. The Hamiltonian thus obtained is generally expressed by the infinite series of the coupling constant.

Next we can clarify that such Hamiltonian is nothing else than the one obtained by usual Heisenberg-Pauli's method using the canonical conjugates connected with the free ones by

$$Q_i(\equiv Q(x/\sigma)) = U^{-1}(\sigma, -\infty) Q^0(x) U(\sigma, -\infty)_{x/\sigma},$$

$$P_i(\equiv P(x/\sigma)) = U^{-1}(\sigma, -\infty) P^0(x) U(\sigma, -\infty)_{x/\sigma}.$$

That is, though the interaction Lagrangian involves higher time-derivatives, we can describe the motions of system with sets of the canonical variables involved in the free Lagrangian. This description is the unique conclusion, if the interaction representation does exist.

We can also extend this theory to the case of the non-local interaction straightforwardly, and understand why the Hamilton formalism does exist in the interaction representation. The Hamiltonian in this case is obtained by throwing away parts of the non-localities in the system, but their other parts are retained. If we construct S -matrix, we obtain the same results in the usual S -matrix descriptions of the non-local interaction.

We can compare the non-local interaction with the non-localized action in this theory and clarify why the non-local interaction can avoid the negative energy difficulty.

From the results of this theory, we can discuss the possibilities of the removals of the divergences in the field theory by introducing the non-local interaction and conclude that we can expect any form factors depending on three points have such possibilities.

In conclusion, we wish to express our cordial thanks to Prof. M. Kobayasi for his kind guidances and encouragements and also to Dr. H. Umezawa and Mr. Y. Takahashi for their valuable discussions.

Appendix

The proof of $-\frac{1}{2}[F(\partial), \epsilon(x-x')] \Delta(x-x') = [F(\partial)]_x \delta(x-x')$

We begin to consider certain product of the differential operators $\partial_{\mu_1} \cdots \partial_{\mu_r}$, and using the definitions

$$\partial_\mu = \partial_\mu^s - n_\mu(n\partial), \quad n_\mu = (0, 0, 0, i),$$

we can expand it as follows:

$$\partial_{\mu_1} \cdots \partial_{\mu_r} = \sum_{\substack{k=0 \\ \text{cyclic}}}^r (-1)^k n_{\mu_1} \cdots n_{\mu_k} \partial_{\mu_{k+1}}^s \cdots \partial_{\mu_r}^s (n\partial)^k.$$

While we have the relations

$$-\frac{1}{2}[(n\partial)^k, \varepsilon(x-x')] \Delta(x-x') = \begin{cases} -\frac{(n\partial)^k - (\Delta-x^2)^{k/2}}{\square - x^2} \delta(x-x') & \text{for } k \text{ even,} \\ -\frac{1}{\square - x^2} \left[(n\partial)^k - \frac{(n\partial)}{(\Delta-x^2)^{1/2}} (\Delta-x^2)^{k/2} \right] \delta(x-x') & \text{for } k \text{ odd.} \end{cases}$$

in virtue of the characters of Green function.

Then we have

$$\begin{aligned} -\frac{1}{2}[\partial_{\mu_1} \cdots \partial_{\mu_r}, \varepsilon(x-x')] \Delta(x-x') &= -\frac{1}{\square - x^2} \sum_{k=0}^r (-1)^k n_{\mu_1} \cdots n_{\mu_k} \partial_{\mu_{k+1}}^s \cdots \partial_{\mu_r}^s \\ &\quad \times \left[(n\partial)^k - \frac{1}{2} \left\{ (1 + (-1)^k) + (1 - (-1)^k) \frac{(n\partial)}{(\Delta-x^2)^{1/2}} \right\} \right] \delta(x-x') \\ &= -\frac{1}{X_{\kappa}^+ X_{\kappa}^-} \left[\partial_{\mu_1} \cdots \partial_{\mu_r} - \frac{1}{X_{\kappa}^- + X_{\kappa}^+} \{ X_{\kappa}^+ (\partial - n X_{\kappa}^-)_{\mu_1} \cdots (\partial - n X_{\kappa}^-)_{\mu_r} \right. \\ &\quad \left. + X_{\kappa}^- (\partial + n X_{\kappa}^+)_{\mu_1} \cdots (\partial + n X_{\kappa}^+)_{\mu_r} \} \right] \delta(x-x'). \end{aligned}$$

where $X_{\kappa}^{\pm} = (\Delta - x^2)^{1/2} \pm (n\partial)$.

Consequently, we get

$$\begin{aligned} -\frac{1}{2}[F(\partial), \varepsilon(x-x')] \Delta(x-x') \\ = -\frac{1}{X_{\kappa}^+ X_{\kappa}^-} \left[F(\partial) - \frac{1}{X_{\kappa}^+ + X_{\kappa}^-} \{ X_{\kappa}^- F(\partial - n X_{\kappa}^-) + X_{\kappa}^+ F(\partial + n X_{\kappa}^+) \} \right] \delta(x-x') \end{aligned}$$

for general differential operator $F(\partial)$, and the proof is verified.

As the important results from this equation, we indicate some examples as follows:

$$F(\partial) = \partial_{\mu} \partial_{\nu} : [\partial_{\mu} \partial_{\nu}] = n_{\mu} n_{\nu},$$

$$F(\partial) = \partial_{\lambda} \partial_{\mu} \partial_{\nu} : [\partial_{\lambda} \partial_{\mu} \partial_{\nu}] = n_{\lambda} n_{\mu} \partial_{\nu} + n_{\mu} n_{\nu} \partial_{\lambda} + n_{\nu} n_{\lambda} \partial_{\mu} + 2n_{\lambda} n_{\mu} n_{\nu} (n\partial),$$

$$F(\partial) = f(\square) : [f(\square)] = -\frac{f(\square) - f(x^2)}{\square - x^2},$$

$$\begin{aligned} F(\partial) = f(\square) g(\partial) : [f(\square) g(\partial)] &= -g(\partial) \frac{f(\square) - f(x^2)}{\square - x^2} \\ &\quad - f(x^2) \frac{1}{X_{\kappa}^+ X_{\kappa}^-} \left[g(\partial) - \frac{1}{X_{\kappa}^+ + X_{\kappa}^-} \{ X_{\kappa}^+ g(\partial - n X_{\kappa}^-) + X_{\kappa}^- g(\partial + n X_{\kappa}^+) \} \right] \end{aligned}$$

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On the Interaction of Additive Electrons with the Polarization in Ionic Crystals, II

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§ 5. Discussions on the first method

(1) *Relations to Kubo's method*

As has been stated in the preceding section, the method which has been obtained there on the assumption of applicability of adiabatic approximation to the additive and crystal electrons is equivalent to that which Kubo²⁾ has derived intuitively on the basis of classical electrostatics, but this holds only when we concern energy. While Kubo described polarizations with two constants α and α_0 , we characterized dielectric properties with three constants α , α_0 and e^* . Nevertheless we came to the result that the minimized energy depends on α , α_0 only and not on e^* (see e.g. (4.36) and (4.40)). That is a rather natural result, because α and α_0 are intimately connected with energy while e^* is of a mechanical nature. Consequently we expect e^* to play a rôle in mechanical processes such as displacements of ions, lattice vibrations and half-breadths of absorption spectra.

In fact, e^* appears in (4.35'), the equation which determines displacement polarization q_2 . On the other hand Kubo resolves two sorts of polarizations according to the ratio

$$q_1 : q_2 = \chi_0 : \chi - \chi_0,$$

but this is justified only in the special case when $c=0$ as is seen from (3.2), (3.3) and (3.4), that is, when

$$(\alpha_0 + 2/3)e^* = 1.$$

In that case (4.21) takes the form

$$\{K + 1/\alpha_0 \cdot V(\mathbf{r}) - 1/\alpha_0 \cdot \varphi_2(\mathbf{r})\} \psi(\mathbf{r}, q_2) = \varepsilon \psi(\mathbf{r}, q_2),$$

and (4.35') becomes

$$q_2 = 1/4\pi \cdot (1 - \alpha_0/\alpha) (\tilde{E}_e + \tilde{E}_q),$$

which are quite in agreement with Kubo's procedure.

Thus our first method leads to different results from those derived on Kubo's method, so far as the quantity other than energy is concerned.

§ 6. The second method for treating the motion of an additive electron

While we have selected, in § 4, the form (4.6) for the wave function $\Phi(\mathbf{r}, \mathbf{r}_j; \mathbf{X}_m)$ assuming the applicability of adiabatic approximation between \mathbf{r} and \mathbf{r}_j , we now take the Hartree's form in which Φ separates into two wave functions, of \mathbf{r} and \mathbf{r}_j , respectively:

$$\Phi(\mathbf{r}, \mathbf{r}_j; \mathbf{X}_m) = \phi(\mathbf{r}_j; \mathbf{X}_m) \phi(\mathbf{r}; \mathbf{X}_m). \quad (6.1)$$

The extremum condition of the expression

$$E(\phi, \phi; \mathbf{X}_m) = \iint \phi \psi H \phi \psi d\mathbf{r}_j d\mathbf{r}, \quad (6.2)$$

which is equivalent to the eigen-equation (4.3), is then reduced to a set of two equations:

$$\delta E(\phi, \phi; \mathbf{X}_m) / \delta \phi = 0, \quad \int \phi^2 d\mathbf{r}_j = 1, \quad (6.3)$$

$$\delta E(\phi, \phi; \mathbf{X}_m) / \delta \psi = 0, \quad \int \psi^2 d\mathbf{r} = 1. \quad (6.4)$$

Using thus determined ϕ and ψ , (6.2) can be calculated as a function of \mathbf{X}_m :

$$E(\phi(\cdot; \mathbf{X}_m), \psi(\cdot; \mathbf{X}_m); \mathbf{X}_m) = E(\mathbf{X}_m).$$

The variation process (4.5) with the above obtained $E(\mathbf{X}_m)$ will give the equilibrium positions of the ions.

The set of processes (6.3), (6.4) and (4.5) is nothing but the variational procedure of $E(\phi, \psi; \mathbf{X}_m)$ as to all of ϕ , ψ and \mathbf{X}_m , the order of the procedure being immaterial. In the following we shall minimize E as regards to ϕ and \mathbf{X}_m , and then solve the variational problem in ψ .

Inserting (4.1) into (6.2) we obtain

$$\begin{aligned} E(\phi, \psi; \mathbf{X}_m) &= \int \phi(\mathbf{r}_j) \widetilde{H}_p(\mathbf{r}_j; \mathbf{X}_m, \psi) \phi(\mathbf{r}_j) d\mathbf{r} \\ &+ \int \phi(\mathbf{r}) (K + V) \phi(\mathbf{r}) d\mathbf{r}, \end{aligned} \quad (6.5)$$

where

$$\widetilde{H}_p(\mathbf{r}_j; \mathbf{X}_m, \psi) = H_e + \widetilde{V}_e + \widetilde{V}_n + W_e + W_n, \quad (6.6)$$

$$\widetilde{V}_e(\mathbf{r}_j; \psi) = \int \phi(\mathbf{r}) V_e(\mathbf{r}_j, \mathbf{r}) \phi(\mathbf{r}) d\mathbf{r},$$

$$\widetilde{V}_n(\mathbf{X}_m; \psi) = \int \phi(\mathbf{r}) V_n(\mathbf{X}_m, \mathbf{r}) \phi(\mathbf{r}) d\mathbf{r}. \quad (6.7)$$

Carrying out variational process (6.3), we are led to

$$\widetilde{H}_p(\mathbf{r}_j; \mathbf{X}_m, \psi) \phi(\mathbf{r}_j) = E_p \phi(\mathbf{r}_j)$$

with E_p as a parameter. This is the wave equation for the crystal electrons when there are fixed charge distribution Q and the electronic charge distribution $\tilde{\rho}_e = -\psi^2$ (ψ and \mathbf{X}_m being frozen). Further, by minimizing thus obtained expression

$$E(\phi(\mathbf{r}_j; \mathbf{X}_m, \psi), \psi; \mathbf{X}_m) = E(\psi; \mathbf{X}_m)$$

with respect to \mathbf{X}_m , we obtain the equilibrium positions of the nuclei, ψ being again frozen.

Consequently, if we can describe the essential aspect of this situation by polarization alone, in other words, if we can select the trial function ϕ and the ion displacements \mathbf{X}_m such that they correspond to the macroscopic state \mathbf{q}_1 and \mathbf{q}_2 , the first integral of (6.5) can be replaced as

$$\int \phi(\mathbf{r}_j) \tilde{H}_p(\mathbf{r}_j; \mathbf{X}_m, \psi) \phi(\mathbf{r}_j) d\mathbf{r}_j = U(\mathbf{q}_1, \mathbf{q}_2, \tilde{\mathbf{E}}_0), \quad (6.8)$$

by considering the meanings of \tilde{H}_p in (6.6), (6.7) and of U in § 2. Of course the zero point of H_c is suitably chosen so that the constant which should appear on the right hand side of (6.8) vanishes. $\tilde{\mathbf{E}}_0$ in U of (6.8) means

$$\tilde{\mathbf{E}}_0 = \tilde{\mathbf{E}}_e + \mathbf{E}_e, \quad (6.9)$$

where $\tilde{\mathbf{E}}_e$, the electrostatic field due to the charge distribution $-\psi^2$, is given in the same form as in (4.27). The difference between (4.10) and (6.8) consists in the electric fields \mathbf{E}_e and $\tilde{\mathbf{E}}_e$ corresponding to H_p and \tilde{H}_p ; this takes away, in the latter case, the difficulty of the divergent integral which appeared in the former.

By (6.5) and (6.8) we have

$$\begin{aligned} E(\phi, \psi; \mathbf{X}_m) &= E(\phi, \mathbf{q}_1, \mathbf{q}_2) \\ &= \int \phi(\mathbf{r}) (K + V) \phi(\mathbf{r}) d\mathbf{r} + U(\mathbf{q}_1, \mathbf{q}_2; \tilde{\mathbf{E}}_0(\phi)). \end{aligned} \quad (6.10)$$

If we solve

$$\left. \begin{aligned} \delta E(\phi, \psi, \mathbf{X}_m) / \delta \phi &= 0, \quad \text{that is,} \quad \delta E(\phi, \mathbf{q}_1, \mathbf{q}_2) / \delta \mathbf{q}_1 = \delta U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{E}_0) / \delta \mathbf{q}_1 = 0, \\ \delta E(\phi, \psi, \mathbf{X}_m) / \delta \mathbf{X}_m &= 0, \quad \text{that is,} \quad \delta E(\phi, \mathbf{q}_1, \mathbf{q}_2) / \delta \mathbf{q}_2 = \delta U(\mathbf{q}_1, \mathbf{q}_2, \mathbf{E}_0) / \delta \mathbf{q}_2 = 0, \end{aligned} \right\} \quad (6.11)$$

with E given by (6.10), we obtain

$$\mathbf{q}_1 = \frac{b-c}{ab-c^2} (\tilde{\mathbf{E}}_0 - \nabla \phi_p) = \frac{b'-c'}{a'b'-c'^2} (\tilde{\mathbf{E}}_0 + \nabla \times \mathbf{A}), \quad (6.12)$$

$$\mathbf{q}_2 = \frac{a-c}{ab-c^2} (\tilde{\mathbf{E}}_0 - \nabla \phi_p) = \frac{a'-c'}{a'b'-c'^2} (\tilde{\mathbf{E}}_0 + \nabla \times \mathbf{A}), \quad (6.13)$$

by (2.9) and (2.9'). Operating $\nabla \times$ on the first equation of (6.12) and (6.13), and considering $\nabla \times \tilde{\mathbf{E}}_0 = 0$, we are led to

$$\nabla \times \mathbf{q}_1 = \nabla \times \mathbf{q}_2 = 0, \quad \nabla \times \mathbf{P} = 0, \quad (6.14)$$

or by (2.6) to

$$\nabla \times \mathbf{A} = 0. \quad (6.15)$$

Inserting (6.15) into (6.12) and (6.13), we have

$$\mathbf{q}_1 = \frac{b' - c'}{a'b' - c'^2} \tilde{\mathbf{E}}_0, \quad (6.12')$$

$$\mathbf{q}_2 = \frac{a' - c'}{a'b' - c'^2} \tilde{\mathbf{E}}_0 = \left\{ \frac{3}{(x_0 + 2)e^*} \right\} \frac{1}{4\pi} \left(1 - \frac{x_0}{x} \right) \tilde{\mathbf{E}}_0, \quad (6.13')$$

Comparing (6.13') with (4.35) and (4.35'), we see that the relation between the displacement polarization and the wave function of the additive electron takes the same form whether one may take the approximation of § 4 or that of this section.

Substituting (6.12'), (6.13') and (6.15) into (2.3') and using (3.8), we have

$$\begin{aligned} U(\mathbf{q}_1(\psi), \mathbf{q}_2(\psi), \tilde{\mathbf{E}}_0(\psi)) &= U(\psi) \\ &= -\frac{1}{2} \frac{a' + b' - 2c'}{a'b' - c'^2} \int \tilde{\mathbf{E}}_0^2 dv = -\frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int \tilde{\mathbf{E}}_0^2 dv. \end{aligned} \quad (6.16)$$

By (6.10), (6.16) and (6.9), the energy E is expressed as follows:

$$\begin{aligned} E(\psi, \mathbf{q}_1(\psi), \mathbf{q}_2(\psi)) &= E(\psi) \\ &= \int \psi(\mathbf{r}) (K + V) \psi(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int (\tilde{\mathbf{E}}_e^2 + 2\tilde{\mathbf{E}}_e \cdot \mathbf{E}_Q) dv \\ &\quad - \frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int \mathbf{E}_Q^2 dv. \end{aligned}$$

When the additive electron is outside of the crystal and has energy zero, the first and the second integrals vanish, there remaining only the third integral which is the polarization energy due to the fixed charge distribution Q . Choosing this state as energy zero we have, instead of the above, the following expression:

$$E(\psi) = \int \psi(\mathbf{r}) (K + V) \psi(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int (\tilde{\mathbf{E}}_e^2 + 2\tilde{\mathbf{E}}_e \cdot \mathbf{E}_Q) dv, \quad (6.17)$$

or by the same transformation as in § 4, the alternative form:

$$E(\psi) = \int \psi(\mathbf{r}) \left(K + \frac{1}{x} V \right) \psi(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int \tilde{\mathbf{E}}_e^2 dv. \quad (6.17')$$

Finally we have to solve

$$\delta E(\psi) / \delta \psi = 0, \quad \int \psi^2 d\mathbf{r} = 1, \quad (6.18)$$

which leads to the simultaneous equations

$$\left\{ \left(K + \frac{1}{x} V(\mathbf{r}) + \left(1 - \frac{1}{x} \right) \tilde{\varphi}_e(\mathbf{r}) \right) \psi(\mathbf{r}) = \lambda \psi(\mathbf{r}), \right. \quad (6.19)$$

$$\Delta \tilde{\varphi}_e(\mathbf{r}) = +4\pi \{ \psi(\mathbf{r}) \}^2, \quad (6.20)$$

with a parameter λ .

Let us now turn to a non-equilibrium state B' to which the system has come immediately after an optical transition from the state A . We use the notations shown in Fig. 3. By Franck-Condon principle and eq. (6.13'), we have

$$q_2' = q_2 = \frac{a' - c'}{a'b' - c'^2} \tilde{E}_0. \quad (6.21)$$

We have to replace the quantities \tilde{E}_0 , \tilde{q}_1 , \tilde{q}_2 and ψ in (6.10) by the corresponding primed quantities and then to minimize the expression with respect to q_1' and ψ' , keeping the value q_2' to that give by (6.21).

Solving the variational equation

$$\delta E'(\psi', q_1', q_2') / \delta q_1' = \delta U'(q_1', q_2', \tilde{E}_0') / \delta q_1' = 0, \quad (6.22)$$

we have, with the help of (2.9) and (2.9'),

$$-\tilde{E}_0' + aq_1' + cq_2' + \nabla \varphi_p' = 0, \quad (6.23)$$

$$-\tilde{E}_0' + a'q_1' + c'q_2' - \nabla \times \mathbf{A}' = 0. \quad (6.23')$$

Operating $\nabla \times$ on the both sides of (6.23) and taking into account that $\nabla \times \tilde{E}_0 = \nabla \times \tilde{E}_0' = 0$ identically, we see that $\nabla \times q_1'$ vanish, which, in turn, assures vanishings of $\nabla \times \nabla \times \mathbf{A}'$ and $\nabla \times \mathbf{A}'$, as one can easily verify by operating $\nabla \times$ on (6.23'). Thus we have

$$q_1' = 1/a' \cdot (\tilde{E}_0'(\psi') - c'q_2'(\psi')). \quad (6.24)$$

With q_1' and q_2' given by (6.24) and (6.21), respectively, and taking into account vanishing of $\nabla \times \mathbf{A}'$, the energy U' can be expressed in an integral quadratic form in \tilde{E}_0 and \tilde{E}_0' , if one uses (2.2') in which all the quantities should be attached primes corresponding to the state B' . Next we insert thus obtained U' into (6.10), then decompose \tilde{E}_0 and \tilde{E}_0' in the manner shown in (6.9), finally choose the same zero point of energy as we have done in the state A . The coefficients can be written in terms of x and x_0 by using (3.8). Thus we get to the following expressions for the energy E' of (6.10) :

$$\begin{aligned} E'(\psi'; \psi) = & \int \psi'(r) (K + V) \psi(r) dr \\ & - \frac{1}{8\pi} \left(1 - \frac{1}{x_0}\right) \int \tilde{E}_e'^2 dv - \frac{1}{4\pi} \left(1 - \frac{1}{x}\right) \int \tilde{E}_e' \cdot \mathbf{E}_Q dv \\ & - \frac{1}{4\pi} \left(\frac{1}{x_0} - \frac{1}{x}\right) \int \tilde{E}_e' \cdot \tilde{E}_e dv + \frac{1}{8\pi} \left(\frac{1}{x_0} - \frac{1}{x}\right) \int \tilde{E}_e^2 dv \end{aligned} \quad (6.25)$$

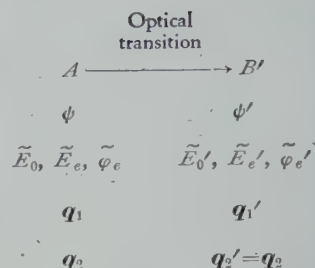


Fig. 3. Notations of the various quantities in the states A and B' .

$$\begin{aligned}
&= \int \phi'(\mathbf{r}) \left(K + \frac{1}{x} V \right) \phi'(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi} \left(1 - \frac{1}{x_0} \right) \int \tilde{\mathbf{E}}_e'^2 d\mathbf{v} \\
&\quad - \frac{1}{4\pi} \left(\frac{1}{x_0} - \frac{1}{x} \right) \int \tilde{\mathbf{E}}_e' \cdot \tilde{\mathbf{E}}_e d\mathbf{v} + \frac{1}{8\pi} \left(\frac{1}{x_0} - \frac{1}{x} \right) \int \tilde{\mathbf{E}}_e^2 d\mathbf{v} \quad (6.25')
\end{aligned}$$

$$\begin{aligned}
&= \int \phi'(\mathbf{r}) \left(K + \frac{1}{x} V \right) \phi'(\mathbf{r}) d\mathbf{r} - \frac{1}{8\pi} \left(1 - \frac{1}{x} \right) \int \tilde{\mathbf{E}}_e'^2 d\mathbf{v} \\
&\quad + \frac{1}{8\pi} \left(\frac{1}{x_0} - \frac{1}{x} \right) \int (\tilde{\mathbf{E}}_e' - \tilde{\mathbf{E}}_e)^2 d\mathbf{v}. \quad (6.25'')
\end{aligned}$$

We have only to solve

$$\delta E'(\psi', \phi) / \delta \psi' = 0 \quad (6.26)$$

with E' of (6.25) or (6.25'). Using the latter expression we obtain, after appropriate transformations of the integrals, the simultaneous equations

$$\left\{ K + \frac{1}{x} V(r) + \left(\frac{1}{x_0} - \frac{1}{x} \right) \tilde{\varphi}_e(r) + \left(1 - \frac{1}{x_0} \right) \tilde{\varphi}_e'(r) \right\} \psi'(r) = \lambda' \psi'(r), \quad (6.27)$$

$$4\tilde{\varphi}_e'(r) = +4\pi \{ \psi'(r) \}^2 \quad (6.28)$$

with λ' as a parameter.

We can state the methods obtained in this section as follows: In order to know the energy E and the wave function ϕ of the equilibrium state A , we insert trial functions ϕ into (6.17) or (6.17') and then carry out (6.18) with thus calculated $E(\phi)$; or alternatively, one may solve the equations (6.19) and (6.20) simultaneously. In order to know E' and ψ' of the nonequilibrium state B' immediately after the optical transition from the state A , one has to calculate (6.25) or (6.25') with the known ϕ of the state A and the trial function ψ' of the state B' , and then to carry out (6.26). One may choose an alternative method, namely to solve (6.27) and (6.28) selfconsistently.

Comparing (6.17) and (6.17') with (4.36) and (4.36'), (6.19) with (4.38), (6.25) and (6.25') with (4.40) and (4.40'), and finally (6.27) with (4.42), we see at once that some of the factors $1/x_0$ in § 4 is replaced by unity in this section. All the terms which appear only in § 6 contain the factor $(1 - 1/x_0)$, and therefore disappear in § 4, owing to the replacement $(1 - 1/x_0) \rightarrow (1/x_0 - 1/x_0)$. In the case $x_0 = 1$, both methods agree exactly; this is a natural result because the difference between the two methods consists in the treatments of the electronic polarization.

It is worth while to note that in the state B' too, we have two simultaneous equations (6.27) and (6.28) to solve, while in the method of § 4 we have only to solve the ordinary Schrödinger equation (4.42), though in actual applications we shall choose the more convenient variation method (6.26). In other words, the characteristics of many-body problems (in our case the additive and the crystal electrons) manifest themselves even in the final equations (6.27) and (6.28); this is the most essential difference between the method of this section and that of § 4, and is closely related to the problem which we shall discuss in § 8.

It is also worth mentioning that the effective charge e^* appears together with α and α_0 in the course of deductions, but that it is not contained in the final energy expressions. This is to be attributed to the same reason as has been stated in § 5 (1).

As for the last integral of positive definite form in (6.25''), one can interpret it in exactly the same manner as in (4.40'').

The generalization of the above formulation to the case of n additive electrons is obvious: in (6.17) for example, one has only to replace $K \rightarrow \sum_i^n K_i$, $V(r) \rightarrow \sum_i^n V(r_i) + \sum_{i < k}^n 1/r_{ik}$ and take \tilde{E}_e as the electric field due to these n electrons.

§ 7. The applications of the second method to some simple centers

The method discussed in the preceding section, as well as the usual one deduced in § 4, is based on a continuum dielectric model, so that we cannot expect it to give quantitatively accurate results when applied to the problems of color-centers and impurity centers. Nevertheless it is very interesting and worth while to study semiquantitatively the results of applications to some simple centers, especially the differences between the *optical* and *thermal* processes, and the dependences of various quantities on the dielectric constants α and α_0 . For the most essential purpose of this paper consist in studying the effects of two sorts of polarizations on the motion of the additive electrons. In the following we shall apply the method of § 6 to conduction electrons, polarons, F -centers and F' -centers with a simple model, and calculate the energies of various thermal and optical processes as functions of α and α_0 .

(i) *The self-trapped electron as the first approximation to the polaron state*

Nowadays the concept of the self-trapped state of an electron in an ionic crystal, the existence of which was suggested by Landau,⁷⁾ has to be replaced by that of the polaron state in which the electron moves through the crystal under interaction with the phonon field with an effective mass not so far from that of the conduction electron.⁸⁾ Nevertheless the former is a natural starting point or the first approximation for the latter, if we apply the method of § 6 which does not take into account the kinetic energies of the nuclei. Thus it seems desirable, and even necessary when we deal with the thermal ionizations of the F - and F' -centers, to calculate the energy of the self-trapped state with our static approximation.

Taking an everywhere smooth localized function

$$\phi = (a^3/56\pi)^{1/2} (1 + ar/2) \exp(-ar/2)$$

as the trial function for the self-trapped electron, and considering $V=0$, we can calculate (6.17'), with the help of (4.27), as

$$E(\phi) = E(a) = 3/56 \cdot a^2 - 5373/50176 (1 - 1/\alpha) a.$$

After minimization (6.18), we have

$$a = 1791/1792 (1 - 1/\alpha),$$

$$E_p = -0.0535 (1 - 1/\alpha)^2.$$

The average radius of the electron cloud is given by

$$\bar{r} = \int r \psi^2 d\mathbf{r}.$$

The values of the energies E and the average radii \bar{r} , of this state and all the other states of the additive electron which we shall describe in the following, are given in Tab. 1 for a typical case $\alpha=5$ and $\alpha_0=2$, the interrelations of these states being illustrated schematically in Fig. 4(a) and 4(b).

(ii) *The conduction electron*

According to the stand-point of our method the conduction electron should be considered to be accompanied by the electronic polarization which it produces around itself, but *not* by the displacement polarization if one wishes to distinguish this state from the polaron state. If we take, however, for ψ a plane wave which extends throughout the crystal, the electric field \tilde{E}_e vanishes and no polarization is produced. On the other hand we can, as the first approximation to the conduction electron, take a localized ψ with electronic polarization around it, and in the next approximation consider the resonance of all these states ψ 's which are located on the lattice points. This localized ψ and the energy E_c' of the conduction electron as the first approximation can be obtained by replacing α by α_0 in the results of (i), or more formally by using (6.25') and (6.26') with $V=0$, $\tilde{E}_e=0$ (the state now considered belongs to B' of § 6). Thus we have

$$-\chi = E_c' = -0.0535(1 - 1/\alpha_0),$$

where χ is the work function of the conduction electron.

The above explained picture of the polaron and the conduction electron as the first approximation may seem somewhat curious or artificial, and actually there is no such clear-cut distinction between the both states as is suggested by the work of Fröhlich,⁽⁸⁾ Pelzer and Zienau. But the present treatments, especially with respect to the above stated distinction, will prove very important in the following discussions of the ionization processes of the F - and F' -centers.

(iii) *The F -center*

In ionic crystals consisting of singly valenced positive and negative ions such as alkali-halides and silver-halides, we can take, as simply as possible,

$$V = -1/r$$

for the potential due to the single negative ion vacancy. The energies of the various states of the F -center are calculated as follows.

(a) The ground state in equilibrium. Assuming the ground state to be of the hydrogen $1s$ -type:

$$\psi(\mathbf{r}) = (a^3/8\pi)^{1/2} \exp(-a\mathbf{r}/2),$$

(6.17') can be calculated as

$$E(\psi) = E(a) = a^2/8 - \{1/2 - 11/32(1 - 1/\alpha)\}a.$$

Minimization as to α gives

$$\alpha = 2 - 11/8(1 - 1/x),$$

$$E_F = -1/8 \cdot \{2 - 11/8(1 - 1/x)\}^2.$$

The thermal dissociation energy W_F for releasing the electron from the ground state of the F^- -center to the polaron state is given by

$$W_F = E_p - E_F = 1/8 \{2 - 11/8(1 - 1/x)\}^2 - 0.0535(1 - 1/x)^2.$$

(b) Trapping of the conduction electron by a negative ion vacancy with optical transition to the ground state. The state immediately after the transition belongs to B' of § 6, so that we must use (6.25'), where we have to take $\tilde{E}_e = 0$, because the electron has not been in the neighbourhood of the vacancy before the transition. With the same form of the trial function ψ' as in (a), we have

$$E'(\psi', 0) = E'(\alpha') = \alpha'^2/8 - \alpha'/2x - 5/32(1 - 1/x_0)\alpha',$$

the minimum of which is given by

$$\alpha' = 2/x + 5/8(1 - 1/x_0),$$

$$E_F'(E_C') = -1/2 \{1/x + 5/16(1 - 1/x_0)\}^2.$$

Here the notation E_F' means the ground state immediately after the optical transition, the state before which is shown in the bracket— E_C' , that is, the conduction electron in this case. The photon energy emitted in the process is equal to the difference of E_C' and $E_F'(E_C')$.

(c) The photoionization of the F^- -center. Immediately after the optical transition there are produced a conduction electron of energy E_C' and a negative vacancy, not in equilibrium, of energy $E_-(E_F)$. We apply (6.25') to the latter system using \tilde{E}_e of (a), and obtain

$$E_-(E_F) = 1/8\pi(1/x_0 - 1/x) \int \tilde{E}_e^2 dv = 5/16(1/x_0 - 1/x) \{1 - 11/16(1 - 1/x)\}.$$

The energy of the total system is $E_C' + E_-(E_F)$, and the photoionization energy I_F is given by

$$I_F = E_C' + E_-(E_F) - E_F.$$

(d) The excited state in equilibrium. Assuming the excited state to be of the hydrogen $2p$ -type:

$$\psi(\mathbf{r}) = (\beta^5/32\pi)^{1/2} r \exp(-\beta r/2) \cos \theta,$$

We can calculate the integral of \tilde{E}_e^2 conveniently by the decomposition

$$4\pi\tilde{\rho}_e = \rho_0(r)P_0(\cos \theta) + \rho_2(r)P_2(\cos \theta),$$

$$\tilde{\varphi}_e = \varphi_0(r)P_0(\cos \theta) + \varphi_2(r)P_2(\cos \theta),$$

and thus obtain for (6.17'),

$$E(\psi) = E(\beta) = \beta^2/8 - \{1/4x + (1-1/x)501/5120\}\beta.$$

To minimize this we have to take

$$\beta = 1 - 779/1280(1 - 1/x),$$

so that the energy is given by

$$E_F^* = -1/8\{1 - 779/1280 \times (1 - 1/x)\}^2.$$

The thermal dissociation energy W_F^* for releasing the electron from this excited equilibrium state is given by

$$\begin{aligned} W_F^* &= E_P - E_F^* \\ &= -0.0535(1 - 1/x)^2 \\ &\quad + 1/8\{1 - 779/1280 \\ &\quad \times (1 - 1/x)\}^2, \end{aligned}$$

which is positive or negative according as x is smaller or larger than 4.81.

(e) The F' -absorption. For the $2p$ -state immediately after the optical transition from the ground state (a), $(6.25')$ can be calculated as-

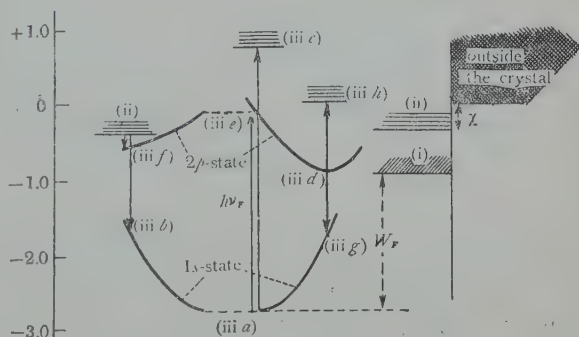
$$\begin{aligned} E'(\phi', \psi) &= \beta'^2/8 - 1/4x \cdot \beta' \\ &\quad - (1 - 1/x_0)501/5120 \cdot \beta' \\ &\quad - (1/x_0 - 1/x)\{\beta'/4 \\ &\quad - 1/4 \cdot \beta'^5/(u + \beta')^5(3u \\ &\quad + \beta')\} + 5/32 \\ &\quad \times (1/x_0 - 1/x)u. \end{aligned}$$

The minimum value $E_F^{*'}(E_F)$ of this expression as to β' , and the corresponding value of β' can be calculated only numerically, the results being shown in Fig. 5 together with those of the states (a) and (d). For the photon

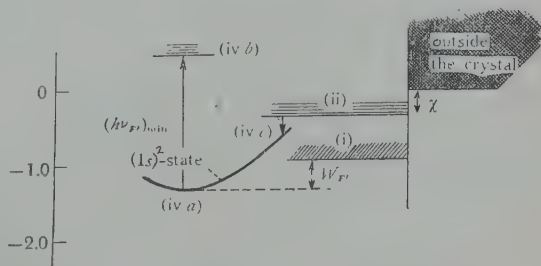
Table 1. The calculated energies and average radii of the various states of an additive electron ($x=5$, $x_0=2$)

States	Notation	Energy (eV)	\bar{r} (Å)
(i)	E_P	-0.93	(2.84)
(ii)	E'_G	-0.36	(4.53)
(iii) a	E_F	-2.76	1.76
b	$E'_F(E'_G)$	-1.73	2.23
c	$E'_G + E'_-(E_F)$	+0.78	(ii)
d	E_F^k	-0.90	5.16
e	$E_F^{*'}(E_F)$	-0.38	4.67
f	$E_F^{*'}(E'_G)$	-0.53	6.67
g	$E'_F(E_F^*)$	-1.71	2.14
h	$E'_G + E'_-(E_F^*)$	+0.05	(ii)
(iv) a	$E_{F'I}$	-4.09 (-1.33)†	2.05
b	$E'_G + E'_F(E_{F'I})$	-2.29 (+0.47)†	1.61, (ii)
c	$E_{F'I}'(E'_G + E_{F'I})$	-3.35 (-0.59)†	2.40

† In the two-electron system it is more convenient to refer the energy to that of the state consisting of an F -center in equilibrium (iii d) and an electron outside the crystal with kinetic energy zero; these values are shown in the brackets.

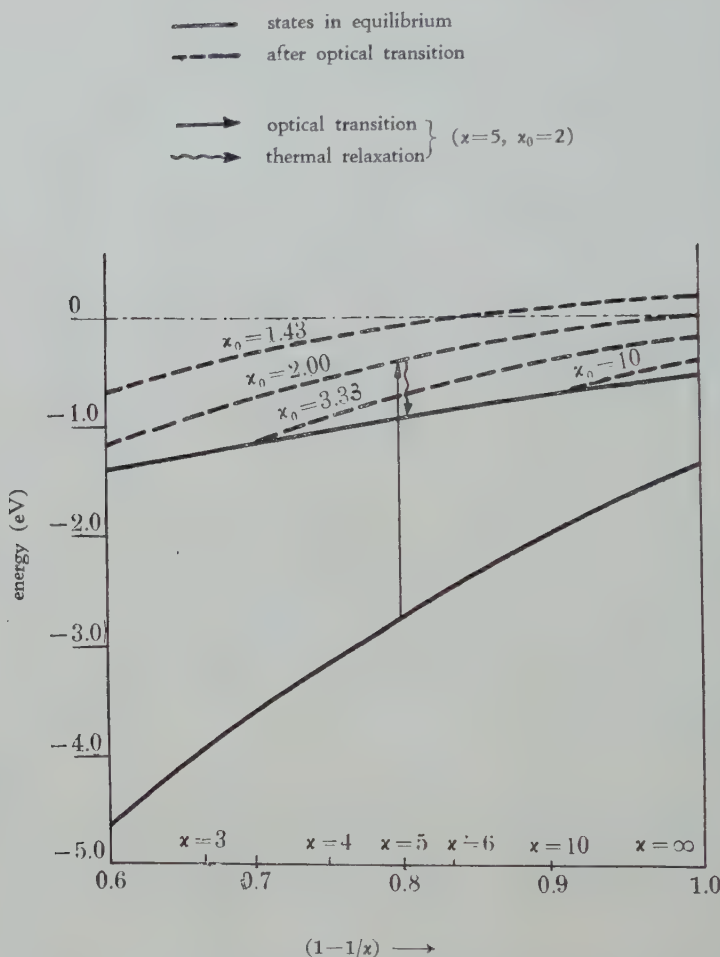


(a) One electron system (F -center)



(b) Two-electron system (F' -center)

Fig. 4. The energy vs. configuration diagrams of F - and F' -centers ($x=5$, $x_0=2$)

Fig. 5. The energies of the ground and excited states of the F -center

energy $h\nu_F$ of the F -absorption, we have

$$h\nu_F = E_F^{I*}(E_F) - E_F.$$

In the same way, the energies can be calculated for the following states:

(f) Immediately after the conduction electron has been trapped in the $2p$ -state of a negative vacancy with an emission.

(g) Immediately after the optical transition from the state (d) to the $1s$ -state.

(h) Photoionization of the state (d).

(iv) The F' -center.

(a) The ground state in equilibrium. As the F' -center is a two electron system, we must add the term $1/r_{12}$ to the expression (6.17') and take \tilde{E}_c as the electric field due to these two electrons. $V(r_1)$ and $V(r_2)$ are the same as in (iii). The trial function is taken as

$$\phi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)$$

with ϕ of the hydrogen $1s$ -type.

Applying (6.17') and (6.18) we obtain the results

$$u = 5/8 + 3/4 \cdot 1/x,$$

$$E_{F'} = -1/4(5/8 + 3/4 \cdot 1/x)^2.$$

For the thermal dissociation energy $W_{F'}$ of the process in which an F' -center loses one electron, producing a polaron and an F -center, we have

$$W_{F'} = (E_F + E_P) - E_{F'}.$$

(b) The photoionization and the F' -absorption. After an F' -center is photoionized, there are produced a conduction electron and an F -center which is not in equilibrium. For the latter system we can apply (6.25') to calculate its energy E' , whose minimum we denote by $E_{F'}'(E_{F'})$. Thus we obtain the threshold photon energy of the F' -absorption:

$$(\hbar\nu_{F'})_{\min.} = E_{F'}'(E_{F'}) + E_G' - E_{F'}.$$

(c) The optical process, in which an F -center traps a conduction electron and produce an F' -center, is discussed in the same manner.

(v) *Comparison with experiments.*

Of the various optical and thermal energies discussed above, those for which the experimental data are available are tabulated in Tab. 2, where the theoretically calculated values (with $x=5$ and $x_0=2$) are compared with the observed values of four alkali-halides. Considering the roughness of the model we have used above, the agreements are quite satisfactory.

Table 2. Comparison of the calculated energy values ($x=5$, $x_0=2$) with the observations (in eV)

		$\hbar\nu_{F'}$	$W_{F'}$	$W_{F'}^*$	$(\hbar\nu_{F'})_{\min.}$	$W_{F'}$	x
Calculations		2.38	1.83	-0.04	1.80	0.40	0.36
Observations	LiCl	3.2					
	NaCl	2.7	1.89	0.075		<1.0††	0.5
	KCl	2.3	1.95		1.3~1.4†	<1.0††	
	KBr	2.0	1.78				

The observed values are taken from the text-book of Mott and Gurney.⁽⁹⁾

† We estimated this by adding the breadth of the F' -band to the threshold energy for the F' -absorption.

†† Mott and Gurney estimated the activation energy of dissociation to be about 1.0 eV from the observed life time of the F' -center at room temperature. Therefore the thermal dissociation energy must be smaller than 1.0 eV.

§ 8. Some consideration on the number of excited states

According to the method of § 4, the optically excited states of any center are to be discussed by the usual eigenvalue problem (4.42'), the operator A being determined beforehand by using the knowledge about the ground state. For the case of the F -center, we have

$$V \sim -1/r, \quad \tilde{\varphi}_e \sim -1/r \quad \text{as } r \rightarrow \infty,$$

whatever form we may use for V in the neighbourhood of the center. The potential part of A , therefore, is asymptotically equal to

$$-1/xr - (1/x_0 - 1/x)1/r = -1/x_0r$$

as r tends to infinity. Now it is a well-known theorem of eigenvalue problems that there are infinitely many discrete eigenstates (trapped states) for the asymptotically Coulomb-like potentials. The number of those states among these to which the optical transitions from the ground states are permitted is also infinite. On the other hand the continuous spectrum of the eigenvalues will, in our case, constitute a conduction band.

The above conclusion holds for any other center whose total charge is positive or zero, as one can easily verify.

The discrepancy between this theoretical conclusion, and experimental facts that for most of the known centers there is only one absorption band (very few even in the other centers) may be explained, as is usually done, by the possible overlappings of rather broad bands, or by the exclusively large oscillator strength of only one transition. From the new stand-point of § 6, however, it is possible to give more well-founded explanation of the experimental facts stated above.

According to this method, the state B' immediately after the optical transition from the state A should be determined by the simultaneous equations (6.27) and (6.28), the former of which is not a usual eigenequation, because the Hamiltonian includes the term $(1 - 1/x_0)\tilde{\varphi}_e'(r)$, this being determined in turn by ψ' , which we are going to solve. This term denotes the potential due to the electronic polarization produce by the charge distribution $-\psi'^2$, namely the *reaction term* from the crystal electrons.

For the solutions of the non-linear eigenvalue problem as is met above, we can no more assert the various theorems we know about the eigenvalues and eigenfunctions of the usual linear operators. Nevertheless it seems to us to be fairly probable that the discrete states are signified by the numbers of nodes of their wave functions, and that the larger the number of the nodes the more spread will be the corresponding wave function, with \tilde{E}_e' and $\int \psi' V \psi' d\mathbf{r}$ converging to zero. Denoting the number of the nodes by n , we thus infer from (6.25') that

$$\begin{aligned} \lim_{n \rightarrow \infty} E'(\psi_n' \psi) &= \lim_{n \rightarrow \infty} \int \psi_n' K \psi_n' d\mathbf{r} + 1/8\pi(1/x_0 - 1/x) \int \tilde{E}_e'^2 dv \\ &\geq 1/8\pi(1/x_0 - 1/x) \int \tilde{E}_e'^2 dv. \end{aligned} \quad (8.1)$$

On the other hand the energy of the photoionized state of the center is given by § 7 (ii) and (6.25') :

$$E'(\psi', \psi) = -0.0535(1 - 1/x_0)^2 + 1/8\pi(1/x_0 - 1/x) \int \tilde{E}_e^2 dv, \quad (8.2)$$

where ψ' is the wave function for a conduction electron.

Comparing (8.1) and (8.2) we see that as the number n increases, the energy of the state ψ'_n will exceed that of ψ' , namely the bottom of the conduction band, before n goes to infinity. The levels for larger n will be buried in the continuum of the conduction band, being deprived of the discreteness. Thus the method of § 6 leads to the conclusion that there are only finite number of discrete states under the conduction band, whether the trapping potential may be asymptotically Coulomb-like or not.

The above inference is, however, not logically strict; it may be, for pessimists, not more than a suggestion. It is a rather difficult problem to investigate the spectra of the non-linear operator of (6.27). It is desirable to seek mathematical means to solve the generalized eigenvalue problems such as this equation, or at least to obtain the knowledge of the qualitative features of their spectra. Such requirements seem to be common to all those problems in which the many-body character plays an essential rôle.

The authors wish to express their thanks to Dr. R. Kubo for his detailed discussions.

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A Tentative Interpretation of Bickford's Observation of the Resonance Absorption in Magnetite below its Transition Point

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An interpretation is given of the double peak resonance absorption observed by Bickford in magnetite below its transition point -160°C . An anisotropy energy of tetragonal symmetry is assumed as in his case, and for weaker fields a magnetic structure of parallel flat domains is assumed. When the magnetization vectors on both sides of each wall oscillate in a certain manner, a surface magnetic charge will be produced on the wall, and this in turn will produce an oscillating magnetic field in each domain. The frequency of oscillation is thus increased.

§ 1. Introduction

Magnetite (Fe_3O_4), cubic of the spinel type, is known, since the work of Weiss and Forrer⁽¹⁾, to undergo a phase transition at -160°C . Below this transition point, it seems that the ferrous and ferric ions in the sublattice 16c take an ordered arrangement of orthorhombic symmetry⁽²⁾. The X-ray analysis of Tombs and Rooksby⁽³⁾ shows that the lattice suffers a slight change at the transition point and becomes rhombohedral below. The magnetic symmetry seems, however, to be tetragonal according to Bickford⁽⁴⁾ who has made an extensive measurement of the resonance absorption over a wide range of temperature with 24000 Mc and 9000 Mc waves; he has shown at the same time that this is roughly consistent with Domenicali's⁽⁵⁾ measurement of the magnetization. The tetragonal easy axis is one of the $[100]$ axes of the cubic upper phase and is established by cooling the crystal with a strong magnetic field applied along that direction.^(*)

* In a recent issue of Rev. Mod. Phys. 25 (January, 1953) p. 75 Bickford presented an evidence that the crystalline structure of the low temperature form is orthorhombic. Later X-ray work by Abraham and Calhoun in Acta Crystallographica verified this (the writer is indebted to Prof. Bickford for a personal communication about these works). It is possible that the double peak resonance is due to a twinning perpendicular to the c -axis (the easy axis of magnetization), as suggested by Calhoun, the magnetic symmetry being orthorhombic; a theoretical treatment of the resonance position would, however, require a consideration of a surface magnetic charge on the twin boundary similar to that given in the present paper. It will be interesting to see whether the ferromagnetic domain structure of the microcrystalline structure is primarily responsible for the observed double peaks.

It came into notice of the writer, at the same time, that D. Polder and J. Smit in Rev. Mod. Phys. 25 (January, 1953), p. 89 has pointed out the same effect of the surface charge on Bloch walls upon resonance frequency as that treated in the present paper.

Bickford used (100) and (110) discs of a diameter of 2 to 3 mm and of a thickness of 0.05 to 0.08 mm, the ratio between them averaging about 25, and measured the resonance absorption for various combinations of biasing and measuring field directions. A striking result he obtained was that for the majority of these combinations double resonance peaks were observed below the transition point. The peaks at higher values of the measuring field strength were explained by him with the assumption of a tetragonal anisotropy energy and a value of g equal to 2.49, but the peaks at lower field strengths remained unexplained. Considering the values of the anisotropy constants he obtained and the field strengths for the latter peaks (see the table given at the end), it can be concluded, at least for the cases where the measuring magnetic field is applied perpendicular to the easy axis, that the magnetization of the specimen was not saturated in these cases. We might therefore consider that the inside of the specimen was divided into magnetic domains. If the magnetization vectors on both sides of a domain wall oscillate in a certain manner, a magnetic charge will be produced periodically on the wall, and this charge will produce a demagnetizing field on both sides of the wall. The frequency of oscillation is therefore increased by the presence of the wall, and in favorable cases one might expect a resonance to occur at a field strength insufficient for saturation.

In the present treatment, it is assumed that the specimen is divided into parallel flat magnetic domains in such cases. This point has to be verified by experiment. But this assumption gives a good explanation of the majority of Bickford's results, while no other hypothesis of the domain structure seems to give as good an explanation. If, for instance, one assumes domains of a cylindrical form, one can not only unable to explain the position of the lower field peak but also one must expect a peak much broader than that observed.

In the case where the applied field has a component along the easy axis, it is difficult to carry out a calculation to the end, because Bickford has not given the magnitude of the magnetization that corresponds to the resonance field strength. One case of the field applied along the easy axis will be treated, however, and it will be shown that the resonance will occur at the observed value of 350 Oe if the corresponding magnetization is assumed to be 80% of the saturation magnetization. Other cases might be treated in a similar way.

§ 2. The resonance peak at a higher field strength

We shall begin with the calculation of this, though Bickford has already done it. He assumed that the magnetization was saturated along the direction of the applied field and used Kittel's formula to calculate the position of the resonance peak:

$$\nu = (g\beta/\hbar) \{ [H_z + (N_y + N_y^e - N_z) M_s] [H_z + N_x^e M_s] \}^{1/2}.$$

Here N_y^e and N_x^e are the effective demagnetization coefficients perpendicular and parallel to the disc plane; they are connected in a certain way with the anisotropy energy. The assumption of the saturation is not always valid, however, and moreover Kittel's formula does not hold in some cases even with saturation, as shown below. Therefore, the results

obtained in this paper are a little different from his.

The anisotropy energy will be assumed in the same form as Bickford's :

$$E_a = -Ka_1^2 - K'a_1^4 - K''(a_2^4 + a_3^4). \quad (1)$$

The corresponding equivalent magnetic field, \mathbf{H}_a , can be obtained in a simple way as follows. We take the variation of (1) :

$$\delta E_a = -2Ka_1 \delta a_1 - 4K'a_1^3 \delta a_1 - 4K''(a_2^3 \delta a_2 + a_3^3 \delta a_3).$$

This must be equal to the quantity

$$-\mathbf{H}_a \delta \mathbf{M} = -H_{a1} M_s \delta a_1 - H_{a2} M_s \delta a_2 - H_{a3} M_s \delta a_3.$$

Therefore we have

$$H_{a1} = ca_1 + c'a_1^3, \quad H_{a2} = c''a_2^3, \quad H_{a3} = c''a_3^3, \quad (2)$$

where

$$c = 2K/M_s, \quad c' = 4K'/M_s, \quad c'' = 4K''/M_s. \quad (3)$$

More rigorously, we have to add indeterminate terms λa_1 , λa_2 , λa_3 to the respective right hand sides of (2), since $a_1^2 + a_2^2 + a_3^2 = 1$. They have, however, no influence on the following calculations.

The equation of motion can be written as

$$\dot{\mathbf{M}} = \gamma [\mathbf{M}, \mathbf{H} + \mathbf{H}_a + \mathbf{H}_d], \quad \gamma = g\beta/\hbar, \quad (4)$$

neglecting the damping term. Here \mathbf{H} is the applied external field and \mathbf{H}_d the demagnetizing field :

$$\mathbf{H}_d = (-N_1 M_1, -N_2 M_2, -N_3 M_3). \quad (5)$$

Here the three axes are taken to be the principal axes of the disc. If y is taken perpendicular to the plane of the disc, as we shall do in the following, then $N_1 = N_3$. Bickford gives the following value of $N_2 - N_1$:

$$N_2 - N_1 = 0.89 \times 4\pi = 11.2 \quad (\text{in c.g.s. Gauss system}).$$

We shall later need the independent values of N_2 and N_1 , for which we shall assume 11.7 and 0.5, respectively. The value of M_s given by Bickford for -190°C is 510 gauss.

The equilibrium orientation of \mathbf{M} is determined by

$$[\mathbf{M}, \mathbf{H} + \mathbf{H}_d + \mathbf{H}_a] = 0. \quad (6)$$

For the discussion of a small oscillation in the neighborhood of the equilibrium position it is convenient to take the variation of eq. (4) :

$$\delta \dot{\mathbf{M}} = \gamma [\delta \mathbf{M}, \mathbf{H} + \mathbf{H}_d + \mathbf{H}_a] + \gamma [\mathbf{M}, \delta \mathbf{H}_d + \delta \mathbf{H}_a]. \quad (7)$$

The constancy of the magnitude of \mathbf{M} , or $\mathbf{M} \cdot \delta \dot{\mathbf{M}} = 0$, is valid in virtue of condition (6).

The equation (7) written in components is as follows :

$$\begin{aligned} \frac{i\omega}{\gamma} \delta a_1 &= (H_3 - N_3 M_s a_3 + N_2 M_s a_3 + c'' a_3^3 - 3c'' a_2^2 a_3) \delta a_2 \\ &\quad - (H_2 - N_2 M_s a_2 + N_3 M_s a_2 + c'' a_2^3 - 3c'' a_3^2 a_2) \delta a_3, \end{aligned} \quad (8_1)$$

$$\begin{aligned} \frac{i\omega}{\gamma} \delta a_2 &= (H_1 - N_1 M_s a_1 + N_3 M_s a_1 + c a_1 + c' a_1^3 - 3c' a_3^2 a_1) \delta a_3 \\ &\quad - (H_3 - N_3 M_s a_3 + N_1 M_s a_3 + c' a_3^3 - c a_3 - 3c' a_1^2 a_3) \delta a_1, \end{aligned} \quad (8_2)$$

$$\begin{aligned} \frac{i\omega}{\gamma} \delta a_3 &= (H_2 - N_2 M_s a_2 + N_1 M_s a_2 + c' a_2^3 - c a_2 - 3c' a_1^2 a_2) \delta a_1 \\ &\quad - (H_1 - N_1 M_s a_1 + N_2 M_s a_1 + c a_1 + c' a_1^3 - 3c'' a_2^2 a_1) \delta a_2, \end{aligned} \quad (8_3)$$

where ω is the circular frequency.

We shall discuss several special cases.

1) (100) disc, the tetragonal axis in the plane of the disc, \mathbf{H} parallel to the tetragonal axis.

Taking y normal to the disc and x along the tetragonal axis, we have

$$\begin{aligned} \delta a_1 &= 0; \\ (i\omega/\gamma) \delta a_2 &= (H + c + c') \delta a_3, \\ (i\omega/\gamma) \delta a_3 &= -(H + (N_2 - N_1) M_s + c + c') \delta a_2, \end{aligned}$$

from which follows

$$(\omega/\gamma)^2 = (H + (N_2 - N_1) M_s + c + c') (H + c + c'). \quad (9)$$

2) The same as above, except that \mathbf{H} is perpendicular to the tetragonal axis. In this case we have

$$\begin{aligned} (i\omega/\gamma) \delta a_1 &= (H + (N_2 - N_1) M_s + c'') \delta a_2, \\ (i\omega/\gamma) \delta a_2 &= -(H + c'' - c) \delta a_1, \\ \delta a_3 &= 0, \end{aligned}$$

and

$$(\omega/\gamma)^2 = (H + (N_2 - N_1) M_s + c'') (H + c'' - c). \quad (10)$$

3) (110) disc, the tetragonal x -axis in the disc plane, \mathbf{H} is perpendicular to the tetragonal axis, i.e., \mathbf{H} is along the $[110]$ axis in the disc plane.

In this case, the y and z axes assumed for (2) make an angle of 45° with the plane of the disc; it is more convenient to transform the anisotropy part of (8) so that the new y -axis becomes normal to the disc. We shall give only the result for the resonance frequency :

$$(\omega/\gamma)^2 = (H + (N_2 - N_1) M_s - c'') (H + c''/2 - c). \quad (11)$$

(9), (10), (11) are special cases of the Kittel's formula. Bickford used them to determine

the value of three anisotropy constants and of g (at -190°C , as we shall always assume it) by comparing them with his three observations with 24000 Mc and one observation for the case 2) with 8960 Mc. The same procedure was followed but slightly different results were obtained.

The present calculation			Bickford
$c = 1.80 \times 10^3$ Oe	$K = 4.34 \times 10^5$ erg/cm ³		4.20×10^5 erg/cm ³
$c' = 0.88$ "	$K' = 1.02$ "		1.23 "
$c'' = -3.10$ "	$K'' = -3.95$ "		-4.04 "
	$g = 2.49$		2.49

A trial was made with $g=2$ but it was found unsuccessful.

4) (110) disc, the tetragonal axis makes an angle of 45° with the disc plane, and \mathbf{H} is along the [100] axis in the disc plane, i.e., \mathbf{H} is perpendicular to the tetragonal axis.

We have again to transform the anisotropy part of (8). The result for the frequency is

$$(\omega/\gamma)^2 = (H + (N_2 - N_1)M_s + c'' - c/2)(H + c'' - c/2) - c^2/4. \quad (12)$$

This is one of the cases where the frequency formula does not reduce to that of Kittel. Bickford observed a resonance at 8630, while the present calculation gives 8570.

5) (100) disc, the tetragonal axis in the disc plane, \mathbf{H} along the [110] axis of the plane, making an angle of 45° with the tetragonal axis.

This case is different from those above in that the magnetization vector is not directed along the applied field. Its direction is determined by (6), which gives

$$(1/\sqrt{2})H(a_1 - a_3) = a_1 a_3 (c + c' a_1^2 - c'' a_3^2), \quad (13)$$

or, writing $a_1 = \cos \theta$, $a_3 = \sin \theta$,

$$H \sin(45^\circ - \theta) = \sin \theta \cos \theta (c + c' \cos^2 \theta - c'' \sin^2 \theta). \quad (13a)$$

Using this condition, the frequency formula can be written for instance as

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(\frac{1}{\sqrt{2}}H + (N_2 - N_1)M_s + c'' a_3^3\right) \left[\left(\frac{1}{\sqrt{2}}H - 2c' a_1^3\right) \frac{a_3}{a_1} + \left(\frac{1}{\sqrt{2}}H - 2c'' a_3^3\right) \frac{a_1}{a_3} \right]. \quad (14)$$

H and θ are calculated to 2320 Oe and 20.8° , while the observed value of H is 2500 and Bickford's calculated value was 3700.

6) (110) disc, the tetragonal axis in the disc plane, \mathbf{H} along the [111] axis of the disc plane.

The direction of the magnetization vector is determined by

$$H(\sqrt{2/3}a_1 - 1/\sqrt{3} \cdot a_3) = a_1 a_3 (c + c' a_1^2 - c''/2 \cdot a_3^2), \quad (15)$$

or

$$H \sin(\theta_0 - \theta) = \cos \theta \sin \theta (c + c' \cos^2 \theta - c''/2 \cdot \sin^2 \theta), \quad \theta_0 = 55^\circ. \quad (15a)$$

The resonance formula can be written as

$$\left(\frac{\omega}{\gamma}\right)^2 = \left(\sqrt{\frac{2}{3}}H + (N_2 - N_1)M_s a_3 - c'' a_3^2\right) \cdot \left[\left(\frac{1}{\sqrt{3}}\frac{a_3}{a_1} + \sqrt{\frac{2}{3}}\frac{a_1^2}{a_3^2}\right)H - (2c' + c'')a_1^2 a_3\right].$$

The result obtained is: $H=3010$, $\theta=30.7^\circ$, while the observed value and the one calculated by Bickford are respectively $3500 \sim 4500$ and 4340 . The value of H obtained here is a little too low.

There are two other cases in Bickford's experiment: (110) disc with the tetragonal axis making 45° to the disc plane and H being along either $[110]$ or $[111]$ axis in the plane. In these cases, we have to expect that the inside of the disc is divided into microcrystalline domains with two different kinds of the crystalline orientation and, correspondingly, two different orientations of the magnetization vector. The actual situation might be complicated. We shall later (§ 3, 3)) treat the first of these cases under a certain simplifying assumption, but the result obtained is not satisfactory.

The results obtained are summarized in Table 1.

§ 3. The resonance peak at a lower field strength

As mentioned in § 1, we assume in the case of unsaturation that the disc consists of parallel flat magnetic domains, though this point has to be investigated by experiment. In the case of a (010) disc with the tetragonal axis along the $[100]$ direction and the applied measuring magnetic field along the $[001]$ direction, the values of the anisotropy constants give $H=5050$ Oe for the saturation to occur, while Bickford observed a resonance at 3000 Oe. It is not clear whether the domain walls are parallel to (001) or (011) or another plane which contains the direction of the easy axis, but possibly they are parallel to (011) since the magnetization vector can most easily be turned in this plane; the threshold value of the potential energy in this plane and that in the plane (001) have a ratio of 1:2. We shall, however, treat both cases in the following.

1) The (010) disc mentioned above.

Let the angle between the disc plane and the domain walls be φ , the walls being supposed to be parallel to the easy axis x , and y -axis being taken normal to the disc plane. The domain thickness will be supposed to be constant and very small compared with the disc thickness. In the case of equilibrium, the magnetization vector is in the plane of the disc but its x -component alternates in plus and minus, while its z -component is the same, as the magnetic field is applied along the z -axis. The domains with plus M_x will be denoted as 1, and those with minus M_x as 2.

When the magnetization vectors $\mathbf{M}^{(1)}$ and $\mathbf{M}^{(2)}$ deviate from their equilibrium directions, a surface charge of the following quantity will be produced on the walls:

$$\pm\sigma = (\partial M_y^{(1)} - \partial M_y^{(2)})\cos\varphi + (\partial M_z^{(1)} - \partial M_z^{(2)})\sin\varphi, \quad (17)$$

$+\sigma$ appearing on walls whose $-z$ side is the domain 1 and $+z$ side the domain 2, and $-\sigma$ on the remaining walls. These surface charges produce in domains 1 a magnetic field of the strength

$$(0, -2\pi\sigma \cos \varphi, -2\pi\sigma \sin \varphi)$$

and in domains 2

$$(0, +2\pi\sigma \cos \varphi, +2\pi\sigma \sin \varphi).$$

These have to be added to the field appearing in the equation (7), the former for the equation of $M^{(1)}$ and the latter for that of $M^{(2)}$. Furthermore, the demagnetizing field and its variation have to be replaced by their mean values :

$$\bar{H}_a = \frac{1}{2} (H_a^{(1)} + H_a^{(2)}), \quad \delta \bar{H}_a = \frac{1}{2} (\delta H_a^{(1)} + \delta H_a^{(2)}) \quad (18)$$

in both equations. Thus we obtain six equations written in components.

The direction of the magnetization vectors in equilibrium with the applied static field is given by

$$\mathbf{H} - N_1 M u_3 = (c + c' u_1^2 - c'' u_3^2) u_3, \quad (19)$$

where $-N_1 M u_3$ cares for the demagnetizing field in the direction of z . The frequency equation is as follows, after eliminating \mathbf{H} by (19) :

$$\begin{aligned} & [(\omega/\gamma)^2 - u_1^2 (c + c' u_1^2 + 4\pi M_s \cos^2 \varphi) (c + c' u_1^2 - 3c'' u_3^2 + N_1 M_s) \\ & \quad - u_3^2 (2c' u_1^2) (c + c' u_1^2 + 4\pi M_s \cos^2 \varphi)] \\ & \cdot [(\omega/\gamma)^2 - u_1^2 (c + c' u_1^2 + N_2 M_s) (c + c' u_1^2 - 3c'' u_3^2 + 4\pi M_s \sin^2 \varphi) \\ & \quad - u_3^2 (2c' u_1^2 - N_1 M_s) (c + c' u_1^2 + N_2 M_s)] \\ & - (4\pi M_s \sin \varphi \cos \varphi)^2 u_1^2 (c + c' u_1^2 + N_2 M_s) \\ & \cdot [u_1^2 (c + c' u_1^2 - 3c'' u_3^2 + N_1 M_s) - u_3^2 (2c' u_1^2 + N_1 M_s)] = 0. \end{aligned} \quad (20)$$

For $\varphi = 90^\circ$ and $\varphi = 0^\circ$ this gives two simple modes of oscillation.

Mode A. $M^{(1)}$ and $M^{(2)}$ are rigidly connected with each other, oscillate in such a way that $\delta M_x^{(1)} = \delta M_x^{(2)}$, $\delta M_z^{(1)} = -\delta M_z^{(2)}$, $\delta M_y^{(1)} = \delta M_y^{(2)}$. The oscillation in the xz -plane and that in the direction of y have a phase difference of $\pi/2$; the magnetization vectors describe an ellipse each. The frequency is given by the second factor of (20), and in the case of $\varphi = 90^\circ$ surface charges are produced on the domain walls so that the frequency is high. This mode is excited by an oscillating field perpendicular to the applied static field.

Mode B. The two magnetization vectors oscillate in the xz -plane symmetrically with respect to the yz -plane and, in the direction of y , antisymmetrically with respect to the same plane. These two oscillations have a phase difference of $\pi/2$. The frequency is given by the first factor of (20), and in the case of $\varphi = 90^\circ$ no surface charge is produced on the domain walls so that the frequency is low. This mode is excited by an oscillating field parallel to the applied static field.

For $\varphi = 0$, both modes have nearly the same frequency, since N_1 and N_2 are nearly equal to 0 and 4π , respectively.

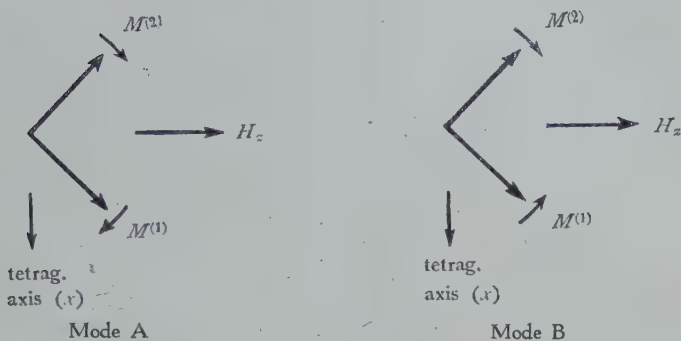


Fig. 1

The result of the calculation for the mode A with 24000 Mc is :

Resonance field	Angle between the tetragonal axis and the magnetization	
$H=2830$ Oe	$\theta=45.5^\circ$	for $\varphi=45^\circ$
3090	50.8	90

Both of these values of H can be compared with the observed value of 3000. For 8960 Mc, a resonance of the mode A is expected at a value of H a little lower than the value of H for saturation (5050), but perhaps it might have been unresolved from another peak at 5600 in Bickford's experiment; in actual crystals, the direction of the magnetization vectors in different domains might be distributed over a fairly large angular range when they are nearly saturated and, consequently, the resonance might be broad.

The resonance for the mode B is expected for 9000 Mc at two values of H near 2000 if $\varphi=90^\circ$, and at a value of H equal to about 4000 if $\varphi=45^\circ$; it does not occur for 24000 Mc. In experiments hitherto done, the oscillating field is always perpendicular; it may be interesting to detect this mode with an oscillating field parallel to the applied static field.

If we could assume that the domains are parallel to the disc plane ($\varphi=0$), two peaks would be expected for 9000 Mc, one close to the saturation value of H and the other at a lower value of H , but no resonance for 24000 Mc.

2) (110) disc, easy axis in the disc plane, H perpendicular to the latter in the same plane.

The principle of the calculation is the same as above, and the result is :

$H=2460$ Oe	$\theta=49.7^\circ$	for $\varphi=90^\circ$
2020	41.0	45

for 24000 Mc. The case of $\varphi=90^\circ$ corresponds to the walls of (110) and is more probable than the case 45° for which the walls are (100), but the observed value 2000 Oe is more favorable for the latter. Other things are about the same as in the case 1).

Table 1. (H in Oe)

Crystal plane of disc	Measuring field direction	Angle between measuring field & tetrag. axis	High-field peak			Low-field peak	
			H_{obs}	(Bic.) H_{calc}	(Nag.) H_{calc}	H_{obs}	(Nag.) H_{calc}
24000 Mc							
1. (100)	[100]	0°	2000	2000*	2000*	350	350* ($\varphi=90^\circ$) $M=0.8M_s$ 2830 ($\varphi=45^\circ$) 3090 ($\varphi=90^\circ$)
2. (100)	[100]	90°	8900	8900*	8900*	3000
3. (100)	[110]	45°	2500	3700	2320 $\theta=20.8^\circ$	900
4. (110)	[100]	0°	1850	2000	2000	300†
5. (110)	[110]	90°	6350	6350*	6350*	2000	2460 ($\varphi=90^\circ$) 2020 ($\varphi=45^\circ$)
6. (110)	[111]	55°	3500 ~4500	4340	3010 $\theta=30.7^\circ$	1200†
7. (110)	[100]	90°	8500	8630	8560	2200
8. (110)	[110]	45°	3600	4100	2460 $\theta=20.3^\circ$	1500†?
9. (110)	[111]	55°	—	3000?	1500?
8960 Mc							
10. (100)	[100]	90°	5600	5600*	5600*	—	—
11. (110)	[110]	90°	3400	3750	3750	—	—
12. (110)	[100]	90° not in plane	4600	5000	5030	—	—

* and () are assumed for calculation, † sharp peak is observed, qualitatively understandable,? difficult to understand.

3) (110) disc, the easy axis makes an angle of 45° with the disc plane (rows 7, 8, 9 of the Table 1).

If H is applied along the [100] axis in the disc plane, we have to consider four kinds of the orientation of the magnetization vector. The inside of the disc is possibly divided by two kinds of walls, one the microcrystalline boundary walls and the other the ferromagnetic domain walls. The former may be perpendicular to the disc plane and parallel to the [100] axis along which H is applied, and the latter may be perpendicular to both, because, if so, no surface charge is produced on these walls in static equilibrium. The calculation of the resonance was not carried out, however, as it is troublesome to solve a three-dimensional equation to determine the orientation of the magnetization vector. It seems, however, that the same explanation is applicable as in 1) and 2) above (compare the sets of two resonance values of H in rows 2, 5, and 7 of Table 1).

The case where H is applied along the [110] axis of the disc plane was treated with the assumption of the domain walls as mentioned above, or with an alternative assumption that the walls of both kinds contain the directions of that [110] axis and the easy axis. This case is simpler because we can expect to have two orientations of the magnetization vectors which can be easily determined. The calculation gives a single peak of resonance at $H=2460$ or 2850 for 24000 Mc, respectively, which are disappointing compared with the

observed values 4100 and 1500.

In the case where \mathbf{H} is applied parallel to the $[111]$ axis of the disc plane, the observation of Bickford gave a single peak at about 1500, which is again difficult to explain, because the calculation would give a value greater than 2000, as above. Perhaps we are simplifying the problem too much, and the phenomenon like that which will be treated in the following case might also play a rôle here.

4) (100) disc containing the easy axis to which \mathbf{H} is parallel. The observed resonance peak is at 350 Oe.

This case may be interpreted in terms of unequal thicknesses of the domains in which the magnetization vector is parallel and antiparallel to the applied field. The ratio of these thicknesses will be denoted as $(1+s)/2 : (1-s)/2$. Then the magnetization of the sample is sM_s .

In order to simplify the calculation, we assume that $\varphi = 90^\circ$, which will give at least a qualitative result. The equation (8) for these parallel and antiparallel domains, (+) and (-), respectively, can be written, after replacing the terms arising from demagnetizing field by their mean values, as

$$\begin{aligned} (i\omega/\gamma) \delta a_2^{(\pm)} &= (H - N_1 \bar{M}_1 \pm (c + c')) \delta a_3^{(\pm)} \pm N_1 M_s \delta \bar{a}_{2,3}, \\ (i\omega/\gamma) \delta a_3^{(\pm)} &= -(H - N_1 \bar{M}_1 \pm (c + c')) \delta a_2^{(\pm)} \mp N_2 M_s \delta \bar{a}_{2,3}, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \bar{M}_1 &= M_s(1+s)/2 - M_s(1-s)/2 = sM_s, \\ \delta \bar{a}_{2,3} &= \delta a_{2,3}^+(1+s)/2 + \delta a_{2,3}^-(1-s)/2. \end{aligned}$$

The surface charge density which appears on the walls is

$$\pm \sigma = \pm (\delta M_3^+ - \delta M_3^-) = \pm M_s (\delta a_3^+ - \delta a_3^-).$$

The field strength due to this in the + or - domains is

$$\begin{aligned} + \text{ domains: } & (1-s) \cdot (0, 0, -2\pi\sigma), \\ - \text{ domains: } & -(1+s) \cdot (0, 0, +2\pi\sigma). \end{aligned}$$

The torque on $\mathbf{M}^{(\pm)}$ has only the y -component, which is $(1 \mp s) M_s 2\pi\sigma$. They have to be added to the respective right hand side of (21). In this way, the resonance frequency is obtained as follows:

$$\begin{aligned} (\omega/\gamma)^4 - (\omega/\gamma)^2 [(c + c') (c + c' + N_1 M_s) + (H + s(N_2 - N_1) M_s) (H - s4\pi M_s) \\ + (H - sN_1 M_s)^2 + (c + c' + N_2 M_s) (c + c' + 4\pi M_s)] \\ + [(c + c') (c + c' + N_2 M_s) - (H - sN_1 M_s) (H + s(N_2 - N_1) M_s)] \\ \cdot [(c + c' + N_1 M_s) (c + c' + 4\pi M_s) - (H - sN_1 M_s) (H - s4\pi M_s)] = 0. \end{aligned} \quad (22)$$

For $H=0$, $s=0$, we have

$$(\omega/\gamma)^2 = (c + c' + 4\pi M_s)(c + c' + N_2 M_s) = 76.9 \times 10^6 \text{ (calc.) mode A}$$

$$(\omega/\gamma)^2 = (c + c')(c + c' + N_1 M_s) = 7.33 \times 10^6 \text{ (calc.) mode B.}$$

With increasing H , s will change from 0 to 1; at $s=1$ we have

$$(\omega/\gamma)^2 = (c + c' + H)(c + c' + (N_2 - N_1)M_s + H) = 21.4 \times 10^6 \text{ for } H=0$$

$$(\omega/\gamma)^2 = (c + c + N_1 M_s + 4\pi M_s - H)(c + c' + N_1 M_s - H) = 26.3 \times 10^6 \text{ for } H=0,$$

the second frequency corresponding to the oscillation of the magnetization vector in infinitesimal domains of the type —. For 24000 Mc and $g=2.49$, the value of $(\omega/\gamma)^2$ is 47.2×10^6 . Therefore, the mode A has a frequency too high at $s=0$ but too low at $s=1$ and $H=0$. Actually, s would increase from 0 to 1 in a certain small interval of increasing H , and at a certain value of H one might have a resonance. Since the magnetization as a function of H is not given in Bickford's paper, the writer determined the value of s at 350 Oe from the condition of resonance, that is, from Eq. (22). The result is $s=0.8$, which seems to be quite reasonable.

The peak at a lower field strength in other cases where the field has a component along the easy axis might be interpreted in a similar way, except those cases mentioned in 3), though no calculation has been carried out. Perhaps a similar explanation may be applicable to the observation of two peaks by Okamura and Torizuka⁽⁶⁾ above the transition point.

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Theory of the Normal Modes of Vibration in Crystal

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The normal modes of vibration in crystal have been determined by the group-theoretical method, familiar in the theory of molecules. Secular equation for determining the frequencies of normal modes has been reduced to lower degrees by making use of the translational, crystal and K -groups. The frequency-distribution curves have been obtained for a few types of crystals, with a brief reference to 2nd order Raman-effect.

§ 1. Introduction

In order to classify the normal modes of vibration of molecules, group-theoretical method is very powerful. Each mode corresponds to a definite irreducible representation of the point group to which the molecule belongs. The secular equation for determining frequencies is reduced to the equations of lower degrees, each corresponding to a definite irreducible representation. In the present paper we have extended this general treatment to determine the normal modes in crystal.

First we develop the general theory for the case of a simple cubic lattice introducing *translational normal coordinates*, and then we reduce the translational dynamical matrix by the use of the irreducible representations of K -groups in special directions in reciprocal lattice space, introduced by Bouchaert-Smoluchowsky-Wigner.¹⁾ Next we apply these theories to actual crystals, especially to NaCl and diamond, and calculate the distribution functions of frequencies by the method of Houston²⁾ assuming appropriate potential energies between atoms in the crystals. In the same way, after Born and Bradburn³⁾ we analyse the profile of the 2nd order Raman-effect for the diamond crystal by calculating the distribution functions of over-tones and combination-tones.

§ 2. General theory for simple cubic lattice

The position of atoms in a simple cubic lattice is expressed by their integral coordinates (l, m, n) in a suitable rectangular coordinate system. Consider a configuration that an atom at (l, m, n) is displaced by unit amount in x -direction, the other atoms remaining in the original regular positions. We shall denote this configuration by $e(l, m, n : x)$, and similary for the displacements in y and z -directions. Then the assembly

$$e(l, m, n : \xi), \quad \xi = x, y, z, \quad -\infty < (l, m, n) < \infty \quad (2 \cdot 1)$$

constitutes the bases of infinite-dimensional vector space describing the possible displacements of the atoms in the crystal. Now combine these vectors into

$$e(k_x, k_y, k_z; \xi) = \sum_{-\infty < l, m, n < \infty} D_{lmn}^{k_x k_y k_z} e(l, m, n; \xi), \quad \xi = x, y, z,$$

$$D_{lmn}^{k_x k_y k_z} = c \exp\{i(lk_x + mk_y + nk_z)\}, \quad -\pi \leq k_x, k_y, k_z \leq \pi. \quad (2.2)$$

$$\sum_{-\infty < l, m, n < \infty} D_{lmn}^{k_x k_y k_z} D_{lmn}^{k'_x k'_y k'_z} = \delta(k_x, k'_x) \delta(k_y, k'_y) \delta(k_z, k'_z),$$

which make up a normalized orthogonal system.

For the translation p, q, r in x, y, z -directions, the vector $e(l, m, n; \xi)$ is transformed into

$$\mathbf{T}(p, q, r)e(l, m, n; \xi) = e(l+p, m+q, n+r; \xi) \quad (2.3)$$

and consequently

$$\mathbf{T}(p, q, r)e(k_x, k_y, k_z; \xi) = \exp\{-i(pk_x + qk_y + rk_z)\}e(k_x, k_y, k_z; \xi). \quad (2.4)$$

This shows that these three vectors $e(k_x, k_y, k_z; \xi)$, $\xi = x, y, z$, defined for one set of $K(k_x, k_y, k_z)$ value, are the bases of the same one-dimensional irreducible representation of the translational group. The normal coordinates of vibrations in a crystal are the linear combinations of the vectors $e(l, m, n; \xi)$. Thus the secular equation (of an infinite order) for determining frequencies, constructed with $e(l, m, n; \xi)$ as basis, is known to be reducible to those with dimensions not more than three, for each set of K value. The reduction is independent of the form of potential energy functions. The form of the transformation matrix \mathbf{M} is to be the following as seen from (2.2).

	\dots	$k_x k_y k_z : x$	$k_x k_y k_z : y$	$k_x k_y k_z : z$	\dots	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	
$lmn : x$	\dots	$D_{lmn}^{k_x k_y k_z}$	0	0	\dots	
$\mathbf{M} = lmn : y$	\dots	0	$D_{lmn}^{k_x k_y k_z}$	0	\dots	
$lmn : z$	\dots	0	0	$D_{lmn}^{k_x k_y k_z}$	\dots	
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	

(2.5)

For simplicity we have given here only 9 elements of $(l, m, n; \xi | k_x, k_y, k_z; \eta)$, $\xi, \eta = x, y, z$ explicitly. With this matrix \mathbf{M} , we shall next transform the infinite matrix \mathbf{A} of the quantity $V = (\mathbf{A} \mathbf{v}, \mathbf{v})$. Here V is the potential energy which is assumed to be quadratic form of the vector

$$\mathbf{v} = \sum_{l, m, n} e(l, m, n; x) u(l, m, n) + e(l, m, n; y) v(l, m, n) + e(l, m, n; z) w(l, m, n). \quad (2.6)$$

We assume that V involves only central force depending upon the distances between atoms, and further interaction between the nearest neighbours and the second nearest

neighbours are taken into consideration for the sake of simplicity. If u and β are force constants for the respective interaction, V is expressed as

$$V = \frac{u}{2} \sum_{\text{nearest}} (u_i - u_k)^2 + \sum_{\text{2nd nearest}} \beta (u_j - u_l)^2, \quad (2.7)$$

where u_i is a displacement vector of i -th atom. We rewrite this displacement quantity by the components u, v, w and put into (2.7). Then we obtain a quadratic form of potential energy V . The elements of matrix A are easily known and it can easily be shown that $(l, m, n : x | l, m, n : x)$ element, the diagonal element of the matrix, is $u + 4\beta$, and while the non-vanishing elements off diagonal such as $(l, m, n : x | l+1, m, n : x)$ is $-\frac{u}{2}$. Therefore V consists of terms $(u + 4\beta)u(l, m, n)^2 - 2\frac{u}{2}u(l, m, n) \times u(l+1, m, n) \dots$. When A is transformed by M , A is decomposed into a number of three dimensional matrices $\mathfrak{D}(k_x, k_y, k_z)$, each labelled by one set of K value. After some calculations we obtain the matrix $\mathfrak{D}(k_x, k_y, k_z)$ in the following form :

$$\mathfrak{D}(k_x, k_y, k_z) = \begin{pmatrix} A(k_x k_y k_z) & B(k_x k_y) & B(k_x k_z) \\ B(k_y k_x) & A(k_y k_z k_x) & B(k_y k_z) \\ B(k_z k_x) & B(k_z k_y) & A(k_z k_x k_y) \end{pmatrix}, \quad (2.8)$$

$$A(k_x k_y k_z) = (u + 4\beta) - u \cos k_x - 2\beta(\cos k_x \cos k_y + \cos k_x \cos k_z),$$

$$B(k_x k_y) = B(k_y k_x) = 2\beta \sin k_x \sin k_y,$$

$$-\pi \leq k_x, k_y, k_z \leq \pi. \quad (2.9)$$

(2.9) is just the same equation as one which was investigated by Born and others⁴⁾ from the infinite set of differential equation. But this group-theoretical method has the merits of making the form of eigenvectors (normal modes of vibration) obvious. We can obtain frequencies by solving this 3-dimensional secular equation and express the eigen-vector corresponding to each eigen-value by liner combination of vectors $e(k_x, k_y, k_z : \xi)$. We have to notice that these solutions can be obtained solely from the most fundamental operations of the translational group characteristic to the crystal; in this sense these eigenvectors are to be called *translational normal vectors of vibrations*. So far we have been concerned with the translational symmetry of the lattice. The full symmetry of the lattice is represented by a space group. So that we have to examine the representation problem concerning to the point group O_h , which is obtained by deviding the space group by the invariant sub-group T .

For one element of point group O_h , say, a rotation $\frac{\pi}{2}$ about z axis referring to a certain fixed lattice point; $R_z\left(\frac{\pi}{2}\right)$, we have

$$R_z\left(\frac{\pi}{2}\right)e(l, m, n : x) = e(-m, l, n : y). \quad (2.10)$$

Then following equation holds

$$\begin{aligned} R_z\left(\frac{\pi}{2}\right)e(k_x, k_y, k_z; x) &= \sum D_{lmn}^{k_x k_y k_z} R_z\left(\frac{\pi}{2}\right)e(l, m, n; x) \\ &= \sum D_{lmn}^{k_x k_y k_z} e(-m, l, n; y) = \sum D_{-mln}^{-k_y k_x k_z} e(-m, l, n; y) = e(-k_y, k_x, k_z; y). \end{aligned} \quad (2.11)$$

We notice the transformation (2.11) is exactly the same as (2.10); that is to say, the operation of a rotation of the group O_h on the translational normal vectors $e(K; \xi)$ can be obtained by changing the arguments K, ξ into K', ξ' in such a way that the unit vector in the direction $\hat{\xi}$ at the point K in K -space is transformed by the same operation into another unit vector in the direction $\hat{\xi}'$ at the point K' . Thus we can find the representation for the elements of group O_h . The dimension of representation at a general point $k_x \neq k_y \neq k_z$ in K -space is as large as 144. Now we resolve the spurs of representation of elements of O_h which operate vectors $e(k_x, k_y, k_z; \xi)$, and have

$$3A_{1g,u} + 3A_{2g,u} + 6E_{g,u} + 9T_{1g,u} + 9T_{2g,u}. \quad (2.12)$$

From this fact, we see that the reduction of the secular equation to less than 3 degrees will be impossible, if we do not make some simplifying assumption for potential energy. We tried to transform this 144 dimensional matrix for our specially assumed potential but in vain. Generally speaking, in the case that n atoms are included in unit cell of crystal, and they are all different from each other, they can not be combined each other by translational operations, and the secular equation for one K value would be $3n$ dimensions (for example 6 for NaCl), namely the transformed matrix $\mathfrak{D}(k_x, k_y, k_z)$ would be $3n$ dimensions. Resolution for factor group are also n times of (2.11) only. In conclusion, the secular equation can not be reduced to dimensions less than $3n$, by space group. But the reduction is sometimes possible at a special point or along special lines in K -space.

B.-S.-W. introduced the theory of K -group. For instance, if we collect only those elements of O_h which leaves the vector $e(k_x, k_y, k_z; \xi)$ invariant, they constitute a sub-group of O_h , which is called (k_x, k_y, k_z) -group, one of K -groups. Out of these whole K -groups we pick up three K -groups only, namely (k, k, k) -group, $(k, k, 0)$ -group, and $(k, 0, 0)$ -group which are interesting for us. Operating each element belonging to each K -group upon the vectors $e(k, k, k; \xi)$, $e(k, k, 0; \xi)$ and $e(k, 0, 0; \xi)$, we obtain the following representations and resolutions by irreducible representations for individual K -group.

(1) (k, k, k) -group

Element	E	$C_3(x, y, z)$	$C_3^2(x, y, z)$	$JC_2(x-y)$	$JC_2(x-z)$	$JC_2(y-z)$
Class	(1)	(2)	(2)	(3)	(3)	(3)
Rep.	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
Res.	$A_1 + A_3$					(2.13)

(2) $(k, k, 0)$ -group

Element	E	$C_2(x, y)$	$JC_4^2(z)$	$JC_2(x-y)$
Class	(1)	(2)	(3)	(4)
Rep.	$\begin{pmatrix} 1 & \\ & 1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \\ & & 1 \end{pmatrix}$
Res.	$\sum_1 + \sum_3 + \sum_4$			

(2.14)

 (3) $(k, 0, 0)$ -group

Element	E	$C_4^2(x)$	$C_4(x)$	$C_4^3(x)$	$JC_4^2(z)$	$JC_4^2(y)$	$JC_2(yz)$	$JC_2(y-z)$
Class	(1)	(2)	(3)	(3)	(4)	(4)	(5)	(5)
Rep.	$\begin{pmatrix} 1 & \\ & 1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & -1 \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \\ & & 1 \end{pmatrix}$
Res.	$4_1 + 4_3$							

(2.15)

Here $C_2(z, y, z)$, $C_2(x-y)$, $C_4(z)$ are elements of group O_h , and they show a rotation $2\pi/3$ about the diagonal axis of x, y, z axes, a rotation π about the axis which bisect x, y , axes and a rotation $\pi/2$ about z axis..... J is an inversion with respect to the origin of the coordinates. The notation $A_1, A_2, \sum_1, \sum_3, \sum_4, 4_1, 4_3$, represent the irreducible representations of K -groups named by $B.-S.-W.$, out of which $A_2, 4_3$ are ones of 2-dimensions, and others of 1-dimension.

The above relation shows no repetition of the same irreducible representation and from this ground, we understand that secular equation is to be reduced down up to one dimension along the above-mentioned 3 directions in K -space. The transformation matrix for obtaining the bases which give the irreducible representations have the following forms :

$$\begin{aligned} & \mathbf{M}(k, k, k) & \mathbf{M}(k, k, 0) & \mathbf{M}(k, 0, 0) \\ & \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1/\sqrt{2} & \sqrt{3}/2 \\ 1 & -\sqrt{2} & 0 \\ 1 & 1/\sqrt{2} & -\sqrt{3}/2 \end{pmatrix}, & \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}, & \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}. \end{aligned} \quad (2.16)$$

Matrix $\mathbf{M}(k, 0, 0)$ shows that $\mathcal{D}(k, 0, 0)$ has already been reduced to 1 dimension. If we transform matrices $\mathcal{D}(k, k, k)$, and $\mathcal{D}(k, k, 0)$ by matrices $\mathbf{M}(k, k, k)$ and $\mathbf{M}(k, k, 0)$, they are completely reduced down as the following equation shows.

$$\begin{aligned} & \mathbf{M}^{-1}(k, k, k) \mathcal{D}(k, k, k) \mathbf{M}(k, k, k) & \mathbf{M}^{-1}(k, k, 0) \mathcal{D}(k, k, 0) \mathbf{M}(k, k, 0) \\ & \begin{pmatrix} A+2B & 0 & 0 \\ 0 & A-B & 0 \\ 0 & 0 & A-B \end{pmatrix} & \begin{pmatrix} A'+B' & 0 & 0 \\ 0 & A'-B' & 0 \\ 0 & 0 & C' \end{pmatrix}. \end{aligned} \quad (2.17)$$

$$\mathcal{D}(k_x, k_y, k_z)$$

$$\begin{pmatrix} A(k_x k_y k_z)/m & B(k_x k_y)/m & B(k_x k_z)/m & C(k_x) & 0 & 0 \\ B(k_y k_x)/m & A(k_y k_z k_x)/m & B(k_y k_z)/m & 0 & C(k_y) & 0 \\ B(k_z k_x)/m & B(k_z k_y)/m & A(k_z k_x k_y)/m & 0 & 0 & C(k_z) \\ C(k_x) & 0 & 0 & A(k_x k_y k_z)/M & B(k_x k_y)/M & B(k_x k_z)/M \\ 0 & C(k_y) & 0 & B(k_y k_x)/M & A(k_y k_z k_x)/M & B(k_y k_z)/M \\ 0 & 0 & C(k_z) & B(k_z k_x)/M & B(k_z k_y)/M & A(k_z k_x k_y)/M \end{pmatrix}. \quad (3.2)$$

In this equation, m and M are masses of atoms and constants A , B and C are defined as follows :

$$A(k_x, k_y, k_z) = a + 4\beta - 2\beta \cos k_x (\cos k_y + \cos k_z),$$

$$B(k_x k_y) = 2\beta \sin k_x \sin k_y,$$

$$C(k_x) = -a \cos k_x / \sqrt{mM}.$$

The transformation matrices by K -groups are the simple enlargement of (2.13) (2.14) (2.15), as shown in the following equation :

$$\mathcal{M}(k, k, k)$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1/\sqrt{2} & \sqrt{3}/2 & \cdot & \cdot & \cdot \\ 1 & -\sqrt{2} & 0 & \cdot & \cdot & \cdot \\ 1 & 1/\sqrt{2} & -\sqrt{3}/2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & 1/\sqrt{2} & \sqrt{3}/2 \\ \cdot & \cdot & \cdot & 1 & -\sqrt{2} & 0 \\ \cdot & \cdot & \cdot & 1 & 1/\sqrt{2} & -\sqrt{3}/2 \end{pmatrix},$$

$$\mathcal{M}(k, k, 0)$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3)$$

When \mathcal{D} matrix (3.2) are transformed by \mathcal{M} of (3.3) we obtain the equations

$$\mathbf{M}^{-1}(kkk)\mathfrak{D}(kkk)\mathbf{M}(kkk) \quad \mathbf{M}^{-1}(kk0)\mathfrak{D}(kk0)\mathbf{M}(kk0)$$

$$\begin{pmatrix} (A+2B)/m & C \\ C & (A+2B)/M \end{pmatrix}, \quad \begin{pmatrix} D/m & C(0) \\ C(0) & D/M \end{pmatrix}, \quad (3.4)$$

$$2 \begin{pmatrix} (A-B)/m & C \\ C & (A-B)/M \end{pmatrix}, \quad \begin{pmatrix} (A+B)/m & C(k) \\ C(k) & (A+B)/M \end{pmatrix}, \quad (3.5)$$

$$\begin{pmatrix} (A-B)/m & C(k) \\ C(k) & (A-B)/M \end{pmatrix},$$

$$A = a + 4\beta - 4\beta \cos^2 k,$$

$$A = a + 4\beta - 2\beta \cos^2 k - 2\beta \cos k,$$

$$B = 2\beta \sin^2 k,$$

$$B = 2\beta \sin^2 k,$$

$$C = -a \cos k / \sqrt{mM},$$

$$C = -a \cos k / \sqrt{mM},$$

$$D = a + 4B - 4\beta \cos k.$$

Thus the secular equation are, as we expected, reduced down to 2nd degrees. It is very simple problems to obtain the eigen-frequencies from (3.5).

(2) *Diamond* In the case of diamond which has two like C atoms in a unit cell, the treatment is different to (1), except the operation of translation. The translational matrix \mathbf{M} has the same type as (3.1) completely, and (l, m, n) and $(l+1, m+1, n+1)$ are numbers which represent the coordinate of 2 like C atoms. From the view-point of translational operations, however, they can be treated as different atoms a and b , because the two atoms are never interchanged by translation. Hereafter we denote atom of a type, and b type. Some calculation leads us to

$$\mathfrak{D}(k_x k_y k_z) = \begin{pmatrix} A(k_x k_y k_z) & B(k_x k_y) & B(k_x k_z) & C(k_x k_y k_z) & D(k_x k_y k_z) & D(k_y k_z k_x) \\ B(k_y k_x) & A(k_y k_z k_x) & B(k_y k_z) & D(k_z k_x k_y) & C(k_y k_z k_x) & D(k_x k_y k_z) \\ B(k_z k_x) & B(k_z k_y) & A(k_z k_x k_y) & D(k_y k_z k_x) & D(k_x k_y k_z) & C(k_z k_x k_y) \\ C(k_x k_y k_z) & D(k_z k_x k_y) & \bar{D}(k_y k_z k_x) & A(k_x k_y k_z) & B(k_x k_y) & B(k_x k_z) \\ D(k_z k_x k_y) & \bar{C}(k_y k_z k_x) & \bar{D}(k_x k_y k_z) & B(k_y k_x) & A(k_y k_z k_y) & B(k_y k_z) \\ \bar{D}(k_y k_z k_x) & \bar{D}(k_x k_y k_z) & \bar{C}(k_z k_x k_y) & B(k_z k_x) & B(k_z k_y) & A(k_z k_x k_y) \end{pmatrix}, \quad (3.6)$$

$$A(k_x k_y k_z) = 2a + 4\beta - 2B \cos 2k_x (\cos k_y + \cos 2k_z),$$

$$B(k_x k_y) = 2\beta \sin 2k_x \sin 2k_y,$$

$$C(k_x k_y k_z) = -a \{ \cos(k_y + k_z) e^{ik_x} + \cos(k_y - k_z) e^{-ik_x} \},$$

$$D(k_x k_y k_z) = -a \{ \cos(k_y + k_z) e^{ik_x} - \cos(k_y - k_z) e^{-ik_x} \},$$

and \bar{C} , \bar{D} are conjugate complex of C , D .

The representation by the rotational operations are quite different from (1). For instance in the case of rotation $\pi/2$ about z axis is

$$\begin{aligned} R_z(\pi/2)e_a(k_x, k_y, k_z; z) &= \sum_a R_z(\pi/2) D_{lmn}^{k_x k_y k_z} e_a(l, m, n; x) \\ &= \sum_b D_{lmn}^{k_x k_y k_z} e_b(-m, l, n; y) = \sum_b D_{-mln}^{-k_y k_x k_z} e_b(-m, l, n; y) = e(-k_y, k_x, k_z; y). \end{aligned} \quad (3.7)$$

As the equation shows, the interchange of a with b often happens, and the representation by \mathbf{K} -groups are not simple multiples of (2.13), (2.14), and (2.15). They are,

(1) (k, k, k) -group

Element	E	$C_2(xyz)$	$C_3^2(xyz)$	$JC_2(x-y)$	$JC_2(x-z)$	$JC_2(y-z)$
Class	(1)	(2)	(2)	(3)	(3)	(3)
Rep.	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$
Res.	$2A_1 + 2A_3$					(3.8)
Trans. Mat.	$\mathbf{M}(k, k, k)$					

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cdot & 1/\sqrt{2} & \cdot & \sqrt{3}/2 & \cdot \\ 1 & \cdot & -\sqrt{2} & \cdot & \cdot & \cdot \\ 1 & \cdot & 1/\sqrt{2} & \cdot & -\sqrt{3}/2 & \cdot \\ \cdot & 1 & \cdot & 1/\sqrt{2} & \cdot & \sqrt{3}/2 \\ \cdot & 1 & \cdot & -\sqrt{2} & \cdot & \cdot \\ \cdot & 1 & \cdot & 1/\sqrt{2} & \cdot & -\sqrt{3}/2 \end{pmatrix}$$

(2) $(k, k, 0)$ -group

Element	E	$C_2(xy)$	$JC_4^2(z)$	$JC_2(x-y)$
Class	(1)	(2)	(3)	(4)
Rep.	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} & & 1 \\ & 1 & \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$,
Res.	$2\Sigma_1+2\Sigma_3+\Sigma_2+\Sigma_4$, (3.9)			
Trans. Mat.	$M(k,k,0)$			

$$\frac{1}{2} \begin{pmatrix} 1 & . & 1 & . & 1 & 1 \\ 1 & . & 1 & . & -1 & -1 \\ . & \sqrt{2} & . & \sqrt{2} & . & . \\ 1 & . & -1 & . & -1 & 1 \\ 1 & . & -1 & . & 1 & -1 \\ . & -\sqrt{2} & . & \sqrt{2} & . & . \end{pmatrix}$$

(3) $(k, 0, 0)$ -group

Element	E	$C_4^2(x)$	$C_4(x)$	$C_4^3(x)$
Class	(1)	(2)	(3)	(3)
Rep.	$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}$	$\begin{pmatrix} & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \end{pmatrix}$	$\begin{pmatrix} & & & & 1 & \\ & & & & & 1 \\ & & & & & & -1 \\ 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \end{pmatrix}$
	$J C_4^2(z)$	$J C_4^2(y)$	$J C_2(yz)$	$J C_2(y-2)$
	(4)	(4)	(5)	(5)
	$\begin{pmatrix} & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \end{pmatrix}$	$\begin{pmatrix} & & 1 & & & \\ & & & -1 & & \\ & & & & 1 & \\ 1 & & & & & \\ & -1 & & & & \\ & & 1 & & & \end{pmatrix}$	$\begin{pmatrix} 1 & & & & & \\ & & -1 & & & \\ & & & 1 & & \\ -1 & & & & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -1 & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \end{pmatrix}$

$$\text{Res.} \quad A_1 + A_2^1 + 2A_3, \quad (3 \cdot 10)$$

Trans. Mat. $M(k, 0, 0)$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & . & . & . & . \\ . & . & 1 & 1 & . & . \\ . & . & . & . & 1 & 1 \\ 1 & -1 & . & . & . & . \\ . & . & 1 & -1 & . & . \\ . & . & . & . & 1 & -1 \end{pmatrix}$$

The transformed matrix $M^{-1} \mathcal{D} M$ have the following forms:

$$(1) \quad \begin{pmatrix} A+2B & C+2D \\ \bar{C}+2\bar{D} & A+2B \end{pmatrix}, \quad 2 \begin{pmatrix} A-B & C-D \\ \bar{C}-\bar{D} & A-B \end{pmatrix}, \quad (3 \cdot 11)$$

$$A=2\alpha+4\beta(\sin 2k)^2, \quad B=2\beta(\sin 2k)^2,$$

$$\begin{aligned}
 (2) \quad & C = -u(\cos 2k e^{ik} + e^{-ik}), \quad D = -u(\cos 2k e^{ik} - e^{-ik}), \\
 & \begin{pmatrix} A+B+C+D & -\sqrt{2} E \\ \sqrt{2} E & F-C \end{pmatrix}, \quad \begin{pmatrix} A+B-C-D & \sqrt{2} E \\ -\sqrt{2} E & F+C \end{pmatrix}, \quad (3.12) \\
 & (A-B-C+D), \quad (A-B+C-D); \\
 & A = 2u + 4\beta - 2\beta \cos 2k (\cos 2k + 1), \quad B = 2\beta (\sin 2k)^2, \\
 & C = -u(\cos 2k + 1), \quad D = u(1 - \cos 2k), \quad E = -iu \sin 2k, \\
 & F = 2u + 4\beta - 4\beta \cos 2k.
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad & 2 \begin{pmatrix} B+C & -D \\ D & B-C \end{pmatrix}, \quad (A+C), \quad (A-C), \quad (3.13) \\
 & A = 2u + 8\beta \sin^2 k, \quad B = 2u + 4\beta \sin^2 k, \quad C = -2u \cos k, \\
 & D = -2iu \sin k.
 \end{aligned}$$

§ 4. Calculation of distribution functions of vibrational frequencies by Houston

Houston calculated the approximate distribution functions of the frequencies on a simple cubic crystal. We applied his method to more complex crystals. We calculated the frequencies applying our group theoretical method of reduction along the 3 directions in \mathbf{K} -space.

For NaCl we solve the secular equation of (3.4), (3.5), and $\mathfrak{D}(k, 0, 0)$, and represent the eigen-frequencies by the function of \mathbf{K} . The ratio of mass we assume for simplicity $m: M = 1:2$. Here Q is some constant times of eigen-frequencies. After some calculation, we have

$$\begin{aligned}
 Q_1^2 &= 1/2 \{ (3 + 1.2 \sin^2 k) \pm [(3 + 1.2 \sin^2 k)^2 - 4(3.6 \sin^2 k + 0.32 \sin^4 k)]^{1/2} \}, \quad (4.1) \\
 Q_2^2 &= 1/2 \{ (3 + 0.3 \sin^2 k) \pm [(3 + 0.3 \sin^2 k)^2 - 4(2.4 \sin^2 k + 0.02 \sin^4 k)]^{1/2} \},
 \end{aligned}$$

$$\begin{aligned}
 Q_3^2 &= 1/2 \{ (3.6 - 0.6 \cos k) \pm [(3.6 - 0.6 \cos k)^2 - 4(0.88 - 0.96 \cos k + 0.08 \cos^2 k)]^{1/2} \}, \\
 Q_4^2 &= 1/2 \{ (3.9 - 0.6 \cos^2 k - 0.3 \cos k) \pm [(3.9 - 0.6 \cos^2 k - 0.3 \cos k)^2 \\
 &\quad - 4(3.38 - 0.52 \cos k - 3.02 \cos^2 k + 0.08 \cos^3 k + 0.08 \cos^4 k)]^{1/2} \}, \quad (4.2)
 \end{aligned}$$

$$\begin{aligned}
 Q_5^2 &= 1/2 \{ (3.3 - 0.3 \cos k) \pm [(3.3 - 0.3 \cos k)^2 - 4(2.42 - 0.44 \cos k - 1.98 \cos^2 k)]^{1/2} \}, \\
 Q_6^2 &= 1/2 \{ (3.3 - 0.3 \cos k) \pm [(3.3 - 0.3 \cos k)^2 - 4(0.02 \cos^2 k - 0.44 \cos k + 0.42)]^{1/2} \}, \\
 Q_7^2 &= 1/2 \{ (3.6 - 0.6 \cos k) \pm [(3.6 - 0.6 \cos k)^2 - 4(2.88 - 0.96 \cos k - 1.92 \cos^2 k)]^{1/2} \}. \quad (4.3)
 \end{aligned}$$

And similarly, for diamond, we have

$$\begin{aligned}
 Q_1^2 &= (4 + 0.8 \sin^2 2k) \pm 2(3 \cos^2 2k + 1)^{1/2}, \quad (4.4) \\
 Q_2^2 &= 0.2 \sin^2 2k, \quad 0.2 \sin^2 2k + 8,
 \end{aligned}$$

$$\begin{aligned}
Q_3^2 &= 0.4 \sin^2 k, \\
Q_4^2 &= 0.4 \sin^2 k + 8, \\
Q_5^2 &= (-0.2 \cos^2 2k - 1.3 \cos 2k + 5.5) \pm [(-0.2 \cos^2 2k - 1.3 \cos 2k + 5.5)^2 \\
&\quad + 4(0.16 \cos^3 2k + 0.32 \cos^2 2k + 244/50 \cos 2k - 268/50)]^{1/2}, \\
Q_6^2 &= (-0.2 \cos^2 2k + 0.7 \cos 2k + 3.5) \pm [(-0.2 \cos^2 2k + 0.7 \cos 2k + 3.5)^2 \\
&\quad - 4(0.24 \cos^3 2k - 0.52 \cos^2 2k - 0.48 \cos 2k + 0.76)]^{1/2}, \\
Q_7^2 &= -0.8 \cos^2 k - 4 \cos k + 4.8, \\
Q_8^2 &= -0.8 \cos^2 k + 4 \cos k + 4.8, \\
Q_9^2 &= 1/2 \{ (8.8 - 0.8 \cos^2 k) \pm [(8.8 - 0.8 \cos^2 k)^2 \\
&\quad - 4(4/25 \cos^4 k - 88/25 \cos^2 k + 84/25)]^{1/2} \}.
\end{aligned} \tag{4.5}$$

Houston showed the approximate distribution functions are given by the equation.

$$N(\nu) = \frac{4\pi}{280} \{ 89 F(k, 0, 0) + 92 F(k, k, 0) + 99^* F(k, k, k) \}.$$

After laborious numerical work, we obtained the distribution functions for NaCl and diamond as shown in Fig. (a) and (b).

The second order Raman effect in crystal are explained by superpositions of combination-tones and over-tones of ground frequencies calculated in \mathbf{K} -space. We could obtain the similar distribution curves as Born and Bradburn did.

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Statistical Theory of Hindered Rotation in Molecular Crystals, II

— Quantum Effects —

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It is an aim of this paper to explain the quantum effects on the phase transformations due to molecular hinderance in solid hydrogen halides. Although the transition points of these substances are situated in so high temperatures that the behaviour of the system might be regarded as almost classical, there exist the effects of quantum correction, which can be seen in the isotope effects.

For the purpose of studying this effects we proceed on the line of the Hartree approximation, where the Hamiltonian contains the kinetic operators in addition to the intermolecular potential ones. The location of the transition point and the jump in the specific heats are obtained, where the nature of phase transformation is of the second kind. In these obtained expressions the moment of inertia of the molecule is contained.

Making use of the transition points of HBr and HI, we expect the transition points of DBr and DI as 91,8°K and 71,7°K respectively, in qualitative agreement with the results observed by K. Clusius and G. Wolf:

$$\text{DBr: } 93,5^\circ\text{K}, \quad \text{DI: } 77,3^\circ\text{K}.$$

And further the transformation heats prove to increase, as the quantum effect decreases, which is in agreement with the observations. The critical value of the ratio of the hindering potential to the moment of inertia, beyond which any ordered state is unable to be realized, is also obtained.

§ 1. Introduction

The solid hydrogen halides change their phases at the temperatures marked by

$$\begin{aligned} \text{HCl: } 98,4^\circ\text{K}, \quad \text{HBr: } 116,9^\circ\text{K}, \quad 113,6^\circ\text{K}, \quad 89,8^\circ\text{K}, \\ \text{HI: } 125,7^\circ\text{K}, \quad 70,1^\circ\text{K}, \end{aligned}$$

respectively¹⁾. According to the Kirkwood theory of rotational phase transformation²⁾, there should exist the dielectric anomaly at the point of rotational phase transformation, just as the antiferromagnetics show the peak of the magnetic susceptibilities at the Curie point. The points which just correspond to the above situation proved to be

$$\text{HCl } 98,4^\circ\text{K}, \quad \text{HBr } 89,8^\circ\text{K}, \quad \text{HI } 70,1^\circ\text{K}$$

from the measurements of dielectric constants by Smith and Hitchcock³⁾.

The intermolecular potential of solid hydrogen halides is now obscure. Kirkwood approximated the potential of two neighbouring molecules by

$$(J/2)\cos\gamma_{ij}, \quad \cos\gamma_{ij} = \cos\theta_i \cos\theta_j + \sin\theta_i \sin\theta_j \cos(\varphi_i - \varphi_j), \quad (1)$$

where (θ_k, φ_k) is the orientational angle of the symmetry axis of molecule. Based on this

model we discussed the effect of short range order on the rotational phase transformation in the previous paper¹⁾. Recently, T. Oguchi²⁾ discussed the same problem based on the simplified form of dipole interaction. We think, however, it is possible to obtain some insights into the physical situation in the rotational phase transformation by use of (1), for which is probably responsible the large repulsive force between two neighbouring halide ions.

The treatments of hindered rotation made by Kirkwood and others⁶⁾ proceed on the line of classical theory, of which only one exception is the original theory of Pauling⁷⁾. However, Pauling's theory is very naïve in its character and the quantum statistical treatment of hindered rotation stands far from the classical theory of Kirkwood.

How much effects has the quantum mechanical behaviour of molecules on the rotational phase transformation? To see this order of magnitude, it might be well to compare the transition temperature T_c with the rotational degeneracy temperature θ defined by

$$k\theta = \hbar^2/2I, \quad (2)$$

where k is the Boltzmann constant, \hbar the Planck constant divided by 2π and I the moment of inertia of the molecule. The θ -values of hydrogen halide molecules are given by⁷⁾

$$\text{HCl } 15.43^\circ\text{K}, \quad \text{HBr } 11.9^\circ\text{K}, \quad \text{HI } 9.2^\circ\text{K}.$$

We see that T_c is as high as about ten times θ , and hence the system proves to be almost classical at the transition temperature.

However, θ/T_c is about ten percent and so we cannot allow to consider that the quantum effect is negligibly small at T_c . K. Clusius and G. Wolf⁸⁾ investigated the thermal properties of solid deuterium halides and they found the transition temperatures shift from those of hydrogen halides. According to their results, the temperatures of phase transformation are as follows:

$$\begin{array}{lll} \text{DCl: } 105.0^\circ\text{K}, & \text{DBr: } 120.3^\circ\text{K}, & 93.5^\circ\text{K}, \\ \text{DI: } 128.3^\circ\text{K}, & 77.3^\circ\text{K}. & \end{array}$$

The transition temperatures of deuterium halides shift to the high temperature side compared with those of hydrogen halides. It is also notable that the number of phase transition of DBr is two, while that of HBr is three. This isotope effect should be interpreted from the quantum mechanical behaviour which survives in the molecules at the transition temperature.

Confining our attention to the rotational transformation, we shall discuss the quantum effect on the phase transformation. For the sake of simplicity we assume the intermolecular potential given by (1) and the Hartree approximation. Hence, our theory may be regarded as the quantum mechanical analogue of the Kirkwood theory. However, since it seems to be difficult to obtain the Slater sum in a closed form, the Slater sum is expanded in power series of s , which measures the degree of order, and so the calculated results are valid only in the neighbourhood of the transition point. But, we attain to the right estimation of the transition point and the jump in the specific heats at the transition point.

§ 2. Evaluation of the Slater sum

Let us assume that the ordered state of the system is realized by the antiparallel configuration of N heteropolar molecules and the polar axis of our polar coordinates system, which specifies the orientation of molecules, is taken as parallel to this antiparallel direction. Then, all lattice points are divided into two sublattices: the parallel and antiparallel sublattices, whose lattice points are denoted by i and j respectively. The common index to both sublattices is denoted by k . The total Hamiltonian of the system, \mathcal{H} , is given by

$$\mathcal{H} = \xi \sum_k A_k + \sum_{\langle i,j \rangle} (J/2) \cos \gamma_{ij} \quad (3)$$

where

$$A_k = - \left(\frac{1}{\sin \theta_k} \frac{\partial}{\partial \theta_k} \sin \theta_k \frac{\partial}{\partial \theta_k} + \frac{1}{\sin^2 \theta_k} \frac{\partial^2}{\partial \varphi_k^2} \right), \quad (3a)$$

$$\xi = \hbar^2 / 2I, \quad (3b)$$

and $\langle ij \rangle$ denotes two neighbouring pair.

On the assumption of the Hartree approximation, the density matrix of the whole system ρ_N is given by

$$\rho_N = \prod_{k=1}^N \rho_1(k) \quad (4)$$

where (k) denotes the index dependent only on the state of the k -th molecule and we assume that ρ_1 is normalized as

$$\text{Tr } \rho_1(k) = 1. \quad (5)$$

The average energy of the system, U ;

$$U = \text{Tr } \rho_N \mathcal{H} \quad (6)$$

is then written

$$U = \xi \sum_k \text{Tr } \rho_1(k) A_k + (J/2) \sum_{\langle i,j \rangle} \text{Tr } \rho_1(i) \rho_1(j) \cos \gamma_{ij}, \quad (6')$$

where we used (4) and (5). In the same way the entropy of the system, $S^{(0)}$:

$$S = -k \text{Tr } \rho_N \ln \rho_N \quad (7)$$

is written

$$S = -k \sum_k \text{Tr } \rho_1(k) \ln \rho_1(k). \quad (7')$$

Since the free energy of the system, $F(\rho_1)$, which may be regarded as a functional of ρ_1 , is

$$F(\rho_1) = U(\rho_1) - TS(\rho_1), \quad (8)$$

the minimum condition of $F(\rho_1)$:

$$\partial F(\rho_1) / \partial \rho_1 = 0 \quad (9)$$

with the restricting condition of (5) leads to

$$\rho_1(i) = \exp \beta [\lambda - \xi A_i - (J/2) \sum_j \text{Tr } (\cos \gamma_{ij} \cdot \rho_1(j))]_j, \quad (10)$$

and a similar equation for $\rho_1(j)$,

$$\beta = 1/kT,$$

where $Tr(\cdots)_j$ denotes the diagonal sum over (j) , which is the state level of a molecule neighbouring with the i -th molecule, and the undetermined multiplier λ is determined from the normalization condition (5) as

$$\exp(-\beta\lambda) = Tr \exp \beta [-\xi A_i - (J/2) \sum_j Tr (\cos \gamma_{ij} \cdot \rho_1(j))_j], \quad (11)$$

being the Slater sum of the i -th molecule.

Using (1) and regarding a property of the density matrix :

$$Tr ((A \text{ function of } \theta_i, \varphi_i), \rho_1(i)) = \langle (A \text{ function of } \theta_i, \varphi_i) \rangle_{AV}, \quad (12)$$

we have

$$Tr (\cos \gamma_{ij} \rho_1(j))_j = -s \cos \theta_i, \quad (13)$$

$$\begin{aligned} s &= -Tr (\cos \theta_j \cdot \rho_1(j)) \\ &= Tr (\cos \theta_i \cdot \rho_1(i)), \end{aligned} \quad (13a)$$

where we assumed

$$\langle \sin \theta_j \cos \varphi_j \rangle_{AV} = \langle \sin \theta_i \cos \varphi_i \rangle_{AV} = 0,$$

$$\langle \sin \theta_j \sin \varphi_j \rangle_{AV} = \langle \sin \theta_i \sin \varphi_i \rangle_{AV} = 0$$

in our coordinates system. Accordingly, if we assume that a molecule is surrounded by z molecules effectively, whose number is of course equal to the number of nearest neighbouring molecules when they are parallel to each other, (10) and (11) are written

$$\rho_1(i) = \exp [\beta\lambda - \beta\xi A_i + (\beta z J s / 2) \cos \theta_i], \quad (14)$$

$$\exp(-\beta\lambda) = Tr \exp [-\beta\xi A_i + (\beta z J s / 2) \cos \theta_i] \quad (15)$$

respectively. And (13a) is written

$$s = -\beta \partial \lambda / \partial (\beta z J s / 2) s \quad (16)$$

(see Appendix I). The similar equation for $\rho_1(j)$ is obtained in the same way but it might be omitted, since we can proceed on without it.

The density matrix ρ_1 is expressed in terms of a parameter s and it is determined from (16). Hence, our problem reduces to the evaluation of the Slater sum $\exp(-\beta\lambda)$. However it seems to be difficult to obtain the expression of $\exp(-\beta\lambda)$ in a closed form, and so we expand this in the power series of s .

For this purpose we use the Bloch equation for ρ_1 :

$$\partial \rho_1 / \partial \beta = (\lambda - \xi A + (z J s / 2) \cos \theta) \cdot \rho_1 \quad (17)$$

obtained from (14). Introducing $G(\beta)$ defined by

$$\rho_1(\beta) = \exp(\beta\lambda) \exp(-\beta\xi A) \cdot G(\beta), \quad (18)$$

we obtain the equation for $G(\beta)$ by substituting (18) into (17) as follows⁽¹⁰⁾ :

$$\partial G(\beta)/\partial\beta = (\varepsilon J s/2) \exp(\beta \xi A) \cdot \cos \theta \cdot \exp(-\beta \xi A) G(\beta). \quad (19)$$

Integrating (19) with respect to β and considering $G(0) = 1$, we have an inhomogenous integral equation

$$G(\beta) = 1 + (\varepsilon J s/2) \int_0^\beta \exp(\beta' \xi A) \cdot \cos \theta \cdot \exp(-\beta' \xi A) \cdot G(\beta') d\beta'. \quad (20)$$

By use of the iteration method, (20) is expanded in the power series of s and $\rho_1(\beta)$ is written

$$\begin{aligned} \exp(-\beta \lambda) \rho_1(\beta) &= \exp(-\beta \xi A) + (\varepsilon J s/2) \exp(-\beta \xi A) \int_0^\beta \exp(\beta' \xi A) \cdot \cos \theta \cdot \\ &\cdot \exp(-\beta' \xi A) d\beta' + (\varepsilon J s/2)^2 \exp(-\beta \xi A) \cdot \int_0^\beta \exp(\beta' \xi A) \cdot \cos \theta \cdot \\ &\cdot \exp(-\beta' \xi A) \cdot \int_0^{\beta'} \exp(\beta'' \xi A) \cdot \cos \theta \cdot \exp(-\beta'' \xi A) d\beta'' d\beta' + \dots \end{aligned} \quad (21)$$

Let us now take the representation which diagonalizes A . As is well known, the diagonal element of $\cos \theta$ is zero in this representation, accordingly in the trace of (21) the terms of odd power of s vanish identically and so

$$\begin{aligned} \exp(-\beta \lambda) &= \text{Tr} \{ \exp(-\beta \xi A) + (\varepsilon J s/2)^2 \exp(-\beta \xi A) \\ &\cdot \int_0^\beta \exp(\beta' \xi A) \cdot \cos \theta \cdot \exp(-\beta' \xi A) \cdot \int_0^{\beta'} \exp(\beta'' \xi A) \cdot \cos \theta \cdot \\ &\cdot \exp(-\beta'' \xi A) d\beta'' d\beta' + O((\varepsilon J s/2)^4) \}. \end{aligned} \quad (22)$$

By use of the matrix element of $\cos \theta$:

$$\begin{aligned} \langle l', m' | \cos \theta | l-1, m \rangle &= \langle l-1, m | \cos \theta | l', m' \rangle \\ &= \left\{ \frac{(l+|m|)(l-|m|)}{(2l+1)(2l-1)} \right\}^{1/2} \delta_{l'l'} \delta_{m'm}, \end{aligned} \quad (23)$$

(22) is evaluated as

$$\begin{aligned} \exp(-\beta \lambda) &= D_0(\beta \xi) + (\beta \varepsilon J s/2)^2 (1/6 \beta \xi^2) \\ &+ \left(\frac{\beta \varepsilon J s}{2} \right)^4 \left[\frac{1}{18} \left(\frac{1}{2\beta \xi^2} \right)^2 - \frac{1}{15} \cdot \frac{11}{9} \left(\frac{1}{2\beta \xi^2} \right)^3 + \frac{1}{10} D_1(\beta \xi) \left(\frac{1}{2\beta \xi^2} \right)^2 \right. \\ &\left. + \frac{1}{15} D_2(\beta \xi) \left(\frac{1}{2\beta \xi^2} \right)^3 \right] + \dots \end{aligned} \quad (24)$$

after somewhat tedious calculations (see Appendix II), where

$$D_0(\zeta) = \sum_{l=0}^{\infty} (2l+1) \exp(-\zeta l(l+1)), \quad (24a)$$

$$D_1(\zeta) = \sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)(2l-1)(2l+3)} \exp(-\zeta l(l+1)), \quad (24b)$$

$$D_2(\zeta) = \sum_{l=1}^{\infty} \frac{(2l+1)(24l^2+24l-9)}{l^2(l+1)^2(2l-1)^2(2l+3)^2} \exp(-\zeta l(l+1)). \quad (24c)$$

§ 3. The location of the Curie point and the anomaly of the specific heats

The subsequent procedure is now in complete parallelism with I¹⁾. It is convenient to introduce two parameters x and σ defined by

$$\beta z J / 6 = x, \quad (25a)$$

$$6\tilde{\epsilon} / z J = \sigma \quad (25b)$$

respectively. By use of (24), (16) is written

$$A_1(\sigma, x)s - A_3(\sigma, x)s^3 + \cdots = 0, \quad (26)$$

$$A_1(\sigma, x) = \sigma^{-1} - D_0(\sigma x), \quad (26a)$$

$$(9x^2)^{-1}A_3(\sigma, x) = \frac{1}{6\sigma x} - \frac{1}{6\sigma^2 x} + \frac{11}{90} \cdot \frac{1}{\sigma^3 x^2} - \frac{3}{10} D_1(\sigma x) \frac{1}{\sigma^2 x} - \frac{1}{10} D_2(\sigma x) \frac{1}{\sigma^3 x^2} \quad (26b)$$

in terms of x and σ .

As x increases from 0 to ∞ , $A_1(\sigma, x)$ increases from $-\infty$ to

$$A_1(\sigma, \infty) = \sigma^{-1} - 1. \quad (27)$$

Then, if $A_1(\sigma, \infty)$ is positive, (26) has real positive roots approximated by

$$s^2 = A_1(\sigma, x) / A_3(\sigma, x) \quad (28)$$

in a certain region of x for which

$$A_1(\sigma, x) > 0, \quad A_3(\sigma, x) > 0$$

are satisfied. The latter condition proves to be satisfied in an intermediate region between $x=0$ and $=\infty$ after numerical inspection, although $A_3(\sigma, 0)=0^{*1)}$ and $A_3(\sigma, \infty)=-\infty$ if $A_1(\sigma, \infty) > 0$.

As $\tilde{\epsilon}$ increases, σ does, hence $A_1(\sigma, \infty)$ decreases. The condition

$$\sigma_c^{-1} - 1 = 0, \quad \text{or} \quad (\tilde{\epsilon} / J)_c = z / 6 \quad (29)$$

gives the critical value of σ_c , beyond which the ordered state cannot be realized. Let us confine our subsequent considerations to the system for which $\sigma < \sigma_c$ is satisfied. Then, the transition point x_c is determined by

$$\sigma D_0(\sigma x_c) = 1, \quad (30)$$

which is the condition that (26) has $s=0$ as a triple root at the transition point. A small σ proves to give a small x_c from the direct inspection of (30). The more detailed information of σ dependency of x_c is obtained from the numerical calculations of (30) and is plotted in Fig. 1.

* This is due to the fact that we have $D_2(0)=1,2217$ after numerical calculation to the first five terms of this series on the one hand and $11/9=1,22222\cdots$ on the other, and hence we expect $D_2(0)=11/9$ although we cannot give a rigorous proof of it.

Let us consider next the behaviour of the system in the ordered state. The order parameter s is determined by (26) and the calculated results are plotted in Fig. 2, in which $x_c/x = T/T_c$ is regarded. In terms of this s the average energy U is expressed. First, the average kinetic energy K is evaluated as

$$\begin{aligned} K &= N\xi^2 \text{Tr } \rho_1 A \\ &= N\beta\xi^2 \partial \lambda / \partial (\beta\xi^2) \\ &= N\xi^2 \left\{ -\frac{D_0'(\sigma x)}{D_0(\sigma x)} + \frac{3}{2} \cdot \frac{x}{\sigma D_0(\sigma x)} \left(\frac{1}{\sigma x} + \frac{D_0'(\sigma x)}{D_0(\sigma x)} \right) s^2 + O(s^4) \right\}. \end{aligned} \quad (31)$$

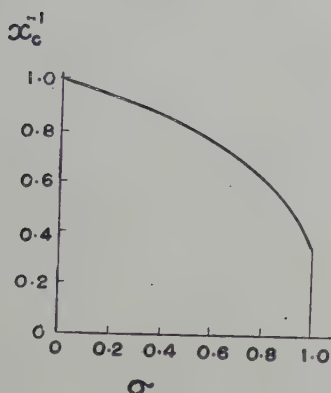


Fig. 1.

Secondly, the average potential energy V is

$$\begin{aligned} V &= (zN/2) (J/2) \text{Tr}(\cos \theta_i \cdot \rho_1(i)) \cdot \\ &\quad \cdot \text{Tr}(\cos \theta_j \cdot \rho_1(j)) \\ &= -(zN/2) (J/2) s^2. \end{aligned} \quad (32)$$

The average energy $U = K + V$ is thus obtained from (31) and (32), and the contribution to it above T_c proves to arise from K exclusively, but this situation is due to the neglect of short range order.

We can now evaluate the specific heats per mol, C_v :

$$C_v/R = -\beta^2 \partial (U/N) / \partial \beta \quad (33)$$

with the help of (31) and (32), where R is the gas constant. And then, in complete parallelism with I, the jump in the specific heats, $\Delta C_v/R$, at the transition point, whose nature is of the second kind as in the classical theory, is obtained as

$$\Delta C_v/R = -\frac{3}{2} \frac{1}{\xi} (\sigma x_c)^3 D_0'(\sigma x_c) \left[\frac{ds^2}{d\beta} \right]_{\beta_c}, \quad (34)$$

where (30) is regarded. By use of (28), (26a) and (30), (34) becomes

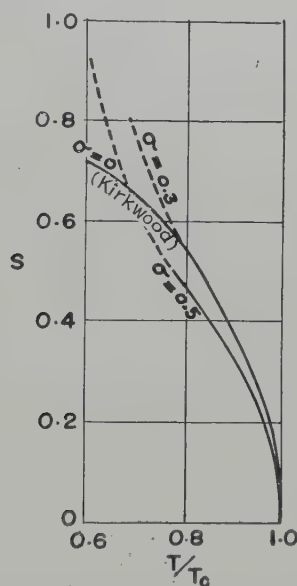


Fig. 2.

$$\Delta C_v/R = 3(\sigma x_c)^3 [D_0'(\sigma x_c)]^2 / 2A_3(\sigma, x_c), \quad (35)$$

which is calculated for various values of σ numerically and its results are plotted in Fig. 3.

§ 4. Discussion

1. Correspondence to the classical theory

As is well known, the quantum theory goes over to the classical one in the limit of $\hbar \rightarrow 0$, which corresponds to $\sigma \rightarrow 0$ in the present theory. Then, we use the following expansion formula for $D_0(\zeta)^{(11),*}$:

$$D_0(\zeta) = 1/\zeta + 1/3 + \zeta/15 + 4\zeta^2/315 + \zeta^3/315. \quad (36)$$

Substituting (36) into (30) and solving for x_c , we have

$$x_c = 1 + 1/3 \cdot \sigma + 8/45 \cdot \sigma^2 + 22/189 \cdot \sigma^3 + \dots \quad (37)$$

In the limit of $\sigma = 0$, (37) becomes

$$T_c = zJ/6k, \quad (38)$$

being identical with Kirkwood's formula exactly.

Unfortunately, we could not obtain the asymptotic expansions of $D_1(\zeta)$ and $D_2(\zeta)$, where the application of the Euler-Maclaurin summation formula seems to break down. However, according to the results of numerical calculations $\Delta C_v/R$ seems to converge to $5/2$, which is identical with $\Delta C_v/R$ in Kirkwood's theory, as σ tends to zero. Accordingly, the correspondence of the present theory to the classical one is satisfactory.

On account of the nature of our approximation, (28) is valid only in the neighbourhood of T_c and this valid interval decreases with decreasing σ . This situation originates from the present perturbation method in which the representation diagonalizing the kinetic part of the Hamiltonian is taken. It will be rather convenient to use the representation diagonalizing the potential part of the Hamiltonian in the cases of almost classical systems. However, even for small σ the present method is effective in the neighbourhood of the transition point, and hence it leads to the correct estimations of T_c and $\Delta C_v/R$ for all possible σ .

2. Comparison with the experiment

As was described in § 1, the thermal behaviour of solid hydrogen halides is regarded as almost classical in the neighbourhood of the transition point. Hence, for the purpose of estimating the magnitude of J , (37) can be used. Then, from (37) J is expressed in terms of T_c and θ as follows:

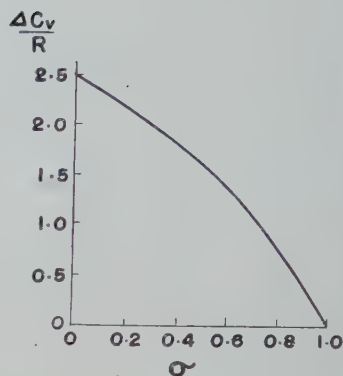


Fig. 3.

* In the reference 11 the first four terms are seen, but the fifth term is newly calculated by the Euler-Maclaurin summation formula.

$$zJ = kT_c \left\{ 6 + 2 \left(\frac{\theta}{T_c} \right) + \frac{2}{5} \left(\frac{\theta}{T_c} \right)^2 + \frac{8}{105} \left(\frac{\theta}{T_c} \right)^3 + \dots \right\}, \quad (39)$$

where we used (2) and (25 a, b). As in Kirkwood's theory the nature of the transition predicted by the present theory is of the second kind. Except HCl which shows the transition of the first kind, HBr and HI are suited for the application of the present theory. Substituting θ and T_c of HBr and HI into (39), we obtain zJ/k , by use of which we compute σ of deuterium halide with the help of θ obtained from the moment of inertia of deuterium halide I , and then we substitute the above obtained zJ/k and σ into (37) to obtain T_c 's of deuterium halide. These results are summarized in Table I.

Table I

	DBr	DI
$zJ/k^\circ K$	563.24	439.50
$I \times 10^{10} \text{ g cm}^2$	6.53	8.54
$\theta^\circ K$	6.18	4.72
σ	0.066	0.064
$(T_c)_{\text{comp.}}$	91.8	71.7
$(T_c)_{\text{obs.}}$	93.5	77.3

The agreement with the observed values is only qualitative but towards the right direction. Considering that the present theory is based on the most naïve approximations, one should content oneself with this small success. It should be noted that the deviation of the predicted T_c from the observed one is considerable in DI.

According to Clusius and Wolf's observations, the transformation heats of hydrogen halide are small compared with those of deuterium halide.

It is explained from the fact that $\Delta C_p/R$ decreases with increasing σ as is seen in Fig. 3.

In conclusion, the writer expresses his cordial thanks to Prof. K. Husimi for calling attention to the Hartree approximation, and to Assist. Profs. T. Tanaka and K. Yosida and Prof. T. Matsubara for their kind interests.

Appendix I

Let us prove

$$\text{Tr } A \cdot \exp(xA + yB) = \frac{\partial}{\partial x} \text{Tr } \exp(xA + yB), \quad (\text{A} \cdot 1)$$

in which A and B are not commutable with each other. In an arbitrary representation,

$$\begin{aligned} \frac{\partial}{\partial x} \text{Tr } \exp(xA + yB) &= \frac{\partial}{\partial x} \sum_m \sum_{k=0}^{\infty} \frac{1}{k!} \langle m | (xA + yB)^k | m \rangle \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{m, m_1, \dots, m_{k-1}} \frac{\partial}{\partial x} \langle m | xA + yB | m_1 \rangle \langle m_1 | xA + yB | m_2 \rangle \dots \langle m_{k-1} | xA + yB | m \rangle \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{m, m_1, \dots, m_{k-1}} \{ \langle m | A | m_1 \rangle \langle m_1 | xA + yB | m_2 \rangle \dots \langle m_{k-1} | xA + yB | m \rangle \\ &\quad + \langle m_1 | A | m_2 \rangle \langle m_2 | xA + yB | m_3 \rangle \dots \langle m | xA + yB | m_1 \rangle \\ &\quad + \dots \dots \\ &\quad + \langle m_{k-1} | A | m \rangle \langle m | xA + yB | m_1 \rangle \dots \langle m_{k-2} | xA + yB | m_{k-1} \rangle \} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \sum_{m, m_1, \dots, m_{k-1}} \langle m|A|m_1\rangle \langle m_1|xA+yB|m_2\rangle \cdots \langle m_{k-1}|xA+yB|m\rangle \\
&= \sum_{m, m_1} \langle m|A|m_1\rangle \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \sum_{m_2, m_3, \dots, m_{k-1}} \langle m_1|xA+yB|m_2\rangle \\
&\quad \times \langle m_2|xA+yB|m_3\rangle \cdots \langle m_{k-1}|xA+yB|m\rangle \\
&= \sum_{m, m_1} \langle m|A|m_1\rangle \langle m_1|\exp(xA+yB)|m\rangle \\
&= \text{Tr } A \cdot \exp(xA+yB).
\end{aligned}$$

Appendix II

We shall derive (24) in the text from (22) in the same. Let us first find the coefficient of $(\varepsilon f s/2)^2$, Q , in (22). In the representation diagonalizing A , whose eigenvalue is denoted by A' ,

$$\begin{aligned}
Q &= \text{Tr} \exp(-\beta \hat{\xi} A) \cdot \int_0^\beta \exp(\beta' \hat{\xi} A) \cdot \cos \theta \cdot \exp(-\beta' \hat{\xi} A) \cdot \int_0^{\beta'} \exp(\beta'' \hat{\xi} A) \\
&\quad \cdot \cos \theta \cdot \exp(-\beta'' \hat{\xi} A) d\beta'' d\beta' \\
&= \sum_{A', A''} \langle A' | \cos \theta | A'' \rangle^2 \exp(-\beta \hat{\xi} A') \int_0^\beta \exp \beta' \hat{\xi} (A' - A'') \int_0^{\beta'} \exp \beta'' \hat{\xi} (A'' - A') \cdot d\beta'' d\beta' \\
&= \sum_{A', A''} \langle A' | \cos \theta | A'' \rangle^2 \left\{ \frac{\beta \exp(-\beta \hat{\xi} A')}{\hat{\xi} (A'' - A')} + \frac{\exp(-\beta \hat{\xi} A'') - \exp(-\beta \hat{\xi} A')}{\hat{\xi} (A'' - A')^2} \right\}. \quad (\text{A} \cdot 2)
\end{aligned}$$

An eigenvalue $A' = l(l+1)$ is $2l+1$ fold degenerate, and hence the summation over these degenerate levels in (A.2) reduces to that over $\langle A' | \cos \theta | A'' \rangle$, whose result is

$$\begin{aligned}
\sum_{m=-l}^l \langle l, m | \cos \theta | l-1, m \rangle^2 &= \sum_{m=-l}^l \frac{l^2 - m^2}{4l^2 - 1} \\
&= l/3, \quad (\text{A} \cdot 3)
\end{aligned}$$

where we used (23) in text and $\sum_{m=-l}^l m^2 = l(l+1)(2l+1)/3$. Then, substituting (A.3) into (A.2), Q is evaluated as

$$Q = \beta/6\hat{\xi}, \quad (\text{A} \cdot 4)$$

which is the contribution from $l=0$ and other terms cancel out with each other.

Let us next find the coefficient of $(\varepsilon f s/2)^4$, $\text{Tr } R$, in which R is an operator defined by

$$\begin{aligned}
R &= \exp(-\beta \hat{\xi} A) \cdot \int_0^\beta \exp(\beta' \hat{\xi} A) \cdot \cos \theta \cdot \exp(-\beta' \hat{\xi} A) \cdot \int_0^{\beta'} \exp(\beta'' \hat{\xi} A) \\
&\quad \cdot \cos \theta \cdot \exp(-\beta'' \hat{\xi} A) \cdot \int_0^{\beta'''} \exp(\beta''' \hat{\xi} A) \cdot \cos \theta \cdot \exp(-\beta''' \hat{\xi} A) \cdot \\
&\quad \cdot \int_0^{\beta''''} \exp(\beta'''' \hat{\xi} A) \cdot \cos \theta \cdot \exp(-\beta'''' \hat{\xi} A) d\beta'''' d\beta''' d\beta'' d\beta'. \quad (\text{A} \cdot 5)
\end{aligned}$$

Now, $Tr R$ gives the diagonal sum over the products of four matrices and the terms

$$[]_{\Lambda' \Lambda''} []_{\Lambda'' \Lambda'''} []_{\Lambda''' \Lambda^{IV}} []_{\Lambda^{IV} \Lambda'}$$

which appear in the diagonal element of these products, prove to be confined to the following three types from the inspection of (23) in text :

Type \ State	Λ'	Λ''	Λ'''	Λ^{IV}
$R_{\Lambda' \Lambda'}^{(1)}$	Λ'	Λ''	Λ'''	Λ''
$R_{\Lambda' \Lambda'}^{(2)}$	Λ'	Λ''	Λ'	Λ''
$R_{\Lambda' \Lambda'}^{(3)}$	Λ'	Λ''	Λ'	Λ'''

The diagonal elements corresponding to these types are obtained in the same way as (A.2) as follows :

$$\begin{aligned}
 R_{\Lambda' \Lambda'}^{(1)} = & \sum_{\Lambda'', \Lambda'''} \langle \Lambda' | \cos \theta | \Lambda'' \rangle^2 \langle \Lambda'' | \cos \theta | \Lambda''' \rangle^2 \left\{ \frac{\beta \exp(-\beta \xi \Lambda')}{\xi^3 (\Lambda'' - \Lambda')^2 (\Lambda''' - \Lambda')} \right. \\
 & + \frac{\beta \exp(-\beta \xi \Lambda'')}{\xi^3 (\Lambda'' - \Lambda')^2 (\Lambda''' - \Lambda'')} + \frac{\exp(-\beta \xi \Lambda'') - \exp(-\beta \xi \Lambda')}{\xi^4 (\Lambda'' - \Lambda')^2} \left(\frac{1}{(\Lambda'' - \Lambda') (\Lambda''' - \Lambda')} \right. \\
 & + \frac{1}{(\Lambda''' - \Lambda') (\Lambda''' - \Lambda'')} + \frac{1}{(\Lambda'' - \Lambda') (\Lambda''' - \Lambda'')} - \frac{1}{(\Lambda''' - \Lambda')^2} \Big) \\
 & \left. + \frac{\exp(-\beta \xi \Lambda''') - \exp(-\beta \xi \Lambda')}{\xi^4 (\Lambda''' - \Lambda')^2 (\Lambda''' - \Lambda'')} \right\}, \quad (A.6)
 \end{aligned}$$

$$\begin{aligned}
 R_{\Lambda' \Lambda'}^{(2)} = & \sum_{\Lambda''} \langle \Lambda' | \cos \theta | \Lambda'' \rangle^4 \frac{1}{\xi^2 (\Lambda'' - \Lambda')^2} \left\{ \frac{\beta^2}{2} \exp(-\beta \xi \Lambda') - \frac{2\beta \exp(-\beta \xi \Lambda')}{\xi (\Lambda'' - \Lambda')} \right. \\
 & \left. - \frac{\beta \exp(-\beta \xi \Lambda'')}{\xi (\Lambda'' - \Lambda')} - 3 \frac{\exp(-\beta \xi \Lambda'') - \exp(-\beta \xi \Lambda')}{\xi^2 (\Lambda'' - \Lambda')^2} \right\}, \quad (A.7)
 \end{aligned}$$

$$\begin{aligned}
 R_{\Lambda' \Lambda'}^{(3)} = & \sum_{\Lambda'', \Lambda'''} \langle \Lambda' | \cos \theta | \Lambda'' \rangle^2 \langle \Lambda'' | \cos \theta | \Lambda''' \rangle^2 \left\{ \frac{(\beta^2/2) \exp(-\beta \xi \Lambda')}{\xi^2 (\Lambda''' - \Lambda') (\Lambda'' - \Lambda')} \right. \\
 & - \frac{\beta \exp(-\beta \xi \Lambda')}{\xi^2 (\Lambda'' - \Lambda') (\Lambda''' - \Lambda')} \left(\frac{1}{\xi (\Lambda'' - \Lambda')} + \frac{1}{\xi (\Lambda''' - \Lambda')} \right) + \frac{\exp(-\beta \xi \Lambda'') - \exp(-\beta \xi \Lambda')}{\xi^4 (\Lambda'' - \Lambda') (\Lambda''' - \Lambda'')} \\
 & \left. - \frac{\exp(-\beta \xi \Lambda''') - \exp(-\beta \xi \Lambda')}{\xi^4 (\Lambda''' - \Lambda')^2 (\Lambda''' - \Lambda'')} \right\}. \quad (A.8)
 \end{aligned}$$

By using

$$\sum_{m=-l}^l \langle l, m | \cos \theta | l+1, m \rangle^2 \langle l+1, m | \cos \theta | l+2, m \rangle^2 = \frac{2}{15} \frac{(l+1)(l+2)}{2l+3} \quad (A.9)$$

obtained from (23) in text and

$$\sum_{m=-l}^l m^4 = (1/5) l(l+1)(2l+1) \{l(l+1) - 1/3\}$$

and then substituting $\mathcal{N} = l(l+1)$ into (A.6), the summation of $R_{\Delta'\Delta}^{(1)}$ over the degenerate levels gives

$$\begin{aligned}
 \sum_{m=-l}^l \langle l, m | R^{(1)} | l, m \rangle &= \frac{2}{15} \frac{(l+1)(l+2)}{2l+3} \left[\frac{\beta \exp(-\beta \hat{\zeta} l(l+1))}{8\hat{\zeta}^3(l+1)^2(2l+3)} \right. \\
 &+ \frac{\beta \exp(-\beta \hat{\zeta} (l+1)(l+2))}{8\hat{\zeta}^3(l+1)^2(l+2)} + \frac{l+3}{16\hat{\zeta}^4(l+1)^3(l+2)^2} \{ \exp(-\beta \hat{\zeta} (l+1)(l+2)) \\
 &- \exp(-\beta \hat{\zeta} l(l+1)) \} + \frac{\exp(-\beta \hat{\zeta} (l+2)(l+3)) - \exp(-\beta \hat{\zeta} l(l+1))}{16\hat{\zeta}^4(2l+3)^2(l+2)^2} \Big] \\
 &+ \frac{2}{15} \frac{l(l-1)}{2l-1} \left[-\frac{\beta \exp(-\beta \hat{\zeta} l(l+1))}{8\hat{\zeta}^3 l^2(2l-1)} - \frac{\beta \exp(-\beta \hat{\zeta} l(l-1))}{8\hat{\zeta}^3 l^2(l-1)} \right. \\
 &+ \frac{l-2}{16\hat{\zeta}^4 l^3(l-1)^2} \{ \exp(-\beta \hat{\zeta} l(l-1)) - \exp(-\beta \hat{\zeta} l(l+1)) \} \\
 &\left. + \frac{\exp(-\beta \hat{\zeta} (l-1)(l-2)) - \exp(-\beta \hat{\zeta} l(l+1))}{16\hat{\zeta}^4(2l-1)^2(l-1)^2} \right], \quad (\text{A.11})
 \end{aligned}$$

where the latter term arises from the transition of the type:

$$(l, m) \rightarrow (l-1, m) \rightarrow (l-2, m) \rightarrow (l-1, m) \rightarrow (l, m)$$

and is restricted to l 's which satisfy $l \geq 2$.

The similar calculations of $Tr R^{(2)}$ and $Tr R^{(3)}$ are in complete parallelism with (A.11). We have only to use

$$\sum_{m=-l}^l \langle l, m | \cos \theta | l-1, m \rangle^4 = \frac{1}{15} \frac{l(4l^2+1)}{(2l+1)(2l-1)}, \quad (\text{A.12})$$

and

$$\sum_{m=-l}^l \langle l, m | \cos \theta | l-1, m \rangle^2 \langle l, m | \cos \theta | l+1, m \rangle^2 = \frac{2}{15} \frac{l(l+1)}{2l+1} \quad (\text{A.13})$$

instead of (A.9). These results are omitted for the sake of brevity.

After some tedious calculations, we obtain

$$\begin{aligned}
 Tr R &= \sum_{\Delta'} (R_{\Delta'\Delta}^{(1)} + R_{\Delta'\Delta}^{(2)} + R_{\Delta'\Delta}^{(3)}) \\
 &= \frac{1}{18} \left(\frac{\beta}{2\hat{\zeta}} \right)^2 - \frac{1}{15} \cdot \frac{11}{9} \frac{\beta}{(2\hat{\zeta})^3} + \frac{1}{10} \sum_{l=1}^{\infty} \frac{(2l+1) \exp(-\beta \hat{\zeta} l(l+1))}{l(l+1)(2l-1)(2l+3)} \left(\frac{\beta}{2\hat{\zeta}} \right)^2 \\
 &+ \frac{1}{15} \sum_{l=1}^{\infty} \frac{(2l+1)(24l^2+24l-9)}{l^2(l+1)^2(2l-1)^2(2l+3)^2} \exp(-\beta \hat{\zeta} l(l+1)) \frac{\beta}{(2\hat{\zeta})^3}. \quad (\text{A.14})
 \end{aligned}$$

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Letters to the Editor

Perturbation Theory of Relativistic Eigenvalue Problem

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In the recent work on the mass-corrections to the fine structure of hydrogen-like atoms, Salpeter has presented the perturbation theory of the relativistic eigenvalue problem for two-particle system.¹⁾ The formulation is essentially based on the facts: there is no external field and thus we have only to do with the coordinates, the motion of the center of gravity being completely separated. Moreover, the unperturbed interaction between particles is assumed to be instantaneous. This reduces rigorously the equation to the one involving only equal time variables for the two particles which corresponds to the Breit equation. His method is, accordingly, not applicable to the cases where there is an external field or the unperturbed interaction is noninstantaneous. We shall offer in this note a primitive but more general perturbation theory including Salpeter's as a special case.

We consider two electrons in an external field which is assumed to be independent of time.²⁾ Such a system is described by the relativistic equation

$$[(i\gamma(P-eA)+M)_a(i\gamma(P-eA)+M)_b-\bar{I}]\psi=0, \quad (1)$$

where P_μ stands for $1/i \cdot \partial/\partial x_\mu$, A is the external field, M the mass operator, \bar{I} the interaction kernel. a and b label the respective electrons. We replace the mass operator M by the electron mass m and consider the remaining terms as being absorbed into \bar{I} .³⁾ We also disregard the effects of pair creation by the external field. Defining the total energy and the relative energy by

$$(P_{a0}+P_{b0})=E \text{ and } 1/2 \cdot (P_{a0}-P_{b0})=\epsilon$$

respectively, (1) is rewritten as

$$(F-I)\psi=0, \quad (1')$$

where $I=\beta^a\beta^b\bar{I}$ and

$$\left. \begin{aligned} F(E, \epsilon) &= [1/2 \cdot E - H_a(\mathbf{P}_a) + \epsilon] \\ &\quad \times [1/2 \cdot E - H_b(\mathbf{P}_b) - \epsilon], \\ H_a(\mathbf{P}_a) &= [\alpha^a(\mathbf{P}_a - e\mathbf{A}) + eA_0 + m\beta^a], \\ H_b(\mathbf{P}_b) &= [\alpha^b(\mathbf{P}_b - e\mathbf{A}) + eA_0 + m\beta^b]. \end{aligned} \right\} \quad (2)$$

Now, we divide the total energy E into the unperturbed energy E_0 and the perturbed $\Delta E = E_1 + E_2 + \dots$, and expand, correspondingly, F as

$$F = F_0 + (\Delta F)' F' + (\Delta F)'' F'',$$

in which $F_0 = F(E_0, \epsilon)$, $F' = 1/2 \cdot (E_0 - H_a(\mathbf{P}_a) - H_b(\mathbf{P}_b))$ and $F'' = 1/4$. The interaction kernel I and the wave function ψ are also divided into the unperturbed part I_0, ψ_0 , and the perturbed $\Delta I = I_1 + I_2 + \dots$, $\Delta\psi = \psi_1 + \psi_2 + \dots$. Substituting these expansions into (1') and picking up the terms of the same order, we can determine the perturbation energy and the wave function successively. First, as the zeroth order terms, we have the unperturbed equation

$$(F_0 - I_0)\psi_0 = 0. \quad (3)$$

The instantaneity of I_0 is not necessary, although one could hardly solve (3) without it.

We proceed next supposing that the solution ψ_0 and the Green's function G_0 of the equation (3) have been obtained by some method. The first order terms of (1') is the equation

$$(F_0 - I_0)\psi_1 + E_1 F' \psi_0 - I_1 \psi_0 = 0, \quad (4)$$

which determines E_1 and ψ_1 : From the inner product of (4) with ψ_0 and the hermite character of I_0 , we have the first order perturbation energy

$$E_1 = \langle \psi_0, I_1 \psi_0 \rangle / \langle \psi_0, F' \psi_0 \rangle. \quad (5)$$

The denominator and the numerator of (5) have a common divergent factor, i.e., $\int d^4X$ in case of no external fields or $\int dX_0$ in case of time-independent external fields, X_μ being the coordinates of the center of gravity. In what follows we suppose such a factor to be dropped. We also obtain from (4)

$$\psi_1 = G_0 I_1 \psi_0 - E_1 G_0 F' \psi_0. \quad (6)$$

In the same way, the second order terms of (1') gives

$$\begin{aligned} E_2 = 1/ \langle \psi_0, F' \psi_0 \rangle \cdot \{ &\langle \psi_0, I_2 \psi_0 \rangle + \langle \psi_0, I_1 G_0 I_1 \psi_0 \rangle \\ &- E_1 \langle \psi_0, I_1 G_0 F' \psi_0 \rangle \} \end{aligned}$$

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$$-E_1(\psi_0, \hat{F}' G_0 I_1 \psi_0) + E_1^2(\psi_0, \hat{F}' G_0 \hat{F}' \psi_0) \\ - E_1^2(\psi_0, \hat{F}'' \psi_0) \} . \quad (7)$$

Thus it is shown that even if the energy appears quadratically in the fundamental equation, the perturbation energy can be calculated to any order by applying the similar procedure as in the nonrelativistic cases where the equation contains the energy linearly.

When there is no external field, all quantities become the functions of the relative energy-momentum, ϵ and $\mathbf{P} = \mathbf{P}_\alpha = -\mathbf{P}_\beta$. In particular, I_0 is the sole function of the relative momentum if it is instantaneous. Then we can define the three-dimensional wave function $\chi(\mathbf{P})$, following Salpeter, by

$$(A_+^a(\mathbf{P}) A_+^b(\mathbf{P}) - A_-^a(\mathbf{P}) A_-^b(\mathbf{P})) \chi(\mathbf{P}) \\ = \int d\epsilon \psi_0(\epsilon) . \quad (8)$$

Here $A_+^a(\mathbf{P})$, $A_-^a(\mathbf{P})$, $A_+^b(\mathbf{P})$ and $A_-^b(\mathbf{P})$ are the usual Casimir operators; e.g., $A_+^a(\mathbf{P}) = [E_\alpha(\mathbf{P}) + H_\alpha(\mathbf{P})] / 2E_\alpha(\mathbf{P})$ where $E_\alpha(\mathbf{P}) = + (m_\alpha^2 + \mathbf{P}_\alpha^2)^{1/2}$, and $m_\alpha = m_\beta = m$ is supposed to have a small negative imaginary part.

The wave function $\chi(\mathbf{P})$ satisfies the equation

$$[E_0 - H_\alpha(\mathbf{P}) - H_\beta(\mathbf{P})] \chi(\mathbf{P}) \\ = - (2\pi i) \int d^4k I_0(k) \psi_0(\epsilon + k) . \quad (9)$$

From (8) and (9), the ϵ -dependence on $\psi_0(\epsilon)$ is uniquely determined as follows:

$$\psi_0(\epsilon) = - (2\pi i)^{-1} \\ \times \frac{E_0 - H_\alpha(\mathbf{P}) - H_\beta(\mathbf{P})}{[\frac{1}{2}E_0 - H_\alpha(\mathbf{P}) + \epsilon][\frac{1}{2}E_0 - H_\beta(\mathbf{P}) - \epsilon]} \chi(\mathbf{P}) , \quad (10)$$

where how to manage the poles in the integration over ϵ is decided according as $H_\alpha(\mathbf{P})$ takes the value $+E(\mathbf{P})$ or $-E(\mathbf{P})$.

We define next $\psi^\Delta(\epsilon)$ by

$$\psi^\Delta(\epsilon) = - (2\pi i)^{-1} \\ \times \frac{E - H_\alpha(\mathbf{P}) - H_\beta(\mathbf{P})}{[\frac{1}{2}E - H_\alpha(\mathbf{P}) + \epsilon][\frac{1}{2}E - H_\beta(\mathbf{P}) - \epsilon]} \chi(\mathbf{P}) . \quad (11)$$

In (11) E is the eigenvalue of (1) and it is to be noted that $\int \psi_0(\epsilon) d\epsilon = \int \psi^\Delta(\epsilon) d\epsilon$. Then ψ^Δ is easily shown to satisfy

$$- (2\pi i)^{-1} (\Delta E) \chi = (F - I_0) \psi^\Delta(\epsilon) . \quad (12)$$

Writing

$$\psi = \psi^\Delta + \psi_\Delta = \psi_0 + \psi_1^\Delta + \psi_2^\Delta + \dots \\ + \psi_{1\Delta} + \psi_{2\Delta} + \dots ,$$

we get from (1) and (12)

$$- (2\pi i)^{-1} (\Delta E) \chi = (\Delta I) \psi - (F_0 - I_0) \psi_\Delta \\ - [(\Delta F) F' + (\Delta E)^2 F''] \psi_\Delta . \quad (13)$$

This is the fundamental equation of Salpeter's theory, from which the perturbation energies can be calculated. On the other hand, the zeroth order terms of (12) are nothing but the unperturbed equation (3). ψ_1^Δ and $\psi_{1\Delta}$ are obtained from (12) and (13) respectively, and each contains the auxiliary function $\chi(\mathbf{P})$, but their sum is free from $\chi(\mathbf{P})$ and coincides with $\psi_1(\epsilon)$ given by (6). Moreover, the equation (12) serves for showing the generality of our method of perturbation; as the first order terms of (12), we have

$$- (2\pi i)^{-1} E_1 \chi = (F_0 - I_0) \psi_1^\Delta + F_1 F' \psi_0 ,$$

which gives the relation

$$(\psi_0, F' \psi_0) = - (2\pi i)^{-1} (\psi_0, \chi) . \quad (15)$$

(ψ_0, χ) on the right hand side of (15) was normalized to unity by Salpeter. In this case, accordingly, E_1 becomes $- (2\pi i) (\psi_0, I_1 \psi_0)$ in accordance with Salpeter's result. The wave function $\chi(\mathbf{P})$, therefore, seems not to play an essential role. From the second order terms of (12) and (15), we can also prove the relation

$$F_1(\psi_0, F'' \psi_0) = - (\psi_0, F' \psi_1^\Delta) , \quad (16)$$

which reduces our second order perturbation energy (7) to Salpeter's.

It is thus obvious that our primitive perturbation theory, which is valid even in the presence of external field, includes completely Salpeter's method which can be used only in the absence of external field.

This perturbation theory can be applied, for example, to the calculation of the triplet splitting of He-atom, in which the nucleus is considered as an external field A_0 , the Coulomb interactions between two electrons as an unperturbed interaction I_0 , the exchange effects of a transverse photon between electrons as I_1 , and so on.

One of us (M.N.) is much obliged to the Yukawa Fellowship of Osaka University for the financial aid.

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- 2) This assumption about an external field is permissible in most practical cases.
- 3) Such terms may be frequently omitted in practical problems.

New Description of Field —Classical Field (continued)—

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Previously one of the authors tried to make a description of the field by the differential form.¹⁾ At that time, he treated the meson fields as the linearized equations, and gave the clear correspondences to our usual formalism. In the second paper we shall discuss the proof of the Lorentz invariance of the field equations described by the differential form, the derivation of the quadratic equation, and the relation to the Lagrangian formalism. And, the appendix contains the explanation of the differential form.

First of all, we shall prove the Lorentz invariance of the eqs. (1), (2), (3), (4) shown in the first paper. For that purpose, since $\omega_i^{(r)}$ clearly is the Lorentz invariant, it suffices to prove that $d\omega_i^{(r)}$ is the Lorentz invariant. Generally speaking, when in the n -dimensional Euclid space (including the Minkowski space and others of the kind) the domain $D_1(x_1, \dots, x_n)$ maps onto a domain $D_2(y_1, \dots, y_n)$ by the one-to-one correspondence which is continuously differentiable, then the relation

$$(d_x \overline{\omega}) = d_y (\overline{\omega}) \quad (1)$$

is proved, where $\overline{\omega}$ means the transformation of the variables and the suffix of d shows the variables performing the differential operation. Namely, we see that the differential operation and the transformation of the variables are commutable. Accordingly, it can be proved that the equation

$$d\omega^{(r)} = \omega^{(r+1)} \quad (2)$$

is the invariant with the Lorentz transformation. So is the case with the electromagnetic field.

Secondly, we shall derive the quadratic equation of the meson field from the linearized equations. For simplicity, we will deal with the free field. The scalar meson field is described as follows by way of the differential form:

$$\begin{cases} d\omega_1^{(0)} = \kappa\omega_0^{(1)}, \\ d\omega_3^{(3)} = \kappa\omega_3^{(4)}, \end{cases} \quad (3)$$

$$\omega_1^{(0)} = \phi, \quad (3')$$

$$\omega_0^{(1)} = V_1 dx + V_2 dy + V_3 dz - V_0 dt, \quad (3'')$$

$$\begin{aligned} \omega_4^{(3)} = & V_0 dx dy dz - V_1 dy dz dt - V_2 dz dx dt \\ & - V_3 dx dy dt, \end{aligned} \quad (4')$$

$$\omega_3^{(4)} = -\phi dx dy dz dt, \quad (4'')$$

where we choose $\epsilon=1$ and bracket out κ . Now, for the differential form ω :

$$\omega = \sum_{i(1), \dots, i(r)} a_{i(1) \dots i(r)} dx_{i(1)} \dots dx_{i(r)},^* \quad (5)$$

we define the dual form $\tilde{\omega}$ by the following:²⁾

$$\tilde{\omega} = 1/\sqrt{|g|} \cdot \sum_{i(1), \dots, i(r)} a_{i(1) \dots i(r)} \vartheta_{i(1)+1} \dots \vartheta_{i(n)}, \quad (6)$$

$$\vartheta_i = g_{i1} dx_1 + \dots + g_{in} dx_n,$$

$$g = |g_{ik}|,$$

$$i(1), \dots, i(r), i(r+1), \dots, i(n): \text{even permutation.}$$

Then, clearly hold the relations:

$$\begin{cases} \omega_3^{(4)} = \tilde{\omega}_1^{(0)}, \\ \omega_4^{(3)} = \tilde{\omega}_0^{(1)}, \end{cases} \quad (7)$$

$$\quad (8)$$

and is described the scalar meson field as follows:

$$\begin{cases} d\omega_1^{(0)} = \kappa\omega_0^{(1)}, \\ d\omega_0^{(1)} = \kappa\omega_1^{(0)}. \end{cases} \quad (9)$$

$$\quad (10)$$

By the use of eqs. (9), (10) and the general relation

$$d(\tilde{\omega}) = \square \tilde{\omega}, \quad (11)$$

it follows that

$$\square \tilde{\omega}_1^{(0)} = \kappa^2 \tilde{\omega}_1^{(0)}, \quad (12)$$

$$\therefore \square \omega_1^{(0)} = \kappa^2 \omega_1^{(0)}. \quad (13)$$

This equation is not anything but the usual equation:

$$(\square - \kappa^2)\phi = 0. \quad (13'')$$

The vector meson field can be treated similarly.

Last, we shall give the relations to the Lagrangian formalism. In the case of the neutral scalar free field the following relation holds:

$$L dx dy dz dt = 1/2 \cdot (d\omega_1^{(0)} d\tilde{\omega}_1^{(0)} + \kappa^2 \omega_1^{(0)} \tilde{\omega}_1^{(0)}). \quad (14)$$

The same can be said in the other cases. And, we can show the correspondence to the variation principle. Moreover, in the case of the electromagnetic free field the Lagrangian is given as below:

$$L dx dy dz dt = 1/2 \cdot \omega^{(2)} \omega^{(2)'} \quad (15)$$

Appendix: For

$$\omega \in A(C_i) \quad i \geq 1$$

* We use the sign $i(r)$ for the sign i_r , in view of the technical limitation on printing.

expressing

$$\omega = \sum_{i(1) < \dots < i(r)} a_{i(1) \dots i(r)}(x) dx_{i(1)} \dots dx_{i(r)} \quad (16)$$

we define $d\omega$ as

$$d\omega = \sum_{i(1) < \dots < i(r)} \sum_{j=1}^n \frac{\partial a_{i(1) \dots i(r)}(x)}{\partial x_j} \times dx_j dx_{i(1)} \dots dx_{i(r)}, \quad (17)$$

and the following commutation relation holds:

$$dx_i dx_j = -dx_j dx_i. \quad (18)$$

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Some Comments on the Theory of V -particle

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June 22, 1953

The assumption was proposed by the author that V^0 has an unknown degeneracy, that is, V^0 has a large number of accessible internal states.¹⁾ These states are thought to be lying with the density of the order of $10^{10}/m_\pi c^2$, and the total number of them is estimated to be very roughly of the order of 10^{10} . This seems to show that V -particles are composite particles. However, as it is difficult to consider V^0 as a composite structure, we cannot think that the unknown degeneracy of V^0 , as a result of the composite structure of V^0 .

There are two reasons why we cannot consider V^0 as a composite particle as, for example, suggested by Sacks.²⁾ First, if we can explain V^0 as a composite particle composed of a nucleon and some pions, the same explanation will lead to the existence of the singly or multiply charged particles which, in other points rather than charge, have almost the same properties as V^0 . However, there is little or no experimental evidence of this sort of particles. Second, if V^0 is a composite particle, it is very unlikely that V^0 is created so easily. As the binding energy forming this complex particle, we may think, is small

compared with its rest mass, its creation probability will be far smaller than the creation probability of the constituent particles as a separate individuals.

Therefore, we must think that we are facing with a new kind of freedom, and that this freedom is the origin of the anomalous degeneracy of V^0 . Yukawa theory of the non-local fields results a degeneracy for the higher mass levels, even if one can successfully eliminate the degeneracy for the lowest or near the lowest mass levels.³⁾ However, we cannot hope for this theory to explain so large a degeneracy as 10^{10} . Some suitable selection rules as proposed by Pais⁴⁾ may reduce the magnitude of the degeneracy needed for V^0 considerably, and these selection rules may not in their origin be independent with the anomalous degeneracy, though in the present stage we can say nothing about explicitly, in what way the selection rules and the degeneracy are related through the internal structure of the particles, and what is the relative weight between them.

One of the criterions for the models of V -particles is the distribution of Q values. According to, for example, Nishijima's model,⁵⁾ the Q value must have one definite value. However, the empirical Q values show some dispersion. If we have more accurate knowledge on the Q values in future, we may be able to select what is true among various models.

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The β - γ Angular Correlation of Sb^{124}

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Recently L. M. Langer et al.¹⁾ reported that the highest energy β group of Sb^{124} is explained by the

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first forbidden ST type* interaction of Fermi theory with $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta a)$ and $\mathfrak{M}(\beta \sigma \times r)$, where \mathfrak{M} 's are the reduced nuclear matrix elements, and the spin change one, on the ground of their experiments. They found the ratios of the nuclear matrix elements to make the Kurie plot straight as follows:

$$\begin{aligned} G_S \mathfrak{M}(\beta r) \\ G_T \mathfrak{M}(\beta \sigma \times r) &= -ix \quad \text{and} \\ \frac{\mathfrak{M}(\beta a)}{\mathfrak{M}(\beta \sigma \times r)} &= \frac{aZ}{2\rho} y, \quad x=y=1. \end{aligned}$$

For brevity, we denote this combination with $(x, y, 1)$, where x and y are real, and Langer's case is written as $(1, 1, 1)$.

Previously the present authors²⁾ explained both the β - γ angular correlation and the β -ray spectrum of $\text{Sb}^{124} \rightarrow \text{Te}^{124*} \rightarrow \text{Te}^{124}$ assuming the pure tensor interaction with the nuclear matrix elements $\mathfrak{M}(\beta a)$, $\mathfrak{M}(\beta \sigma \times r)$ and $\mathfrak{M}(B_{ij}^B)$ and the electric quadrupole transition in the same spin change $3(-) \rightarrow 2(+)$ $-0(+)$ as Langer's case.¹⁾

Therefore we reinvestigate the β - γ angular correlation with ST type. The formulation of Yamada and Morita³⁾ is used and the angular distribution functions $F_{LL'}^M$ for the ST type are newly calculated by the authors.⁴⁾

In the case with the ratio $(1, 1, 1)$ of \mathfrak{M} 's the β - γ angular correlation coefficient

$$a(W) = \{ \mathfrak{M}(\pi) - \mathfrak{M}(\pi/2) \} / \mathfrak{M}(\pi/2)$$

is too small in the absolute value, $a(5) \doteq +0.02$, and has the opposite sign in comparison with the experimental value. And whatever the ratios $(x, y, 1)$ may be the calculated angular correlations are not so large in the absolute value as the experimental one. Therefore these combinations are definitely ruled out.

If the recently obtained β -spectrum¹⁾ is not so different from the old one⁵⁾ — for we have not yet seen the former — the combinations of $\mathfrak{M}(B_{ij}^B)$ and $(1, 1, 1)$ make the Kurie plots straight because $\mathfrak{M}(B_{ij}^B)$ as well as $(1, 1, 1)$ have this property and their interferences do not exist in the correction factor. The good angular correlation can be obtained with this combination by choosing the ratio r suitably, where $\mathfrak{M}(B_{ij}^B)/\mathfrak{M}(\beta \sigma \times r) = ir$, (r is real).

From these considerations we obtain $r \doteq -7.3$ or -99 in $W=5$. The β - γ angular correlations are

drawn in Figs. 1 and 2, and the several curves are shown within the limits of the experimental errors in each case. We consider the tendencies of the curves in Fig. 2 are not good, therefore, the ratio r is probably $-6 > r > -9$. In these cases the theoretically calculated ft values for $\mathfrak{M}(B_{ij}^B)$ are approximately 9×10^{10} which are considerably larger than the usual ones, but they are not inconsistent with the first forbidden transition. Then it seems that the transition is $3-2-0$ and the β -decay is caused by the nuclear matrix elements $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta a)$, $\mathfrak{M}(\beta \sigma \times r)$ and $\mathfrak{M}(B_{ij}^B)$ in the first forbidden ST type.

Using the newly obtained β -spectrum¹⁾ and taking the nuclear finite size effect and the de Broglie wavelength effect, the ratio r might change somewhat, but the main results will not change.

The work is proceeding in the case of $4-2-0$ ^{2, 6)} of the second forbidden ST type, where $\mathfrak{M}(R_{ij}^B)$, $\mathfrak{M}(A_{ij}^B)$, $\mathfrak{M}(T_{ij}^B)$, $\mathfrak{M}(S_{ijk}^B)$ and their interference terms are concerned. When cancellation is not so

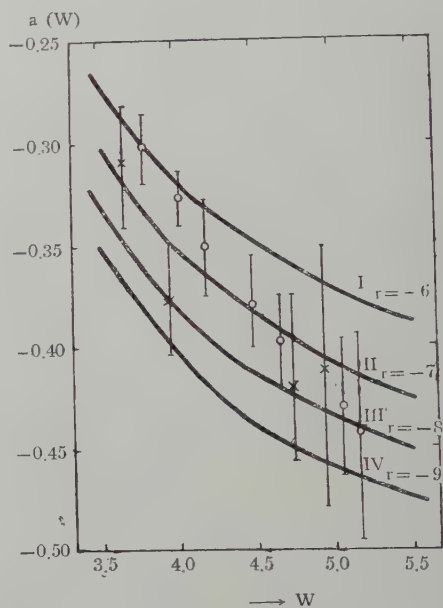


Fig. 1. The angular correlation coefficients for $3(-) \rightarrow 2(+)$ $-0(+)$, in the cases of the combination of $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta a)$, $\mathfrak{M}(\beta \sigma \times r)$, $\mathfrak{M}(B_{ij}^B)$.

I. $r = -6$, II. $r = -7$, III. $r = -8$, IV. $r = -9$.

○ Experimental values of Darby and Opechowski.⁷⁾

x Experimental values of Stevenson and Deutsch.⁷⁾

* This means the combination of scalar and tensor interactions and is not limited by the special values of their coupling constants.

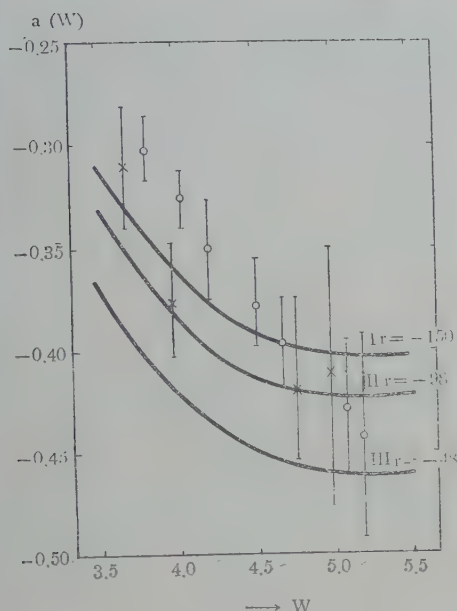


Fig. 2. The angular correlation coefficients for those of Fig. 1.

I. $r = -150$, II. $r = -98$, III. $r = -48$.

large, $\mathcal{M}(R_{ij}^0)$ is like $\mathcal{M}(T_{ij}^0)$ very much, therefore the combinations of $\mathcal{M}(R_{ij}^0)$, $\mathcal{M}(A_{ij}^0)$ and $\mathcal{M}(T_{ij}^0)$ will be perhaps ruled out, as in the case of $\mathcal{M}(A_{ij}^0)$ and $\mathcal{M}(T_{ij}^0)$. If $\mathcal{M}(S_{ijk}^0)$ is taken into account, the result will be similar to the case of the tensor only.²⁾

The authors wish to thank Professor S. Nakamura, Professor L. M. Langer and Mr. M. Umezawa.

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Theory of Rearrangement Collisions

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Recent development of experiments of $d-p$ and $d-n$ reactions has provided powerful means for understanding the nuclear structure, in connection with theories developed by Butler¹⁾ and others. The theory is so constructed on the basis of physical intuition that further elaboration is needed for increasing its mathematical accuracy as well as making clearer its physical meaning. It is the purpose of this paper to construct the theory of the rearrangement collisions such as $d-p$ reactions along the line of the general theory of scattering developed by Lippmann and Schwinger²⁾ and others.³⁾

The theory of rearrangement collisions has been considered only in terms of the Born approximation and of its modification,⁴⁾ but not in a complete general way. Lippmann and Schwinger's theory is concerned only with such a case that the total Hamiltonian is decomposed into free and interaction parts in one way only, namely, that the wave functions of incident and outgoing particles belong to the same orthogonal set. In such cases as $d-p$ reactions and pick up processes, however, the wave function of a deuteron in the initial (final) state evidently belongs to a different orthogonal set from that of outgoing (incident) free nucleons. Unless the total Hamiltonian is decomposed in two different ways in the initial and final states, therefore, the theory is subject to approximation methods such as the impulse approximation. The extension of the theory is required, in order to treat the rearrangement collision rigorously.

Initial and final states are designated by suffices a and b , respectively. The total Hamiltonian H is decomposed in two ways in the respective states.

$$H = H_a^0 + H_a' = H_b^0 + H_b'. \quad (1)$$

The wave functions in respective states when interactions are absent are characterized by

$$(H_a^0 - E_a) \phi_a = 0, \quad (H_b^0 - E_b) \phi_b = 0, \quad (2)$$

with the conservation of energy

$$E_a = E_b. \quad (3)$$

With an incident wave Φ_a outgoing wave $\Psi_a^{(+)}$ is given by the integral equation of Lippmann and Schwinger as

$$\Psi_a^{(+)} = \Phi_a + 1/(E_a + i\epsilon - H_a^0) \cdot H_a' \Psi_a^{(+)} \quad (4a)$$

A similar equation holds for incoming wave $\Psi_b^{(-)}$

$$\Psi_b^{(-)} = \Phi_b + 1/(E_b - i\epsilon - H_b^0) \cdot H_b' \Psi_b^{(-)} \quad (4b)$$

The formal solutions of (4a) and (4b) are obtained as⁵⁾

$$\Psi_a^{(+)} = \Phi_a + 1/(E_a + i\epsilon - H) \cdot H_a' \Phi_a, \quad (5a)$$

$$\Psi_b^{(-)} = \Phi_b + 1/(E_b - i\epsilon - H) \cdot H_b' \Phi_b. \quad (5b)$$

In obtaining transition matrix T_{ba} for the reaction $a \rightarrow b$, an asymptotic form of $\Psi_a^{(+)}$ must be constructed. Since the asymptotic form has to involve a factor $\delta(E_a - H_b^0)$, $(E_a + i\epsilon - H)^{-1}$ is so reduced that the scattered wave possesses a factor $(E_a + i\epsilon - H_b^0)^{-1}$. This can be done as follows.

$$\frac{1}{E_a + i\epsilon - H} = \frac{1}{E_a + i\epsilon - H_b^0} + \frac{1}{E_a + i\epsilon - H_b^0} H_b' \frac{1}{E_a + i\epsilon - H} \quad (6)$$

Substituting (6) into (5a) we obtain

$$\begin{aligned} T_{ba} &= (\Phi_b, H_a' \Phi_a) \\ &+ \left(\Phi_b, H_b' \frac{1}{E_a + i\epsilon - H} H_a' \Phi_a \right) \\ &= (\Psi_b^{(-)}, H_a' \Phi_a). \end{aligned} \quad (7)$$

An alternative expression of (7) is given by

$$\begin{aligned} T_{ba} &= (\Phi_b, H_b' \Psi_a^{(+)}) = (\Phi_b, H_b' \Phi_a) \\ &+ \left(\Phi_b, H_b' \frac{1}{E_a + i\epsilon - H} H_a' \Phi_a \right). \end{aligned} \quad (8)$$

The equality of (7) and (8) can be verified by taking their difference as

$$\begin{aligned} (\Phi_b, (H_a' - H_b') \Phi_a) &= (\Phi_b, (H_b^0 - H_a^0) \Phi_a) \\ &= (E_b - E_a) (\Phi_b, \Phi_a). \end{aligned}$$

The last expression vanishes for $E_b = E_a$, because Φ involves no singularity as Ψ does.* Thus the reciprocity character maintains also in our formalism.

Our formalism can be extended so as to include the symmetry properties of interacting particles and also to the dispersion formula currently employed in the theory of nuclear reactions.

* It must be noticed that the expression

$$\begin{aligned} T_{ba} &= (\Phi_b, H' \Psi_a^{(+)}) \\ &= (\Phi_b, (H - H^0) \Psi_a^{(+)}) \\ &= (E_a - E_b) (\Phi_b, \Psi_a^{(+)}) \text{ for } H_a^0 = H_b^0 \end{aligned}$$

does not vanish for $E_a = E_b$, because $\Psi_a^{(+)}$ possesses a singularity of $(E_a - E_b)^{-1}$.

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On the Conservation of Heavy Particles*

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June 29, 1953

The conservation of heavy particle recently emphasized by Oneda²⁾ is certainly one of the most important problems to be solved in the theory of the interaction of elementary particles.

For it theoretical explanations have been proposed by many authors^{3) 4) 5)} under the guiding principle of restricting the type of the interactions by assuming the invariance of the theory against possible transformations such as the charge conjugation or the time reflection.** Of course such attempts of seeking for selection rules within the framework of the current theory are orthodox ones, but it seems to us that this is a problem to be understood in a relation with a more intrinsic structure of elementary particles. We shall show in this note that the extension of our previous theory to spinor field suggests a way of such approach.

* This note is a direct continuation of our previous one¹⁾ "The Theory of the Structure of Elementary Particles", and the notations are the same with it unless otherwise remarked.

** Recently Oneda and Umezawa⁶⁾ have shown that the conservation of heavy particles can be explained to some extent by assuming, instead of anti, the commutability of field variables describing heavy and light particles.

Starting with a non-local spinor field $\psi_\rho(X_\mu r_\mu)$ ($\rho=1, 2, 3, 4$) which describes the spinor Urmaterie, and defining the spin and the mass operators in an analogous way as that of the scalar case, the spin operator is found to be

$$S^2 = (L + \sigma/2)^2, \quad (1)$$

while the eigenvalue equation for the mass spectrum leads to

$$M^2 \varphi = m^2 \varphi$$

with

$$M^2 = -(1-x^2) \cdot d^2/dx^2 - 3x \cdot d/dx + x^2/(1+x^2) \cdot Q + 3/4 + x\rho_1(\sigma p), \quad (2)$$

where L is the angular momentum, Q its magnitude, p the momentum of the internal motion, and ρ and σ are the usual Dirac matrices.

If S^2 and M^2 compose a complete set of mutually commuting observables with respect to internal coordinates, the specification of elementary particles with the spin and the rest mass is complete, and no other structure constant appears. If, however, another scalar observable that commutes with them exists, it means that the elementary particles possess a structure constant other than the spin and the rest mass. In the case of spinor field, such an observable is provided by contracting $R_{\mu\nu}$ with its dual tensor $\tilde{R}_{\mu\nu}^*$. Some parts of $R_{\mu\nu} \tilde{R}^{\mu\nu}$ are not independent of S^2 and M^2 , and omitting such irrelevant terms, we are left with

$$\Theta = \rho_1. \quad (3)$$

Although Θ is pseudoscalar, it is easy to construct a scalar observable that inherits the essential feature of it,** and we disregard this odd character in the following to simplify the discussion.

Θ has eigenvalues ± 1 . As will be seen from (2), Θ is closely related to the eigenvalue of m^2 , and in classifying all spinor particles into two families according to this eigenvalue, the eigenvalue ± 1 determine the minimum value of the rest mass appearing in these two families. Although eq. (2) has not yet been solved exactly, a preliminary estimation treating $x\rho_1(\sigma p)$ as a small perturbation yields

$$m^2 \min. \approx \begin{cases} 5 & \text{for } \Theta = 1 \\ 0 & \text{for } \Theta = -1. \end{cases}$$

* In the case of scalar, $R_{\mu\nu} \tilde{R}^{\mu\nu}$ vanishes identically.

** Such procedure as an example is to double the components of $\psi_\rho(X_\mu r_\mu)$ into eight, and to introduce an independent set of Pauli matrices other than ρ and σ .

As was discussed in our previous note, m represents the rest mass of elementary particles measured in unit $(\hbar/c\lambda)$. Therefore, by taking λ of the order of the compton wave length of the nucleon, these two can be interpreted as corresponding to the nucleon and lepton family respectively.

Thus, Θ is interpreted as expressing the intrinsic difference of these two families, and the conservation of Θ leads at one to the conservation of heavy particles. A deeper meaning of Θ is found by introducing further

$$\theta = 1/2 \cdot (\Theta + 1). \quad (5)$$

θ takes eigenvalues 1, 0, for nucleon and lepton family respectively, and just correspond to λ introduced by Oneda.²⁾ Therefore, it would be natural to interpret it as mesic charge,* and to assume the invariance of the theory under the "gauge transformation" performed in a relation with it;

$$\psi \rightarrow \psi e^{i\theta\alpha},$$

$$\psi^* \rightarrow \psi^* e^{-i\theta\alpha},$$

where α is an arbitrary constant. From this requirement follows the conservation law of mesic current, which, in our opinion, is nothing but the conservation of heavy particles.

Thus, the conservation of heavy particles seems to be a strong evidence for the existence of a new intrinsic structure in spinor particles, and we hope that our theory might serve as a first step in elucidating it, even if it might not be correct at the ultimate stage.

In conclusion the authors wish to express their sincere gratitude to Prof. S. Sakata for valuable discussions and criticism. Details will appear in a later issue of this journal.

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The Statistical Quantum Mechanics of Time-dependent Phenomena and Viscosity

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June 29, 1953

In his theory of irreversible process, Cox⁽¹⁾ has derived viscous force from the classical statistical mechanics. But it seems that there are some vague points in his paper.

In the present short note, it will be made clear what an approximation have to be done to derive the macroscopic viscous force from the quantum statistical point of view. Throughout this note Dirac's notation will be used as it is simple, and a discrete set of states labelled by a parameter a or a' will be taken. In general, eigenvalues of Hamiltonian of the system are not degenerate in quantum theory. But a macroscopic observer will consider that the system consists of macroscopic states h 's, each of which is composed of microscopic states a 's.

We use the same method as the perturbation theory in quantum mechanics, that is, we expand the quantum density at any time and place with stationary bras and kets. Let us denote the probability of finding the system in the a' -th state as $P_{a'}$, which is a function of time owing to a time dependent perturbation. Quantum density is given by definition:

$$\rho_t = \sum_{a'} |a'\rangle P_{a'} \langle a'|. \quad (1)$$

By this definition, we can obtain results that are independent of the initial phases of the states. Let H be the Hamiltonian of the system, then the Liouville equation in this case will be

$$\begin{aligned} i\hbar \frac{\partial \rho_t}{\partial t} &= \sum_{a'} i\hbar \left\{ \frac{\partial |a'\rangle}{\partial t} P_{a'} \langle a'| + |a'\rangle \frac{\partial}{\partial t} \langle a'| + |a'\rangle \frac{\partial P_{a'}}{\partial t} \langle a'| \right\} \\ &= \sum_{a'} \left\{ H_0 |a'\rangle P_{a'} \langle a'| - |a'\rangle P_{a'} \langle a'| H_0 + i\hbar |a'\rangle \frac{\partial P_{a'}}{\partial t} \langle a'| \right\} \\ &= H_0 \rho_t - \rho_t H_0 + \sum_{a'} i\hbar |a'\rangle \frac{\partial P_{a'}}{\partial t} \langle a'|, \quad (2) \end{aligned}$$

where H_0 means unperturbed Hamiltonian of the system.

The statistical mean value of an observable A is defined as usual:

$$\begin{aligned} \langle A \rangle &= \sum_{a'} P_{a'} \langle a'| A |a'\rangle = \sum_{a'} P_{a'} \langle a'| \xi \rangle \langle \xi | A |a'\rangle \\ &= \sum_{\xi} \langle \xi | A |a'\rangle P_{a'} \langle a'| \xi \rangle = \sum_{\xi} \langle \xi | A \rho_t | \xi \rangle. \quad (3) \end{aligned}$$

Then the rate of change of the average value of A over the ensemble is calculated from (3) as

$$\begin{aligned} \frac{d\langle A \rangle}{dt} &= \sum_{a'} \left\{ \frac{\partial P_{a'}}{\partial t} \langle a'| A |a'\rangle + P_{a'} \frac{\partial}{\partial t} \langle a'| A |a'\rangle \right. \\ &\quad \left. + P_{a'} \langle a'| \frac{\partial A}{\partial t} |a'\rangle + P_{a'} \langle a'| A \frac{\partial}{\partial t} |a'\rangle \right\} \\ &= \sum_{a'} \left\{ P_{a'} \langle a'| \frac{\partial A}{\partial t} |a'\rangle + \frac{1}{i\hbar} \langle A H_0 - H_0 A \rangle \right. \\ &\quad \left. + \frac{\partial P_{a'}}{\partial t} \langle a'| A |a'\rangle \right\}. \quad (4) \end{aligned}$$

This equation can also be derived from (2), operating $\langle \xi | A$ and $| \xi \rangle$ and taking spur of it.

Now

$$\begin{aligned} P_{a'} &= \langle a'| \rho_t |a'\rangle = \sum_a \langle a'| T |a\rangle P_a^0 \langle a| \bar{T} |a'\rangle \\ &= \sum_a P_a^0 P(aa'). \end{aligned}$$

where T satisfies the equation

$$i\hbar \cdot \partial T / \partial t = H T.$$

Let us suppose that initial system, labelled with a , was in a stationary state and P_a^0 is a constant. Then

$$\begin{aligned} \frac{\partial P_{a'}}{\partial t} &= \sum_a P_a^0 \frac{\partial P(aa')}{\partial t} \\ &= \sum_{aa''} P_a^0 P(aa'') P^{-1}(a''a) \frac{\partial P(aa')}{\partial t} \\ &= \sum_{aa''} P_{a''} P^{-1}(a''a) \frac{\partial P(aa')}{\partial t} \\ &= \sum_{a''} P_{a''} \sigma_{aa''a'}, \quad (5) \end{aligned}$$

where $P^{-1}(a'a)$ is the inverse of $P(a'a)$ and we have put

$$\sum_a P^{-1}(a''a) \cdot \partial P(aa') / \partial t = \sigma_{aa''a'}.$$

Then Eq. (4) becomes

$$\begin{aligned} \frac{d\langle A \rangle}{dt} &= \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{i\hbar} \langle A H - H A \rangle \\ &\quad + \sum_{a''} P_{a''} \sigma_{aa''a'} \langle a'' | A |a' \rangle. \quad (6) \end{aligned}$$

The last term in this equation is "the viscous force". Evaluation of $P(aa')$ is beyond the scope

of "statistical quantum mechanics", though it can be evaluated by means of "quantum mechanics"). Other terms in this equation can be determined by ordinary procedure of statistical quantum mechanics. Thus Eq. (6) shows the margin of utility of statistical quantum mechanics.

Now a macroscopic observer will be concerned about the sum $\sum_{a''', h} P_{a'''}$ of many microscopic states a''' belonging to a macroscopic one " h ". This sum is the probability of finding the system in a macroscopic state h . We may use a notation P_h for this sum. In the case when the macroscopic state is specified and nothing is done about the microscopic one, we would have to consider each microscopic state a''' in h is equally probable. Thus we may use a notation \bar{P}_h for P_h/g_h , where g_h is the statistical weight of, or the number of microscopic states in, the macroscopic state h . Further we introduce notations

$$\langle h|A|h \rangle \text{ for } \langle a'|A|a' \rangle \quad (a' \in h),$$

and

$$\sigma_{hk} \text{ for } \sigma_{a'''a'} \quad (a''' \in h, a' \in k).$$

Then the last terms of Eq. (6) can be written as

$$\begin{aligned} & \sum_{a''', a'} P_{a'''} \sigma_{a'''a'} \langle a'|A|a' \rangle \\ &= \bar{P}_h g_h^2 \sigma_{hh} \langle h|A|h \rangle + \sum_k \bar{P}_h g_h g_k \sigma_{hk} \langle h|A|k \rangle \\ &+ \sum_k \bar{P}_k g_k g_h \sigma_{kh} \langle k|A|h \rangle \\ &+ \sum_h \bar{P}_k g_k^2 \sigma_{kk} \langle k|A|k \rangle \\ &+ \sum_{k'} \bar{P}_h g_h g_{k'} \sigma_{hk'} \langle k'|A|k' \rangle \\ &+ \sum_{k'} \bar{P}_{k'} g_{k'} g_k \sigma_{k'k} \langle k'|A|k \rangle + \dots, \end{aligned} \quad (6')$$

where the prime means the sum over k 's except h . Here

$$P_h \approx \bar{P}_h$$

and

$$\langle h|A|h \rangle \approx \langle k|A|k \rangle$$

for nearly equal energies E_h and E_k . (In the above equation \approx means about the same order of magnitude.) This nearly equality of energies between macro-states will be legitimate.

When probability of finding the system in any macroscopic state is negligibly small except one labelled with h , among all terms in the above expression (6'), the first term will be overwhelmingly great than others. Thus we have

$$\begin{aligned} \sum_{a''', a'} P_{a'''} \sigma_{a'''a'} \langle a'|A|a' \rangle &= P_h g_h \sigma_{hh} \langle h|A|h \rangle + \delta \\ &= g_h \sigma_{hh} \langle A \rangle + \delta', \end{aligned}$$

where δ and δ' represent negligibly small quantities. Then Eq. (6) becomes

$$\begin{aligned} \frac{d\langle A \rangle}{dt} &= \left\langle \frac{\partial A}{\partial t} \right\rangle + \frac{1}{i\hbar} \langle AH - HA \rangle \\ &+ g_h \sigma_{hh} \langle A \rangle + \delta'. \end{aligned} \quad (7)$$

Now $g_h \sigma_{hh}$ is the macroscopic coefficient of viscosity, reciprocal of "relaxation time" or "betsensation time".

A full account will be published later.

- 1) R. T. Cox, Rev. Mod. Phys. 22 (1950), 238; 24 (1952), 312.
- 2) P. A. M. Dirac, *The Principles of Quantum Mechanics*, (The third edition) (1947), (Oxford, at the Clarendon Press)
- 3) P. A. M. Dirac, l. c., p. 175, eq. (26).

Interaction between Spin Waves and Conduction Electrons

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We have calculated the relaxation time that the energy of microwave absorbed by the spin system of ferromagnetic 3d-electrons dissipates to the system of 4s-conduction electrons in ferromagnetic conductors. We take the Heitler-London model for 3d-electrons, which follow adiabatically the motions of ions. On the other hand, it can be assumed that 4s-electrons are free. The spin system of 3d-electrons can be treated by the spin wave approximation, which is valid when the temperature is comparatively low, i.e., $T/T_c \ll 1$. Since conduction electrons play a important role at low temperature, it does not lose a essential part of the problem even if this approximation is adopted.

Through magnetic dipolar interactions between 3d- and 4s-electrons, there occurs the collision process between spin waves and conduction electrons. The spin wave quantum decays in the collision with a 4s-electron. The 4s-electron is deflected and change its orbital and Zeeman energies. Thus the energy of the microwave absorbed by the spin wave is

transferred to the 4s-electron through this collision process. If only the spin wave quanta of zero momentum are excited by the perturbation of the microwave field, this process can not occur owing to the energy and momentum conservation laws. However, Kittel and Abrahams¹⁾ have shown that the spin-spin relaxation time is shorter than that of the other relaxations. Since the equilibrium of the spin wave system can be established very fast, the spin wave quanta of all the wave numbers take part in our process.

Treating the s-electron system with the method of second quantization, we can set the interactions between spin waves and conduction electrons in the form

$$H_I = (S/N)^{1/2} \sum_k \sum_{k'} \sum_{\lambda} G^{xx}(k', k) \times \{ e^{i\phi_{\lambda}} (l_{\lambda} - \bar{a}_{k'} c_{\lambda} b_k + l_{\lambda} + \bar{b}_{k'} c_{\lambda} a_k) \delta_{k', -K_{\lambda} - k} + e^{-i\phi_{\lambda}} (l_{\lambda} + \bar{a}_{k'} c_{\lambda} b_k + l_{\lambda} - \bar{b}_{k'} c_{\lambda} a_k) \delta_{k', +K_{\lambda} - k} \}, \quad (1)$$

where \bar{a}_k and a_k are the respective operators of creation and annihilation of the s-electron with spin up in the state k , \bar{b}_k and b_k refer to those of the s-electron with spin down, and \bar{c}_{λ} and c_{λ} to those of the spin wave quantum of wave vector K_{λ} . $l_{\lambda} \pm$ are the Holstein-Primakoff's transformation constants,²⁾ ϕ_{λ} the polar angle of the wave vector K_{λ} , of which the z -axis is the direction of the static field H_0 . After some calculations, we have the integrals of $G^{xx}(K_{\lambda} + k, k)$ as $(2/S) 4\pi\beta M_0 (K_{\lambda}(\omega)/K_{\lambda})^2$, where S is the total spin per atom, M_0 the saturation magnetic moment of the sample, and β the Bohr magneton.

Let ω_{λ}^{ab} and ω_{λ}^{em} be the respective probabilities per unit time that the s-electron system may absorb and emit one quantum of spin wave K_{λ} . ω_{λ}^{ab} and ω_{λ}^{em} can be calculated from (1). The energy \dot{Q} transferred from the spin wave system to the s-electron system per unit time is

$$\dot{Q} = V^2 / (2\pi)^6 \int \int (\omega_{\lambda}^{ab} - \omega_{\lambda}^{em}) E_{\lambda} d^3k dK_{\lambda}, \quad (2)$$

where V is the volume of the sample, E_{λ} the energy of the spin wave quantum K_{λ} , which takes the simple form of $E_{\lambda} = 2Sf d^2 K_{\lambda}^2 + 2\beta H_0$, neglecting the d - d dipolar interactions. We can define the relaxation time of our process τ as follows,

$$\tau = \Delta T / \{ \dot{Q} / (1/C_{\text{cond}} + 1/C_{\text{spin}}) \}, \quad (3)$$

where ΔT is the temperature difference between the

spin wave system and s-electron system, and C_{cond}^* and $C_{\text{spin}}^{(3)}$ the respective heat capacities of the two systems. Putting (2) into (3), we can obtain τ after somewhat long calculations.

$$\tau = \frac{5}{2\pi} \frac{\hbar^5 S N_0 d^2}{\frac{2}{3} Z(S+1) m^3 (\beta \beta H_0)^2 x^2 l_0^{+1/2} T_c} \times \left[\frac{1}{\xi} \left\{ \frac{1}{1.1 \times 10^3 T_c} + \frac{1}{0.113 (S(S+1) Z \frac{2}{3})^{3/2} N_0 x} \right\} \times \left(\frac{1}{\xi} \right)^{1/2} \right], \quad (4)$$

where N_0 is the number of atoms per unit volume, d the lattice constant, z the number of nearest neighbour atoms, m the mass of an electron, x the Boltzmann constant, ξ the reduced temperature, i.e., $\xi = T/T_c$ and $|l_0|^2 \approx 4$. Inserting the values of Nickel, we have $\tau \approx 7 \times 10^{-4}$ sec at 1°K , 3×10^{-4} sec at 3°K , and 4×10^{-6} sec at 300°K . The spin-lattice relaxation time obtained by Kittel and Abrahams¹⁾ are ≈ 1 sec at 1°K , 5×10^{-4} sec at 3°K , and 6×10^{-7} sec at 300°K .

From the above results, it can be concluded that the interaction between spin waves and conduction electrons gives the essential contributions to the relaxation mechanism at low temperature, where the lattice vibrations are almost quenched.

Details of the calculations will be published in the near future. The author wishes to express his sincere thanks to Prof. K. Ariyama and Assistant Prof. T. Takabayasi for their kind interests to this problem and Assistant Prof. S. Nakajima and Mr. K. Sugihara for their valuable discussions.

- 1) E. Abrahams and C. Kittel, Phys. Rev. 88 (1952), 1200.
C. Kittel and E. Abrahams, Rev. Mod. Phys. 25 (1953), 233.
- 2) T. Holstein and H. Primakoff, Phys. Rev. 58 (1940), 1098.
- 3) c. f. Mott and Jones, *Properties of Metals and Alloys*, p. 237.

* As the value of the specific heat of the conduction electron, we adopted the experimental value of the electronic specific heat of Cu. C. f. Seitz, *Modern Theory of Solids*, p. 152.

'Quantum Condition' in the Phase-space Representation of Quantum Mechanics

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Quantum mechanics can be formulated in terms of certain Markoff-like process in phase space as was attempted by Moyal.¹⁾ We shall in this note give some remarks on this formulation, especially clarifying the subsidiary conditions for it.

Take up, for simplicity, a single spinless particle in a force potential $V(x)$. The method of the phase-space ensemble represents a quantum-mechanical state of wave function ψ with a probability distribution of particle in phase space, given by

$$f(x, p) = \frac{1}{(2\pi\hbar)^3} \int \rho\left(x - \frac{y}{2}, x + \frac{y}{2}\right) e^{ip y / \hbar} dy, \quad (1)$$

where p is a particle momentum and $\rho(x, x') = \psi(x)\psi^*(x')$ means the density matrix. The function $f(x, p)$ is real but not positive definite. This fact discloses the unreal nature of such ensemble, but enables it correctly to give the quantum-mechanical expectation values of any dynamical quantities $F(x, p)$ as the mean values over the ensemble, i.e.,

$$\langle F \rangle_{\text{qm}} = \langle F \rangle_f \equiv \int F(x, p) f(x, p) dx dp. \quad (2)$$

The temporal development equation for the distribution function to be determined from the Schrödinger equation is

$$\frac{\partial f}{\partial t} + \frac{p}{m} \nabla f = A[f], \quad (3)$$

where

$$A[f] = \int_{-\infty}^{\infty} J(x, p-p') f(x, p') dp', \quad (4)$$

$$\begin{aligned} &= \nabla f \cdot \nabla p f + \sum_{\lambda=3,5,7,\dots} \sum_{\lambda_1+\lambda_2+\lambda_3=\lambda} \\ &\times \frac{(-1)^{(\lambda-1)/2} (\hbar/2)^{\lambda-1}}{\lambda_1! \lambda_2! \lambda_3!} \\ &\times \frac{\partial^\lambda V}{\partial x_1^{\lambda_1} \partial x_2^{\lambda_2} \partial x_3^{\lambda_3}} \frac{\partial^\lambda f}{\partial p_1^{\lambda_1} \partial p_2^{\lambda_2} \partial p_3^{\lambda_3}}, \end{aligned} \quad (5)$$

$$J(x, p) = \frac{i}{\hbar} \frac{1}{(2\pi\hbar)^3} \int_{-\infty}^{\infty} \left\{ V\left(x + \frac{y}{2}\right) - V\left(x - \frac{y}{2}\right) \right\} e^{ip y / \hbar} dy. \quad (6)$$

The formal interpretation of (3) (with (4)) shows that the ensemble develops as follows: the coordinate x of each point of the ensemble changes continuously with the momentum p , while the momentum changes discontinuously and non-causally with a transition probability J , which itself is perfectly determined by the external potential V , and is independent of the present value of particle momentum. ($J(x, p)$ means the probability with which the momentum jumps by an amount p at the point x .) The various 'transition moments' of momentum components become

$$\begin{aligned} m_{n_1 n_2 n_3} &\equiv \int_{-\infty}^{\infty} p_1^{n_1} p_2^{n_2} p_3^{n_3} J(x, p) dp, \\ &= \begin{cases} -(-1)^{(n-1)/2} \left(\frac{\hbar}{2}\right)^{n-1} \frac{\partial^n V}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}}, & \text{if } n = \sum_i n_i = \text{odd}, \\ 0, & \text{if } n = \text{even}. \end{cases} \end{aligned} \quad (7)$$

Especially, the first moments, i.e., the rate of the average change of momentum, coincide with the classical value:

$$\int_{-\infty}^{\infty} p J(x, p) dp = -\nabla V. \quad (8)$$

The picture, however, cannot be taken as real, since J (though always real) can take negative values, and the integral $\int J(x, p' - p) dp' = m_{000}$ vanishes.

We may also start from the integral form²⁾ for the development of ψ :

$$\psi(x, t) = \int K(x, t | x_0, t_0) \psi(x_0, t_0) dx_0,$$

where K is the transformation kernel for the wave function. This leads to the analogous form for f :

$$f(x, p, t) = \int T(x, p, t | x_0, p_0, t_0) f(x_0, p_0, t_0) dx_0 dp_0, \quad (9)$$

with

$$\begin{aligned} T(x, p, t | x_0, p_0, t_0) &= \frac{1}{(2\pi\hbar)^3} \\ &\times \int K\left(x - \frac{y}{2}, t | x_0 - \frac{y_0}{2}, t_0\right) \\ &\times K^*\left(x + \frac{y}{2}, t | x_0 + \frac{y_0}{2}, t_0\right) \\ &\times e^{i\hbar(p y - p_0 y_0)} dy dy_0. \end{aligned} \quad (10)$$

The function T is real and symmetric, and satisfies the iteration law:

$$T(x_1 p_1 | x_0 p_0 t_0) \\ = \int T(x_1 p_1 | x_1 p_1 t_1) T(x_1 p_1 t_1 | x_0 p_0 t_0) dx_1 dp_1. \quad (11)$$

These show that the ensemble develops according to a sort of Markoff process in phase space admitting negative probabilities, where $T(x_1 p_1 | x_0 p_0 t_0)$ means the transition probability, i.e., the distribution at t conditional in x_0, p_0 at t_0 . Thus T satisfies the same equation as (3), which may be regarded as the 'differential form' obtained by reducing the integral equation (9) or (11) (Smoluchowski equation), but is more general than the usual Fokker-Planck type: Differential coefficients of V in and above third degree, in so far as they do not vanish, play the parts of higher transition moments for momentum components (see Eq. (7)) and bring about "quantum terms" to (3), the terms following $\sum_{j=1}^{\infty}$ in (5). In our process, the probability that the value of momentum changes by a finite amount in a small time interval cannot be regarded as small, essentially differing from the case in usual Brownian processes.

We have seen that a quantum-mechanical state corresponds to an ensemble in phase space, but, conversely, an arbitrary phase-space ensemble does not necessarily correspond to a quantum-mechanical pure state. The "uncertainty relation",

$$\langle (x_k - \langle x_k \rangle)^2 \rangle_f \cdot \langle (p_k - \langle p_k \rangle)^2 \rangle_f \geq \hbar^2/4,$$

which results in by the help of (2), is of course a necessary condition but not a sufficient one. We shall now obtain the conditions that a phase-space ensemble should really correspond to a quantum-mechanical pure state. The density matrix $\rho(x, x')$ for a pure state satisfies the condition, $\int \rho(x, x'') \times \rho(x'' x') dx'' = \rho(x, x')$, or in differential form,

$$\left(\frac{\partial^2 \rho}{\partial x_k \partial x_{k'}} - \frac{1}{\rho} \frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_{k'}} \right) = 0. \quad (12)$$

The relation (12) taken on the diagonal is equivalent to six symmetrized relations:

$$\left[\left(\frac{\partial^2 \rho}{\partial x_k \partial x_{k'}} + \frac{\partial^2 \rho}{\partial x_k \partial x_{k'}} \right) \right. \\ \left. - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_{k'}} + \frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_{k'}} \right) \right]_{x'=x} = 0, \quad (13)$$

plus three antisymmetrized relations:

$$\left(\frac{\partial^2 \rho}{\partial x_k \partial x_{k'}} - \frac{\partial^2 \rho}{\partial x_k \partial x_{k'}} \right) \\ - \frac{1}{\rho} \left(\frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_{k'}} - \frac{\partial \rho}{\partial x_k} \frac{\partial \rho}{\partial x_{k'}} \right) \Big|_{x'=x} = 0. \quad (14)$$

These conditions on $\rho(x, x')$ impose the corresponding conditions on the distribution $f(x, p)$ which is nothing but the Fourier transform of $\rho(x, x')$ along its antidiagonal. Corresponding to (13) we obtain

$$\bar{p}_i \bar{p}_k - \bar{p}_i \bar{p}_k = \frac{\hbar^2}{4} \frac{\partial^2 (\log P)}{\partial x_k \partial x_k}, \quad (15)$$

and to (14) we get

$$\text{curl } \bar{p} = 0, \quad (16)$$

where P, \bar{p} , and $\bar{p}_i \bar{p}_k$ are quantities derived from $f(x, p)$ by

$$P(x) = \int f(x, p) dp, \quad (17)$$

$$\bar{p}(x) = \int p f(x, p) dp / P, \quad (18)$$

$$\bar{p}_i \bar{p}_k = \int p_i p_k f(x, p) dp / P. \quad (19)$$

Consequently, our Markoff-like picture in the phase space represents a quantum-mechanical change of pure state under the subsidiary conditions (15) and (16). Especially, (15) may be called the "quantum condition" for the phase-space formulation.

Details and further considerations will shortly be published.

- 1) J. E. Moyal, Proc. Camb. Phil. Soc. **45** (1949), 99. Also see E. Wigner, Phys. Rev. **40** (1932), 749.
- 2) R. P. Feynman, Rev. Mod. Phys. **20** (1948), 367.

Equivalence between the Formulation of Quantum Mechanics in terms of Quantum Potential and the One in terms of Markoff-like Process

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In a previous paper¹⁾ we have shown the formulation of quantum mechanics in terms of the picture of trajectory ensemble in the configuration space. Alternatively, we can also formulate quantum mechanics in terms of certain Markoff-like process in phase space as was initiated by Wigner²⁾ and Moyal³⁾ and supplemented in our preceding note⁴⁾ (hereafter referred to as I). The effect of quantum fluctuation is represented with the occurring of certain quantum potential in the former picture, while in the latter with Markoff-like transitions in the momentum. We shall now analyse the relationship between both formulations.

Again we take up a single spinless particle in an external potential $V(x)$. The method of the coordinate-space ensemble represents a quantum-mechanical state of wave function

$$\psi = R e^{iS/\hbar}, \quad (R, S: \text{real}) \quad (1)$$

with an ensemble which consists of a probability distribution of a particle in the density

$$P(x) = R(x)^2, \quad (2)$$

the momentum of the particle p being uniquely correlated with its position as

$$\bar{p}(x) = \nabla S(x). \quad (3)$$

The ensemble has therefore a particular phase-space distribution:

$$f_c(x, p) = R(x)^2 \delta(p - \nabla S(x)). \quad (4)$$

The Schrödinger equation ensures that the temporal development of the ensemble is built up through the process in which each point of the ensemble moves along a continuous trajectory with the momentum (3) at each instant, accelerated not only by the potential V but also by an additional 'quantum' potential,

$$V' = -(\hbar^2/2m) \Delta R/R. \quad (5)$$

It is because, by (2) and (3), the Schrödinger equation leads to the continuity equation:

$$\partial P / \partial t + \text{div}(P \bar{p}) / m = 0, \quad (6)$$

and the equation of motion:

$$d\bar{p}/dt = -\nabla(V + V'), \quad (7)$$

or the relation of momentum conservation:

$$\begin{aligned} \frac{\partial(P \bar{p}_i)}{\partial t} + \frac{1}{m} \sum_k \frac{\partial(P \bar{p}_i \bar{p}_k)}{\partial x_k} \\ = -P \frac{\partial V}{\partial x_i} + \sum_k \frac{\partial \sigma_{ik}}{\partial x_k}, \end{aligned} \quad (8)$$

where σ_{ik} means the 'quantum stress'

$$\sigma_{ik} = \frac{\hbar^2}{4m} P \frac{\partial^2 (\log P)}{\partial x_i \partial x_k}. \quad (9)$$

Conversely, any ensemble of trajectories which satisfies (6), (7) (or (8)), and the subsidiary condition:

$$\text{curl } \bar{p} = 0, \quad (10)$$

corresponds to a quantum-mechanical change of state.

On the other hand the method of the phase-space ensemble is stated in I. Now we can produce a coordinate-space ensemble from a phase-space ensemble by 'projecting' the latter on the coordinate space. This means that we introduce the coordinate-space ensemble which consists of the density and momentum fields, $P(x)$ and $\bar{p}(x)$, derived from the phase-space distribution function $f(x, p)$ by I (17) and I (18); in other words we eliminate the momentum dispersion at each space point x in the phase-space ensemble, adopting the mean momentum and the total density at each point x . Then we can show that the projection of a possible phase-space ensemble results in a possible coordinate-space ensemble which corresponds to the same quantum-mechanical state.

First, we project eq. I (3) on the coordinate space, and just obtain the continuity equation (6), the effects of the stochastic transitions in the p cancelling. Next we project the momentum conservation relation for the phase-space ensemble,

$$\begin{aligned} \frac{\partial(\bar{p}_i f)}{\partial t} + \frac{1}{m} \sum_k \bar{p}_i \bar{p}_k \frac{\partial f}{\partial x_k} \\ = \int_{-\infty}^{\infty} \bar{p}_i J(x, p - p') f(x p') dp', \end{aligned} \quad (11)$$

obtainable from I (3). Then the effects of the stochastic transitions in the p induced by V are averaged out and reduced to the classical value, $-P \partial V / \partial x_i$, according to I (8), while the convection term in the phase-space picture transforms to the

convection term in the coordinate-space picture, $\sum_k \partial(P\hat{p}_k\hat{p}_k)/\partial x_k$, plus the extra momentum flow such as is ascribable to the occurrence of the 'quantum stress', taking into account the 'quantum condition' I (15) for the original phase-space distribution. Thus we get just the momentum conservation relation (8) for the coordinate-space ensemble. Furthermore, the second subsidiary condition I (16) for the phase-space ensemble immediately corresponds to the subsidiary condition (10) for the coordinate-space ensemble. Consequently the latter ensemble introduced by means of the contraction of the former ensemble satisfies just the conditions for the correspondence to the quantum-mechanical change of state.

In a previous paper,³⁾ we have taken up the question whether the quantum potential (5) could be analysed into a mechanism like any Markoff process underlying, and stated that such attempt would also lead to difficulties in the interpretation. The problem is now explained more clearly by the foregoing analysis. The quantum potential can be regarded as an *apparent force* appearing as the result of projecting on the coordinate space the phase-space ensemble which satisfies the quantum condition and changes according to a sort of Markoff process. We cannot, however, regard the latter picture as a real one any more than the picture of trajectory ensemble under quantum potential, because of the inevitable appearance of negative probabilities. Furthermore in the phase-space formulation we had to put certain quantum condition ad hoc. It is also characteristic that the probability of momentum transitions is perfectly determined by the external field and is moreover irrespective of the present value of the particle momentum, and further that the transition probability J is an odd function and so the transition moments of even orders vanish while those of odd orders survive (eq. I (7)), in contrast to the usual Brownian processes, where the second order moment plays the essential role, letting the particle velocity tend to the equilibrium distribution. We should only accept our Markoff-like picture as a kinematical one without any attempt to derive it dynamically. Though, at first sight, it might seem tempting to consider some hidden mechanism of irregular disturbances which vanish in the average yet make the particles momentum fluctuate, it would, however, lead to unnatural model because of the features of the transitions as are above stated.

Recently Weizel⁶⁾ attempted to derive the quantum potential from certain stochastic process based on some model. He proceeded in a considerably different

fashion, but the nature of his method can also be seen clearly from our viewpoint that stands upon the systematic formulation of quantum mechanics in terms of the phase-space ensemble.

Details will soon be published.

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- 4) T. Takabayasi, Prog. Theor. Phys. **10** (1953), 119.
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- 6) W. Weizel, ZS. f. Phys. **134** (1953), 264.

Subsidiary Conditions in the Phase-space Representation of Quantum Mechanics

—Supplements to Previous Notes¹⁾—

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It is possible to reformulate quantum mechanics²⁾ within a certain range in terms of Wigner's phase-space distribution function.³⁾ In this formulation it is essential to restrict the phase-space distribution function, in so far as it should correspond to a (quantum-mechanical) *pure state*, by certain subsidiary conditions. In I we have partly obtained these conditions, which should now be supplemented.

A general quantum-mechanical mixed state of a particle can be represented with a density matrix ρ , or $\rho(x|x')$ in the coordinate representation neglecting spin. The ρ should be Hermitian:

$$\rho(x|x') = \rho(x'|x)^*, \quad (1)$$

and normalizable:

$$\int \rho(x|x) dx = 1, \quad (2)$$

and have no negative eigenvalues:

$$\text{Spur}(\rho A^2) \geq 0, \text{ for any Hermitian operator } A.$$

(3)

Alternatively, we can represent a mixed state with a phase space distribution function $f(\mathbf{x}\mathbf{p})$, to be derivable from ρ by

$$f(\mathbf{x}\mathbf{p}) = \frac{1}{(2\pi\hbar)^3} \int \rho\left(\mathbf{x} - \frac{\mathbf{y}}{2}, \mathbf{x} + \frac{\mathbf{y}}{2}\right) e^{i\mathbf{p}\mathbf{y}/\hbar} d\mathbf{y}. \quad (4)$$

Then $f(\mathbf{x}\mathbf{p})$ should satisfy, corresponding to (1),

$$f(\mathbf{x}\mathbf{p}) = \text{real}; \quad (5)$$

corresponding to (2),

$$\int f(\mathbf{x}\mathbf{p}) d\mathbf{x} d\mathbf{p} = 1; \quad (6)$$

and to (3),

$$\langle A^2 \rangle = \int A(\mathbf{x}\mathbf{p})^2 f(\mathbf{x}\mathbf{p}) d\mathbf{x} d\mathbf{p} \geq 0 \quad (7)$$

for any real dynamical quantity A , though f itself is not necessarily positive everywhere. Conversely, any distribution function $f(\mathbf{x}\mathbf{p})$, satisfying (5), (6), and (7), corresponds to a mixed state.

Now the condition that a mixed state should in particular become a pure state is given by

$$\int \rho(\mathbf{x}\mathbf{x}'') \rho(\mathbf{x}'') \rho(\mathbf{x}'\mathbf{x}') d\mathbf{x}'' = \rho(\mathbf{x}\mathbf{x}'). \quad (8)$$

If this is satisfied, there exists a suitable function $\psi(\mathbf{x})$ that makes $\rho(\mathbf{x}\mathbf{x}')$ written as

$$\rho(\mathbf{x}\mathbf{x}') = \psi(\mathbf{x}) \psi(\mathbf{x}')^*. \quad (9)$$

But, satisfied this condition, we have clearly

$$\frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_k'} - \rho \frac{\partial^2 \rho}{\partial x_i \partial x_k'} = 0, \quad (i, k=1, 2, 3); \quad (10)$$

and conversely, if (10) be satisfied for every i and k , we get (9) by integrating (10) and by use of (1). We can therefore adopt (10), in place of (8), as the condition for a mixed state to specialize itself into a pure state. Eq. (10) is of course compatible with the equation of motion for ρ .

We can now get the condition that a phase space distribution $f(\mathbf{x}\mathbf{p})$, satisfying (5), (6), and (7), should particularly correspond to a pure state, by transforming (10) into relations as to f through (4). It is obtained as

$$\begin{cases} (\hat{p}_i f) * (\hat{p}_k f) - f * (\hat{p}_i \hat{p}_k f) \\ = \frac{\hbar^2}{4} \left(f * \frac{\partial^2 f}{\partial x_i \partial x_k} - \frac{\partial f}{\partial x_i} * \frac{\partial f}{\partial x_k} \right), \quad (11) \\ \frac{\partial f}{\partial x_i} * (\hat{p}_k f) - \frac{\partial f}{\partial x_k} * (\hat{p}_i f) \\ = f * \left(\hat{p}_k \frac{\partial f}{\partial x_i} - \hat{p}_i \frac{\partial f}{\partial x_k} \right), \quad (12) \end{cases}$$

where the notation such as $f_1 * f_2$ means the convolution with respect to \mathbf{p} , i.e.,

$$f_1(\mathbf{x}\mathbf{p}) * f_2(\mathbf{x}\mathbf{p}) = \int f_1(\mathbf{x}\mathbf{p}') f_2(\mathbf{x}, \mathbf{p} - \mathbf{p}') d\mathbf{p}'.$$

Eq. (11) is symmetric in i and k and consists of six relations, while (12) is three antisymmetric ones; thus we have nine real conditions on $f(\mathbf{x}\mathbf{p})$.

Next we find that these relations, when 'projected on the coordinate space' (i.e., when integrated throughout over the momentum space), are reduced to

$$\begin{cases} \overline{\hat{p}_i \hat{p}_k} - \overline{\hat{p}_i} \overline{\hat{p}_k} = \frac{\hbar^2}{4} \frac{\partial^2 (\log P)}{\partial x_i \partial x_k}, \quad (13) \\ \text{curl } \overline{\mathbf{p}} = 0, \quad (14) \end{cases}$$

which are just the conditions that we have obtained in I. In these equations,

$$P(\mathbf{x}) = \int f(\mathbf{x}\mathbf{p}) d\mathbf{p}, \quad (15)$$

$$\overline{\hat{p}_i}(\mathbf{x}) = \int \hat{p}_i f(\mathbf{x}\mathbf{p}) d\mathbf{p} / P, \quad (16)$$

$$\overline{\hat{p}_i \hat{p}_k} = \int \hat{p}_i \hat{p}_k f(\mathbf{x}\mathbf{p}) d\mathbf{p} / P, \quad (17)$$

are the distribution moments of momentum components of the 0th, the 1st, and the 2nd orders, respectively; and so (13) means the relation for the dispersion tensor.

If we multiply (11) by \hat{p}_j and then project the result, we get ten relations:

$$\begin{aligned} \overline{\hat{p}_i \hat{p}_j \hat{p}_k} - \overline{\hat{p}_i} \overline{\hat{p}_j \hat{p}_k} \\ = -\frac{\hbar^2}{4} \left\{ \sum_{i,j,k} \overline{\hat{p}_i} \overline{\hat{p}_j} \overline{\hat{p}_k} \log P + \partial_i \partial_j \overline{\hat{p}_k} \right\}, \\ (\partial_i \equiv \partial / \partial x_i, \sum_{i,j,k} \overline{\hat{p}_i} \overline{\hat{p}_j} \overline{\hat{p}_k} = \overline{\hat{p}_i} \overline{\hat{p}_j} \overline{\hat{p}_k} \\ + \overline{\hat{p}_j} \overline{\hat{p}_k} \overline{\hat{p}_i} + \overline{\hat{p}_k} \overline{\hat{p}_i} \overline{\hat{p}_j}), \quad (18) \end{aligned}$$

for the (symmetrical) moment tensor of the third order,

$$\overline{\hat{p}_i \hat{p}_j \hat{p}_k} = \int \hat{p}_i \hat{p}_j \hat{p}_k f(\mathbf{x}\mathbf{p}) d\mathbf{p} / P.$$

On the other hand the same operation on (12) results in no new relation; in fact (12) is derivable from (11) and (14). Continuing similar procedures on (11), we get in succession the relations for successively higher moments. Eqs. (13) and (14) alone are insufficient for the distribution to correspond to a pure state; we must take (14) and (11), or, in place of them, take (14) and all of relations for successively higher moments: (13), (18), ...

This circumstance does not affect the conclusions in II on the equivalence between the formulation of quantum mechanics in terms of the phase-space ensemble and the one in terms of the coordinate-space ensemble: Given a phase-space distribution

$f(xp)$ satisfying (11) and (14), the density and momentum fields P and \bar{p} in the coordinate space produced from f by (15) and (16) satisfy (13) and (14), and so they mean a *possible* ensemble in the coordinate-space ensemble formulation as was stated in II. Conversely, if we are given a coordinate-space ensemble specified with $P(x)$ and $\bar{p}(x)$, we can from them determine successively the quantities, $\bar{p}_i \bar{p}_k, \bar{p}_i \bar{p}_j \bar{p}_k, \dots$, by exploiting (13), (18), \dots . Thus we can uniquely define a phase space distribution function $f(xp)$, such that it takes these values, $P(x), \bar{p}_i(x), \bar{p}_i \bar{p}_k, \bar{p}_i \bar{p}_j \bar{p}_k, \dots$, as its successively higher moments.

To represent the pure state condition (10) in terms of the distribution function, we may also proceed in the following way. From (10) we can easily get

$$\partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3} \rho \partial_i' \rho - \rho \partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3} \partial_i' \rho = 0, \\ (\rho_i^{n_i} \equiv (\partial/\partial x_i)^{n_i}, \partial_i' \equiv \partial/\partial x_i'). \quad (19)$$

Now we can show that we may replace the condition (10) by the condition that every equations of (19)

(including (10) as its lowest relation) should hold on the diagonal, i.e.,

$$[\partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3} \rho \partial_i' \rho - \rho \partial_1^{n_1} \partial_2^{n_2} \partial_3^{n_3} \partial_i' \rho]_{x'=x} = 0, \\ \text{for } \sum_i n_i = 1, 2, 3, \dots \infty. \quad (20)$$

The lowest relation of (20), when symmetrized or antisymmetrized, exactly agrees with (13) or (14) as was stated in I. The next relation.

$$[\partial_i \partial_j \rho \partial_k' \rho - \rho \partial_i \partial_j \partial_k' \rho]_{x'=x} = 0, \quad (21)$$

can be shown just to yield (18). We can further show that the higher order equations of (20) give the relations between higher moments and their space derivatives.

Fuller accounts will soon be published.

- 1) T. Takabayasi, Prog. Theor. Phys. **10** (1953), 119; T. Takabayasi, *ibid.* **10** (1953), 121. These notes are hereafter referred to as I and II respectively.
- 2) J. E. Moyal, Proc. Cambridge Phil. Soc. **45** (1949), 99; also see I.
- 3) E. Wigner, Phys. Rev. **40** (1932), 749.

ERRATUM

The j - j Coupling Shell Model Vol. 8, No. 5, P. 509

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In § 2, $\phi^0(n-sj-2, n-sj-1)$ and $\tau'_{\zeta}(n-sj-2, n-sj-1)$ must be replaced by $\phi^0(n-sj-1, n-sj)$ and $\tau'_{\zeta}(n-sj-1, n-sj)$. In § 3, at 14th line in p. 511, the wording "this eigenfunction has a primitive structure" have to be rewritten in the following precise wording — "this eigenfunction includes in general the eigenfunction of the primitive structure". In § 8, the two configurations $(d_{5/2})^3$ and $(f_{5/2})^{-5}$ of Cl^{15} in p. 514 and V^{51} in p. 515 are misprinted and the right configurations are $(d_{5/2})^3$ and $(f_{7/2})^{-5}$ respectively.

On the Constants of Motion for the Case of Non-localized Interactions

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By generalizing the method formerly developed by the author, the energy-momentum tensor is obtained also for the case of non-localized interactions. It comes out that the interaction part of this tensor is not given simply by $-L'\delta_{\mu\nu}$ as due to Kristensen-Møller and Bloch, and that the total tensor satisfies the equation of continuity as a consequence of the equations of motion. As the result of the space integration of the μ^4 component of the tensor, a general and simple expression of the energy-momentum 4-vector is derived in terms of δ functions and sign functions of the time. This vector turns out to be the constant of motion owing to the continuity equation. The same is true of the case of current density and total charge. Thus, it may be asserted contrary to the opinion of Kristensen-Møller and Bloch that it is always possible to construct not only the constants of collision but also the constants of motion when the invariant Lagrangian is given. The problem is now to consider the physical significance of the expressions and how to perform the quantization upon Heisenberg representation.

§ 1. Introduction and summary

Kristensen-Møller¹⁾ and Bloch²⁾ have developed the form factor theory of non-localized interactions by constructing the S -matrix after the method of Yang and Feldman, and pointed out that the divergence difficulties can thus be avoided. They noted, however, that owing to the non-locality the energy-momentum tensor and the current density vector do not satisfy the local equation of continuity, and so the total energy, momentum and charge cannot be the "constants of motion", being merely the "constants of collision" between infinite past and future.

Recently, on the other hand, Takahashi-Umezawa³⁾ and Katayama⁴⁾ showed, starting from the reexamination of the interaction representation, that the Hamiltonian (in the interaction representation) can be constructed even when the interaction is non-localized. This Hamiltonian is expressed as an infinite series of the interaction constant, although its convergence seems to have not been proved. Previous to this, Nambu⁵⁾ also obtained through investigation of the Lagrangian formalism the expression of the interaction Hamiltonian by the so-called P^*-L method. This method can as well be applied to the case of non-localized interaction.

Under these circumstances, we are confronted with the question whether the Hamiltonian formalism is generally possible in the case of non-localized interaction, and if possible how to understand its physical significance.

Here we treat before everything else the question as to the existence of constants of motion in this case. In § 2, the most general form of the form factor and its properties

are examined. Then the method formerly developed by the author^(*) is generalized in § 3, yielding the energy-momentum tensor $T_{\mu\nu}$ and proving that it satisfies the equation of continuity owing to the equation of motion. The expression of its interaction part $T'_{\mu\nu}$ is complicated depending on the implications of the form factor, and is not of such a simple form $-L'\delta_{\mu\nu}$ as given by Kristensen-Møller and Bloch. By space integration of its $\mu=4$ component, we get the energy-momentum 4-vector G_μ , which actually proves to be the constant of motion as a result of continuity equation. Moreover, its expression is so general and simple, containing symmetrically δ functions and sign functions, that it can easily be generalized for any kind of interactions. The same procedure is developed in § 5 for the current density vector and the total charge, starting from the definition of the former on the basis of gauge-invariance. In this way, we have succeeded in constructing the expression of the current vector that satisfies the continuity equation and that of the total charge which is the constant of motion.**

The above-mentioned situation is obvious from the general relativistic viewpoint. That is to say, according to the general relativity, there is established the following fact: When an invariant Lagrangian, the variation of which by the field quantities yields equations of motion, exists, the energy-momentum tensor can be obtained by the variation of the gravitational quantities ($g_{\mu\nu}$ or $h_{(p)}^\mu$), and the tensor thus derived must of necessity satisfy the continuity equation by virtue of equations of motion.⁽⁷⁾ In other words, *the very presence of an invariant Lagrangian ensures the existence of the constants of motion.* The question that comes to the front is therefore to seek out the physical implications of the expressions thus obtained.

It is to be remarked that every expression obtained in this paper stands on Heisenberg representation. Although it is beyond question that the interaction representation is preferable for many purposes, Heisenberg representation may also be useful when the physical meaning in terms of the corresponding classical conception comes into question. Further, there remain to be investigated the problems as to the quantization of the obtained expressions and their relation to those derived by Takahashi-Umezawa and Katayama and by Nambu.

§ 2. Some remarks on the form factor

Let us take, for example, the case of non-localized interaction between a neutral scalar field and a spinor field. The free Lagrangian density

$$L_0(x) = -(\bar{\psi}\gamma_\alpha\partial_\alpha\psi + M\bar{\psi}\psi) - (1/2)(\partial_\alpha u\partial_\alpha u + x^2 u^2) \quad (1)$$

and the action function of interaction

$$I' = \int dx' dx'' dx''' \bar{\psi}(x') \Phi(x', x'', x''') u(x'') \psi(x''') \quad (2)$$

* This paper is referred to as paper [1] in the following.

** C. Hayashi has also pointed out the possibility to construct them by using Green functions (private communication).

yield the equations of motion :

$$\begin{aligned} -(\gamma_\alpha \partial_\alpha + M)\psi(x) &= -\int \Phi(x, x'', x''')u(x'')\psi(x''')dx''dx''', \\ \partial_\alpha \bar{\psi}(x)\gamma_\alpha - M\bar{\psi}(x) &= -\int \bar{\psi}(x')\Phi(x', x'', x)u(x'')dx'dx'', \\ (\square - x^2)u(x) &= -\int \bar{\psi}(x')\Phi(x', x, x''')\psi(x''')dx'dx'''. \end{aligned} \quad (3)$$

Because of the invariance by a translation, the form factor $\Phi(x', x'', x''')$, which is in general a matrix including Dirac γ 's, is actually function of relative coordinates only. Thus, for example,

$$\Phi = \Phi(x' - x'', x' - x'''), \quad (4)$$

and hence the interaction Lagrangian density is given by

$$L'(x) = \int dx''dx''' \bar{\psi}(x)\Phi(x - x'', x - x''')u(x'')\psi(x''').* \quad (5)$$

While Kristensen-Møller and Bloch put $\Phi = AF(x', x'', x''')$ of which F is a scalar function, it is intended here to search for the most general form of Φ . In the first place, when the spinor field is to be extended invariantly, it is required to put the factor in the form

$$\sum_q \lambda_q (\gamma \partial) \delta(x - x''') = (f_1(\square) + f_2(\square) \gamma \partial) \delta(x - x'''),** \quad (6)$$

as was done in paper [1]. Next, when we must extend the interval between $\bar{\psi}$ and u , the Lorentz invariance of

$$\bar{\psi} \gamma_\alpha^p \psi \cdot \partial_\alpha^p u$$

requires the factor of the form

$$\sum_p \lambda_p' (\gamma \partial)^p \delta(x - x'') = (f_1'(\square) + f_2'(\square) \gamma \partial) \delta(x - x''). \quad (7)$$

Thus, combining the two cases (with the substitution $x - x'' \rightarrow x$, $x - x''' \rightarrow y$), we may write the factor Φ as

$$\Phi(x, y) = \sum_{(p, q)} \lambda_{(p, q)} (\gamma \partial_x)^{p_1} (\gamma \partial_y)^{q_1} \delta(x) \delta(y), \quad (8)$$

where (p, q) means that p $(\gamma \partial_x)$'s and q $(\gamma \partial_y)$'s stand in various orders. When two $(\gamma \partial_x)$'s (or $(\gamma \partial_y)$'s) stand side by side, they become \square_x (or \square_y), which is no more concerned with the ordering; that is, among these orderings there exist equivalent ones. It is therefore to be understood that in (p, q) only the non-equivalent orderings are included. By substitution of (8) in (5), we get

$$L'(x) = \sum_{(p, q)} \bar{\psi}(x) (\gamma \partial_x)^{p_1} (\gamma \partial_y)^{q_1} u(x) \psi(y)_{(y \rightarrow x)}. \quad (9)$$

Bringing all the $(\gamma \partial_y)$'s to the right of $(\gamma \partial_x)$, and taking account of the relation

* We took here, for convenience, $\bar{\psi}$ as a standard (i.e. $L^{(1)}$ of Kristensen-Møller), but the final results will prove to be the same, when we take u or ψ as a standard.

** $\gamma \partial$ means $\gamma_\alpha \partial_\alpha$.

$$(\gamma\partial_x)(\gamma\partial_y) + (\gamma\partial_y)(\gamma\partial_x) = 2(\partial_x\partial_y),^*$$
(10)

we obtain as Φ the more perspective form :

$$\begin{aligned}\Phi(x, y) &= \{F_1(\square_x, \square_y, \partial_x\partial_y)\mathbf{1} + F_2(\square_x, \square_y, \partial_x\partial_y)(\gamma\partial_x) \\ &\quad + F_3(\square_x, \square_y, \partial_x\partial_y) + F_4(\square_x, \square_y, \partial_x\partial_y)(\gamma\partial_x)(\gamma\partial_y)\}\delta(x)\delta(y) \\ &= F_1(x, y) + (\gamma\partial_x)F_2(x, y) + (\gamma\partial_y)F_3(x, y) + (\gamma\partial_x)(\gamma\partial_y)F_4(x, y) \\ &= \sum_{i=1}^4 \Phi_i(x, y),\end{aligned}$$
(11)

where the function F_1 and F_4 result from the terms $p \sim q = 0$ or even of (8), and F_2^* and F_3 from those $p \sim q = \text{odd}$. Φ_1 and Φ_2 mean respectively the extension of scalar and vector coupling, while Φ_3 and Φ_4 that of Konopinski-Uhlenbeck type.** Bloch took Φ_1 only, and Kristensen-Møller put $F_1 = F_2$ except for γ_5 and appropriate factors, but in our opinion it may not always be necessary to cling to the coupling type of local fields as they did. In fact, even if we adopted Φ_1 only, the operators \square_y and $\partial_x\partial_y$ would represent K.U. type.

Finally, from the Hermitian character of the action function I' , there results the condition for Φ :

$$\Phi(x', x'', x''') = \gamma_4 \Phi^+(x''', x'', x') \gamma_4.$$
(12)

In the special case, where $\Phi = \Phi_1$ or $\Phi = F_1 (= \Phi_1) + i(\gamma\partial_x)F_2$, this reduces respectively to

$$\Phi_1(x', x'', x''') = \Phi_1^*(x''', x'', x'),$$
(12')

or

$$F_i(x', x'', x''') = F_i^*(x''', x'', x'); \quad i=1, 2.$$
(12'')

But the condition is not so simple, when Φ contains Φ_3 or Φ_4 ***

§ 3. The energy-momentum tensor

Expressing the differential operators, which appear in the form factor (8) or (11), in general relativistic forms, we can obtain the energy-momentum tensor by the method developed in paper [1]. Let us, for simplicity, take Φ_1 of (11). Then, from the expansion

$$\begin{aligned}\Phi(x, y) &= F(\square_x, \square_y, \partial_x\partial_y) \delta(x) \delta(y) \\ &= \sum_{p, q, n=0}^{\infty} \lambda_{pqn} \square_x^p \square_y^q (\partial_x\partial_y)^n \delta(x) \delta(y),\end{aligned}$$
(13)

follows that

* $\partial_x\partial_y$ means $\partial_{x\alpha}\partial_{y\alpha}$.

** Differentiation of $\bar{\psi}$, being a sort of K.U. type, reduce to any type here mentioned owing to the indefiniteness of Lagrangian, e.g. $\partial_x\bar{\psi}\partial_x\psi \cdot u$ reduces to $\bar{\psi}\square_x\psi \cdot u + \bar{\psi}\partial_x\psi \cdot \partial_x u$.

*** Viewed from this situation, it is more convenient to take u as a standard.

$$I' = \sum \lambda_{pqn} \int d\lambda \bar{\psi}(x) \square^p \partial^n u \cdot \square^q \partial^n \psi \cdot * \quad (14)$$

One of these terms can be written in general invariant form as

$$\mathfrak{S}^{(p, q, n)} = \int \bar{\psi}(x) \square_{x(ge)}^p \cdot \square_{y(ge)}^q \cdot (\partial_x \partial_y)^n u(x) \psi(y) \sqrt{g} dx_{(y \rightarrow x)}, \quad (15)$$

in which**

$$\begin{aligned} \square_{x(ge)} &= \frac{1}{\sqrt{g}} \left(\frac{\partial}{\partial x^\alpha} \left(\sqrt{g} g^{\alpha\beta} \frac{\partial}{\partial x^\beta} \right) \right), \\ \square_{y(ge)} &= \gamma_\rho \gamma_\epsilon \left(h_{(\rho)}^\alpha \frac{\partial}{\partial y^\alpha} + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^\alpha} (h_{(\rho)}^\alpha \sqrt{g}) \right) \\ &\quad \times \left(h_{(\epsilon)}^\beta \frac{\partial}{\partial y^\beta} + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^\beta} (h_{(\epsilon)}^\beta \sqrt{g}) \right), \\ (\partial_x \partial_y)_{(yz)} &= \left(h_{(\rho)}^\alpha(x) \frac{\partial}{\partial x^\alpha} \right) \cdot \left(h_{(\rho)}^\beta(y) \frac{\partial}{\partial y^\beta} + \frac{1}{2} \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^\beta} (h_{(\rho)}^\beta \sqrt{g}) \right). *** \end{aligned} \quad (16)$$

Moreover, in view of the general invariance, it becomes clear that $(\partial_x \partial_y)^n$ must all be put before $\square_{x(ge)}^p$ and $\square_{y(ge)}^q$; in other words, $(\partial_x \partial_y)^n$ must operate after $\square_{x(ge)}^p$ and $\square_{y(ge)}^q$. Variation of $h_{(p)}^\mu$ yields

$$\begin{aligned} T_{\mu\nu}^{(p, q, n)}(x) &= \left\{ -\bar{\psi} \square^p \partial^n u \cdot \square^q \partial^n \psi + \frac{1}{2} (-\partial)^\mu (\bar{\psi} \square^p \partial^n u) \cdot \square^q \psi \right. \\ &\quad - \frac{1}{2} \square^q (-\partial)^\nu (\bar{\psi} \square^p \partial^n u) \cdot \psi \left. \right\} \delta_{\mu\nu} \\ &\quad + \sum_{a=0}^{p-1} \left\{ (\square^a A \cdot \square^{p-a} u + \square^a \partial_\alpha A \cdot \square^{p-1-a} \partial_\alpha u) \delta_{\mu\nu} \right. \\ &\quad - \square^a \partial_\mu A \cdot \square^{p-1-a} \partial_\nu u - \square^a \partial_\nu A \cdot \square^{p-1-a} \partial_\mu u \left. \right\} \\ &\quad + \frac{1}{2} \sum_{b=0}^{q-1} (\square^b \partial_{\mu\alpha} B \cdot \gamma_\alpha \gamma_\nu \square^{q-1-b} \psi + \square^b B \cdot \gamma_\nu \gamma_\alpha \partial_{\mu\alpha} \psi \\ &\quad - \square^b \partial_\mu B \cdot \gamma_\nu \gamma_\alpha \square^{q-1-b} \partial_\alpha \psi - \square^b \partial_\alpha B \cdot \gamma_\alpha \gamma_\nu \partial_\mu \psi) \\ &\quad + \sum_{c=0}^{n-1} (-1)^c \left\{ \frac{1}{2} \partial^c \partial_\alpha (\bar{\psi} \partial^{n-1} \partial_\alpha \square^p u \cdot \partial^{n-1-c} \square^q \psi) \delta_{\mu\nu} \right. \\ &\quad - \frac{1}{2} \partial^c \partial_\mu (\bar{\psi} \partial^{n-1} \partial_\nu \square^p u \cdot \partial^{n-1-c} \square^q \psi) \\ &\quad + \partial^c (\bar{\psi} \partial^{n-1} \partial_\nu \square^q \psi) \cdot \partial^{\mu-1-c} \partial_\mu \square^p u \\ &\quad \left. + \partial^c (\bar{\psi} \partial^{n-1} \partial_\nu \square^p u) \cdot \partial^{\mu-1-c} \partial_\mu \square^q \psi \right\}, \quad (17) \end{aligned}$$

* $\partial^n u \partial^n \psi$ means $\partial_1 a_1 \dots a_n u \cdot \partial a_1 \dots a_n \psi$.

** As for the notations, see paper [1].

*** For the derivation of this expression, see Appendix.

where

$$A \equiv (-\partial)^n (\bar{\psi} \square^q \partial^n \psi), \quad B \equiv (-\partial)^n (\bar{\psi} \square^p \partial^n u)$$

and

$$\partial^0 \dots \partial^{n-1} \dots \partial^{n-1-c} \equiv \partial_{a_1 \dots a_c} \dots \partial_{a_1 \dots a_r a_{c+1} \dots a_{n-1}} \dots \partial_{a_{c+1} \dots a_{n-1}}.$$

Next, as was done in paper [1], take the Fourier component of (17) and, taking into account the relation

$$\sum_{c=0}^{n-1} (-1)^c a^c \partial^{n-1-c} = \frac{\partial^n - (-a)^n}{b+a},$$

carry out the summation with respect to p, q, n . Then we obtain in the end the energy-momentum tensor of the interaction part:

$$\begin{aligned} \tilde{T}'_{\mu\nu}(k) = & \frac{1}{(2\pi)^8} \int dk' dk'' \tilde{\bar{\psi}}(k-k'-k'') \wedge \psi(k'') u(k') \\ & \times \left[\left\{ -\tilde{\Phi}(k', k'') - \frac{1}{2} \tilde{\Phi}(k, k''-k) + \frac{1}{2} F(-k'^2, -k''^2, (k-k') \cdot k') \right\} \delta_{\mu\nu} \right. \\ & - ((k-k')_\mu k'_\nu + (k-k')_\nu k'_\mu - k \cdot k' \delta_{\mu\nu}) \frac{\tilde{\Phi}(k'-k, k'') - F(-k'^2, -k''^2, (k-k') \cdot k'')}{k^2 - 2k \cdot k'} \\ & + \frac{1}{2} (k-2k'')_\mu ((k \cdot \gamma) \gamma_\nu - 2k_\nu'') \frac{\tilde{\Phi}(k', k''-k) - F(-k'^2, -k''^2, (k-k') \cdot k'')}{k^2 - 2k \cdot k''} \\ & + \frac{1}{2} (k_\mu k'_\nu - k \cdot k' \delta_{\mu\nu}) \frac{F(-k'^2, -k''^2, k \cdot k') - \tilde{\Phi}(k', k'')}{k' \cdot (k+k'')} \\ & - k'_\mu k'_\nu \frac{F(-k'^2, -k''^2, (k-k') \cdot k'') - \tilde{\Phi}(k', k'')}{k \cdot k''} \\ & \left. - k'_\nu k'_\mu \frac{F(-k'^2, -k''^2, (k-k'') \cdot k') - \tilde{\Phi}(k', k'')}{k \cdot k'} \right]. \end{aligned} \quad (18)$$

Here,

$$\tilde{\Phi}(k', k'') = F(-k'^2, -k''^2, -k' \cdot k'') \quad (19)$$

is the Fourier component of $\Phi(x, y)$, and the notation \wedge means that the expression in the square bracket must occupy the position between $\bar{\psi}$ and ψ . The symmetrical tensor can be derived by the aid of

$$\Theta'_{\mu\nu} = (T'_{\mu\nu} + T'_{\nu\mu})/2.$$

Now, multiplying (18) scalarly by ik_ν , we get

$$\begin{aligned} ik_\nu \tilde{T}'_{\mu\nu}(k) = & -\frac{1}{(2\pi)^8} \int dk' dk'' \tilde{\bar{\psi}}(k-k'-k'') \wedge \tilde{\psi}(k'') \tilde{u}(k') \\ & \times [i(k-k'-k'')_\mu \tilde{\Phi}(k', k'') + ik'_\mu \tilde{\Phi}(k'-k, k'') + ik'_\mu \tilde{\Phi}(k', k''-k)] \end{aligned} \quad (20)$$

or in x -representation

$$\begin{aligned}
 \partial_\nu T'_{\mu\nu}(x) = & - \left\{ \int \partial_\mu \bar{\psi}(x) \Phi(x, x'', x''') u(x'') \phi(x''') dx'' dx''' \right. \\
 & + \int \bar{\psi}(x') \Phi(x', x'', x) u(x'') \partial_\mu \phi(x) dx' dx'' \\
 & \left. + \int \bar{\psi}(x') \Phi(x', x, x''') \partial_\mu u(x) \phi(x''') dx' dx''' \right\}. \quad (20')
 \end{aligned}$$

On the other hand, the divergence of the free field tensor

$$\begin{aligned}
 T_{\mu\nu}^{(0)}(x) = & - (1/2) (\bar{\psi} \gamma_\nu \partial_\mu \psi - \partial_\mu \bar{\psi} \gamma_\nu \psi) + \{ (1/2) (\bar{\psi} \gamma_\alpha \partial_\alpha \psi - \partial_\alpha \bar{\psi} \gamma_\alpha \psi) + M \bar{\psi} \psi \} \delta_{\mu\nu} \\
 & - \partial_\mu u \partial_\nu u + (1/2) (\partial_\alpha u \partial_\alpha u + x^2 u^2) \delta_{\mu\nu} \quad (21)
 \end{aligned}$$

becomes

$$\partial_\nu T_{\mu\nu}^{(0)}(x) = -\partial_\mu \bar{\psi} (-\gamma_\alpha \partial_\alpha + M) \psi - (\partial_\alpha \bar{\psi} \gamma_\alpha - M \bar{\psi}) \partial_\mu \psi - (\square - x^2) u \cdot \partial_\mu u, \quad (22)$$

whose right-hand side is, owing to the equations of motion (3), just the same as that of (20') except for the reversal of sign. Thus, the total energy-momentum tensor

$$T_{\mu\nu} = T_{\mu\nu}^{(0)} + T'_{\mu\nu} \quad (23)$$

satisfies the equation of continuity :

$$\partial_\nu T_{\mu\nu} = 0. \quad (24)$$

In this way, we can construct in general the energy-momentum tensor satisfying (24), although its expression is of rather complicated form conditioned by the transformation character of the fields and the form of Φ .

§ 4. The total energy and momentum

In this section, the space integration of $\mu 4$ component of the tensor above obtained will be performed to get the energy-momentum 4-vector $G'_\mu(\vec{G}', iW')$. From the familiar equation

$$G'_\mu(x_0) = i \int T'_{\mu 4}(x) d\vec{x} = \frac{1}{2\pi} \int dk_0 \tilde{T}'_{\mu 4}(0, k_0) \exp(-ik_0 x_0)$$

we get the following extremely simple formula :

$$\begin{aligned}
 G'_\mu(x_0) = & i \int dx' dx'' dx''' \bar{\psi}(x') \wedge \phi(x''') u(x'') \left[-\{ \partial(x_0 - x_0') \right. \\
 & + \delta(x_0 - x_0'') + \delta(x_0 - x_0''') \} \delta_{\mu 4} \\
 & + \frac{i}{2} \left\{ \varepsilon(x_0 - x_0') \frac{\partial}{\partial x'_\mu} + \varepsilon(x_0 - x_0'') \frac{\partial}{\partial x''_\mu} \right. \\
 & \left. \left. + \varepsilon(x_0 - x_0''') \frac{\partial}{\partial x'''_\mu} \right\} \right] \Phi(x', x'', x'''). * \quad (25)
 \end{aligned}$$

* $\varepsilon(x_0)$ means sign function.

The characteristic features of this expression are as follows:

i) Perfect symmetry holds between x' , x'' , and x''' , irrespective of the starting Lagrangian density $L'(x)$.

ii) It is applicable to any Φ , i.e. the matrix character of Φ is of no concern.*

iii) Corresponding to the degree of extension similar terms make their appearance in the expression; notice that when $\Phi(x', x'', x''') = \Phi(x', x'') \delta(x' - x''')$, the terms $\delta(x_0 - x_0''')$ and $\varepsilon(x_0 - x_0''') \partial / \partial x_\mu'''$ simply drop out from (25).

iv) In the local case $\Phi(x', x'', x''') = \delta(x' - x'') \delta(x' - x''')$, and only in this case the simple formula

$$G'_\mu = - \int L'(x) d\vec{x} \delta_{\mu 4}$$

is valid. (Thus in this case $\vec{G}' = 0$ while, what is remarkable, $\vec{G}' \neq 0$ in general.)

v) Owing to (24), the total vector

$$G_\mu(x_0) = G_\mu^{(0)}(x_0) + G'_\mu(x_0) \quad (26)$$

is the constant of motion:

$$dG_\mu(x_0)/dx_0 = 0. \quad (27)$$

The properties i) ~ iv) suggest that when in general n field quantities $\varphi_1(x^{(1)}), \dots, \varphi_n(x^{(n)})^{**}$ are in interaction, i.e.

$$I' = \int \dots \int \prod_{r=1}^n \varphi_r(x^{(r)}) d\vec{x}^{(r)} O \Phi(x^{(1)}, \dots, x^{(n)}), \quad (28)$$

we can write the vector as

$$G'_\mu(x_0) = - \int \dots \int \prod_{r=1}^n \varphi_r(x^{(r)}) d\vec{x}^{(r)} O \sum_{r=1}^n \left\{ \delta(x_0 - x_0^{(r)}) \delta_{\mu 4} - \frac{i}{2} \varepsilon(x_0 - x_0^{(r)}) \frac{\partial}{\partial x_\mu^{(r)}} \right\} \Phi(x^{(1)}, \dots, x^{(n)}), \quad (29)$$

where O means an appropriate large matrix, and $O\Phi$ must occupy a proper position among φ 's. This formula might be of use, for example, in the case of Fermi-interaction.

Similarly, the angular momentum defined by

$$M_{ij}(x_0) = i \int (x_i T_{j4} - x_j T_{i4}) d\vec{x} = \frac{1}{2\pi} \int dk_0 \left(\frac{\partial \tilde{T}_{j4}(k)}{\partial k_i} - \frac{\partial \tilde{T}_{i4}(k)}{\partial k_j} \right) \bigg|_{k=0} \exp(-ik_0 x_0)$$

may be calculated, and will prove to be the constant of motion containing likewise δ 's and ε 's.

§ 5. The current density and the total charge

Kristensen-Møller stated that the current density

* Actually, we verified (25) for the case $\Phi = \Phi_1 + \Phi_2$.

** Of course $\bar{\psi}$ and ψ of the same spinor field are contained in them.

$$s_{\mu}^{(0)}(x) = ie\bar{\psi}\gamma_{\mu}\psi \quad (30)$$

does not satisfy the continuity equation and so the total charge

$$Q^{(0)}(x_0) = e \int \psi^* \psi d\vec{x} \quad (31)$$

can only be the constant of collision. But also in this case the expression of total charge which is the constant of motion can be constructed. When the Lagrangian density contains higher derivatives of $\bar{\psi}$ and ψ , the current density 4-vector must be defined as

$$s_{\mu} = ie \sum_{r, m=0}^{\infty} (-1)^m \left(\partial_{a_r} \bar{\psi} \cdot \partial_{\beta_m} \frac{\partial L}{\partial (\partial_{\mu a_r, \beta_m} \bar{\psi})} - \partial_{\beta_m} \frac{\partial L}{\partial (\partial_{\mu a_r, \beta_m} \psi)} \cdot \partial_{a_r} \psi \right)^*, \quad (32)$$

and this in turn satisfies the equation

$$\partial_{\mu} s_{\mu} = 0 \quad (33)$$

because of the requirement of gauge invariance of L .

When we put, for example,

$$\Phi(x, y) = F(\square_x, \partial_x \partial_y) \delta(x) \delta(y), \quad (34)$$

the equation (32) gives

$$\tilde{s}_{\mu}'(k) = \frac{e}{(2\pi)^3} \int dk' dk'' \tilde{\bar{\psi}}(k - k' - k'') \wedge \psi(k'') u(k') k'_{\mu} \frac{\tilde{\Phi}(k', k'' - k) - \tilde{\Phi}(k', k'')}{k \cdot k'}, \quad (35)$$

and multiplication of ik_{μ} yields in x -representation

$$\begin{aligned} \partial_{\mu} s_{\mu}' &= ie \left(\int \bar{\psi}(x') \Phi(x', x'', x) u(x'') \psi(x) dx' dx'' \right. \\ &\quad \left. - \int \bar{\psi}(x) \Phi(x, x'', x''') u(x'') \psi(x''') dx'' dx''' \right). \end{aligned} \quad (36)$$

This is identical with the right-hand side of $\partial_{\mu} s_{\mu}^{(0)}$ except for the sign reversed. Thus for

$$s_{\mu} = s_{\mu}^{(0)} + s_{\mu}' \quad (37)$$

the equation (33) is verified.

Next,

$$\begin{aligned} Q'(x_0) &= \frac{1}{i} \int s_4'(x) d\vec{x} \\ &= e \int dx' dx'' dx''' \bar{\psi}(x') \wedge \psi(x''') u(x'') \frac{1}{2} (\varepsilon(x_0 - x_0') - \varepsilon(x_0 - x_0''')) \dot{\Phi}(x', x'', x'''). \end{aligned}$$

This quantity (interaction charge) has also a general form irrespective of character of Φ . It vanishes in the special case where $\Phi(x', x'', x''') = \Phi(x', x'') \delta(x' - x''')$; in other words, when there is no positron theoretical extension, the interaction charge disappears. Finally, the total charge

* $\partial_{\mu a_r, \beta_m}$ means $\partial_{\mu a_1 \dots a_r, \beta_1 \dots \beta_m}$.

$$Q(x_0) = Q^{(0)}(x_0) + Q'(x_0) \quad (39)$$

comes out to be the constant of motion owing to (33) :

$$dQ(x_0)/dx_0 = 0. \quad (40)$$

Appendix

General transformation property of current density 4-vector

Let us consider a spinor field in (local) interaction with the electromagnetic field. The Lagrangian density is given by

$$L = L^E + L^F + L^I \\ = -(1/4)f_{\alpha\beta}f_{\alpha\beta} - \{ (1/2)(\bar{\psi}\gamma_\alpha\partial_\alpha\psi - \partial_\alpha\bar{\psi}\cdot\gamma_\alpha\psi) + M\bar{\psi}\psi \} + s_\alpha\varphi_\alpha, \quad (A.1)$$

where the electromagnetic potential satisfies the Lorentz condition

$$\partial_\alpha\varphi_\alpha = 0, \quad (A.2)$$

and

$$s_\alpha = ie\bar{\psi}\gamma_\alpha\psi = (\vec{s}, i\rho).$$

Deriving the canonical energy-momentum tensor and symmetrizing, there results

$$\theta_{\mu\nu} = \theta_{\mu\nu}^E + \theta_{\mu\nu}^{FI}, \\ \theta_{\mu\nu}^E = -f_{\mu\alpha}f_{\nu\alpha} + (1/4)f_{\alpha\beta}f_{\alpha\beta}\delta_{\mu\nu}, \quad (A.3) \\ \theta_{\mu\nu}^{FI} = -(1/4)\{\bar{\psi}\gamma_\nu D_\mu\psi - D_\mu^*\bar{\psi}\cdot\gamma_\nu\psi + \bar{\psi}\gamma_\mu D_\nu\psi - D_\nu^*\bar{\psi}\cdot\gamma_\mu\psi\} + (L^F + L^I)\delta_{\mu\nu},$$

where

$$D_\mu = \partial_\mu - ie\varphi_\mu, \quad D_\mu^* = \partial_\mu + ie\varphi_\mu,$$

and the term $L^F + L^I$ disappears on account of the equations of motion.⁸⁾ The equation of continuity reads

$$\partial_\nu\theta_{\mu\nu} = \partial_\nu\theta_{\mu\nu}^E + \partial_\nu\theta_{\mu\nu}^{FI} = f_{\mu\nu}s_\nu - f_{\mu\nu}s_\nu = 0.$$

Next, we decompose $\theta_{\mu\nu}^{FI}$ as follows :

$$\theta_{\mu\nu}^{FI} = \theta_{\mu\nu}^E + \theta_{\mu\nu}^I,$$

where

$$\theta_{\mu\nu}^E = -(1/4)\{\bar{\psi}\gamma_\nu\partial_\mu\psi - \partial_\mu\bar{\psi}\cdot\gamma_\nu\psi + \bar{\psi}\gamma_\mu\partial_\nu\psi - \partial_\nu\bar{\psi}\cdot\gamma_\mu\psi\} + L^F\delta_{\mu\nu}, \quad (A.4)$$

$$\theta_{\mu\nu}^I = (1/2)(s_\mu\varphi_\nu + s_\nu\varphi_\mu) - s_\alpha\varphi_\alpha\delta_{\mu\nu}. \quad (A.5)$$

Then

$$\partial_\nu\theta_{\mu\nu}^E = -f_{\mu\nu}s_\nu - (1/2)\partial_\nu(s_\mu\varphi_\nu + s_\nu\varphi_\mu) + \partial_\mu(s_\alpha\varphi_\alpha) \\ = s_{\mu\nu}\varphi_\nu + (1/2)\partial_\nu(s_\mu\varphi_\nu + s_\nu\varphi_\mu), \quad (A.6)$$

and

$$\partial_\nu\theta_{\mu\nu}^I = (1/2)\partial_\nu(s_\mu\varphi_\nu + s_\nu\varphi_\mu) - \partial_\mu(s_\alpha\varphi_\alpha), \quad (A.7)$$

with

$$s_{\mu\nu} \equiv \partial_\mu s_\nu - \partial_\nu s_\mu.$$

On the other hand, when the equation of motion of extended spinor field are given by

$$\int \bar{\psi}(x') K(x' - x) dx' = \bar{\psi} O^+, \quad \int dx' K(x - x') \psi(x') = O \psi,^* \quad (\text{A.8})$$

we arrive at, with the help of Eq. (5.12) of paper [1], the general formula

$$\begin{aligned} \partial_\nu \theta_{\mu\nu}^P = & \partial_\mu (\bar{\psi} O \psi) - (1/4) \{ \partial_\alpha (\bar{\psi} O^+) \cdot \gamma_\alpha \gamma_\mu \psi - \bar{\psi} O^+ \gamma_\mu \gamma_\alpha \partial_\alpha \psi - \partial_\alpha \bar{\psi} \cdot \gamma_\alpha \gamma_\mu O \psi \\ & + \bar{\psi} \gamma_\mu \gamma_\alpha \partial_\alpha (O \psi) + \partial_\mu (\bar{\psi} O^+) \cdot \psi - \bar{\psi} O^+ \partial_\mu \psi - \partial_\mu \bar{\psi} \cdot O \psi + \bar{\psi} \partial_\mu (O \psi) \}. \end{aligned} \quad (\text{A.9})$$

Applying this formula to the present case, where $O(x) \equiv i \gamma_\alpha \varphi_\alpha(x)$, we get just the same expression as (A.6).

Now, the question is how the interaction tensor (A.5) can be derived by making use of the method described in paper [1]. If we assume s_α to be a usual vector, as is the case with the potential φ_α , the tensor derived from the action function

$$\mathfrak{S}'' = \int g^{\alpha\beta} s_\alpha \varphi_\beta \sqrt{g} dx$$

would become

$$\theta_{\mu\nu}'' = (s_\mu \varphi_\nu + s_\nu \varphi_\mu) - s_\alpha \varphi_\alpha \delta_{\mu\nu},$$

which differs by the factor 1/2 from (A.5). On the contrary, when we use the form

$$\mathfrak{S}' = \int h_\alpha^{(p)} s_p \varphi_\alpha \sqrt{g} dx, \quad (\text{A.10})$$

we get just the right answer.

Thus we must conclude that the quantity of the form $j_\alpha = \bar{\psi} \gamma_\alpha \psi$ is not a vector from the general relativistic viewpoint, but the quantity $h_\alpha^{(p)} j_p$ and $h_\alpha^{(p)} j_p$ are to be regarded as contravariant and covariant vector, respectively. In other words, each j_p is actually a scalar as the component of a vector along each local coordinate axis.** This conclusion can be deduced also from different angles.

With this in mind, it is easily observed that the general expression of

$$\bar{\psi} \gamma_\alpha \psi \cdot \partial_\alpha u$$

takes the form

$$\bar{\psi} h_\alpha^{(p)} \gamma_p \psi \cdot \partial_\alpha u.$$

Then combining this and that of $\bar{\psi} \gamma_\alpha \partial_\alpha \psi$ in paper [1],*** and using the relation (10), we finally obtain the third formula of (16).

* $O(x)$ is in general a matrix, and O^+ is its Hermitian conjugate. Dirac field is included as a special case: $K(x) = -(\gamma \partial + M) \delta(x)$.

** Because $h_\alpha^{(\epsilon)} h_\alpha^{(p)} j_p = \delta_{\epsilon p} j_p = j_\epsilon$.

*** i.e. the formula (5.3) of paper [1].

Note added in proof :

After sending my manuscript, I had an occasion to read the paper of J. Rzewuski (Nuovo Cimento, **10** (1953), 182) treating the similar problem. His conclusion comes out the same as mine. Thus, in his notation, the formula (29) can be written in covariant form as

$$G_{\mu'}(\sigma)_{12} = \sum_{i=1}^n \left\{ \int_{\sigma} d\sigma_{\mu}(x^{(i)}) \varphi_{\delta}(x^{(i)}) - \int_{\sigma_1}^{\sigma} dx^{(i)} \varphi_{\delta}(x^{(i)}) \partial_{\mu}(i) \right\} \\ \times \int_{\sigma_1}^{\sigma_2} \cdots \int_{\sigma_1}^{\sigma_2} \prod_{r=1}^n \varphi_r(x^{(r)}) dx^{(r)} O\varphi(x^{(1)}, \dots, x^{(n)}). \quad (29')$$

But, according to his method, it seems in general not so simple to derive the explicit expression of the energy-momentum tensor (or current density).

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- 8) Cf. G. Wentzel, "Quantum Theory of Fields" (1949), p. 189.

On the Equivalence of the Particle Formalism and the Wave Formalism of Meson, II

— Case of Interacting Meson and Nucleon Fields —

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The β -matrix formalism is applied to the interaction of the charged meson with the nucleon. Harish-Chandra's I -formalism is used for introducing the invariant coupling terms, and the scattering matrix S is constructed covariantly by Umezawa-Takahashi's method in order to avoid the troublesome "surface terms". This S -matrix contains a normal independent direct coupling of nucleons besides the usual meson exchange couplings. It is shown that this additional coupling is due to the direct coupling included implicitly in the Lagrangian, and that the possibility, stated by Klein, of cancelling it by the corresponding term appearing in the whole interaction Hamiltonian is lost. The neutral and the charge symmetrical meson are also discussed briefly.

§ 1. Introduction

In the previous paper¹⁾, the interaction of the meson with the electromagnetic field is treated covariantly in the β -formalism, and if we use Fujiwara's formalism²⁾ it is shown that the scattering matrix S agrees with that in the wave formalism in spite of the remarkable difference of structure. In the case of the interaction of the meson with the photon, the interaction Lagrangian \mathcal{L}_{int} contains the meson field bilinearly, hence Fujiwara's formalism is applicable. But Harish-Chandra's I -formalism³⁾ is more convenient for the case of the interaction of the meson with the nucleon, because \mathcal{L}_{int} contains the meson field linearly. With the aid of this formalism, we can introduce elegantly any of the invariant coupling terms permitted in the wave formalism without referring to the special representations of the β -matrices.

The object of the present investigation is to treat the meson-nucleon system in Tomonaga-Schwinger's covariant formulation, and to show that the so-called "wave" aspect of meson can also be treated by the Duffin-Kemmer field. The interaction Hamiltonian of the interaction representation has the additional surface dependent terms besides the negative of the interaction Lagrangian, as is the case in the meson-photon system. On physical grounds, this surface dependence should not appear in any final results of calculation and it is shown that by a suitable modification explicit calculation can be done avoiding these troublesome terms. For this purpose, it is most convenient to use Umezawa-Takahashi's⁴⁾ rule which is easily confirmed in the following in the same way as Kinoshita's⁵⁾. In our treatment, an additional normal independent direct coupling

appears in the S-matrix besides the usual meson exchange coupling of nucleons. This difference is due to the difference of the starting Lagrangian. In the Duffin-Kemmer field the charged meson is most easily treated, but the extension to the neutral or the charge symmetrical meson is also done by introducing a suitable matrix θ .

In § 2 the mathematical preliminaries of our calculation, the I -formalism and its connection with Fujiwara's formalism, are discussed briefly. In § 3, the interaction of the charged meson with the nucleon is treated covariantly. Meson parts of the S-matrix are calculated explicitly. It is pointed out that these parts contain the normal independent direct couplings. In § 4 the origin of these direct couplings is discussed. In § 5 the extension to the neutral or the charge symmetrical meson is made briefly.

§ 2. Mathematical preliminaries and the free meson field

In this section, we summarize our basic formulas without proofs, details of which can be seen in the cited paper by Harish-Chandra. Also, the connection with Fujiwara's formalism is discussed. For simplicity, the matrices β_μ are considered to be already reduced to three irreducible representations, 10-, 5- and 1-row representations, and it is also assumed to be Hermitian. There is no loss of generality to make this assumption. In this section, we consider only the free meson field. Notations are the same as in I.

Free meson $\psi(x)$ satisfies the following equation of motion :

$$(\beta\partial + x)\psi(x) = 0. \quad (1)$$

The β_μ 's satisfy the following commutation relations :

$$\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda = \delta_{\mu\nu}\beta_\lambda + \delta_{\mu\lambda}\beta_\nu. \quad (2)$$

We define the adjoint $\bar{\psi}(x)$ of $\psi(x)$ by

$$\bar{\psi}(x) = \psi^*(x)\eta_A, \quad (3)$$

where $\psi^*(x)$ is the complex conjugate of $\psi(x)$ and

$$\eta_\mu = 2\beta_\mu^2 - E, \quad (\text{no summation}) \quad (4)$$

where E is the unit matrix. $\bar{\psi}(x)$ satisfies the following equation of motion :

$$\partial\bar{\psi}(x)\beta - x\bar{\psi}(x) = 0. \quad (5)$$

We introduce a one-row vector matrix Γ_μ^* such that $\Gamma_\mu^*\psi(x)$ transforms as a vector for any Lorentz transformation. Then $\bar{\psi}(x)\Gamma_\mu$, where Γ_μ is the Hermitian conjugate of Γ_μ^* , transforms as a vector, too. Write P_μ or P_μ^* in place of Γ_μ^* or Γ_μ in the scalar case, and R_μ or R_μ^* in the vector case.

These satisfy the following relations from their definitions and the commutation relations (2),

Scalar

Vector

$$\left. \begin{aligned}
 P_\mu^* P_\nu &= \delta_{\mu\nu} \\
 P_\mu^* l_\rho P_\nu &= 0 \\
 P_\mu^* \beta_\rho l_\sigma &= \delta_{\mu\rho} l_\sigma^* \\
 \beta_\sigma \beta_\rho P_\mu &= \delta_{\mu\rho} l_\sigma^* \\
 P_\mu^* \beta_\sigma &= P_\sigma^* \beta_\mu = 1/4 \cdot \delta_{\mu\sigma} P_\rho^* \beta_\rho \\
 \beta_\sigma P_\mu &= \beta_\mu P_\sigma = 1/4 \cdot \delta_{\mu\sigma} \beta_\rho P_\rho
 \end{aligned} \right\}, \quad (6s)$$

$$\left. \begin{aligned}
 R_\mu^* R_\nu &= \delta_{\mu\nu} \\
 R_\mu^* l_\rho R_\nu &= 0 \\
 R_\mu^* \beta_\rho l_\sigma &= \delta_{\rho\sigma} R_\mu^* - \delta_{\mu\sigma} R_\rho^* \\
 \beta_\sigma \beta_\rho R_\mu &= \delta_{\rho\sigma} R_\mu - \delta_{\mu\sigma} R_\rho \\
 R_\mu^* \beta_\sigma &= -R_\sigma^* \beta_\mu \\
 \beta_\sigma R_\mu &= -\beta_\mu R_\sigma
 \end{aligned} \right\}, \quad (6v)$$

We may be able to take P_μ^* (or R_μ^*) to be 1×5 (or 1×10) matrix, but it may as well be considered as 1×16 (or 1×16) one if all other elements are zero. So, we consider hereafter \bar{P} , P_μ^* and R_μ^* (or l , P_μ and R_μ) as 1×16 (or 16×1), and β_μ 's as 16×16 matrices, which are reduced to three submatrices already. When we multiply P_ν^* by P_μ from the left, we understand it always as a direct product $P_\mu \times P_\nu^*$, but omit the product symbol \times as no confusions arise. Put

$$\left. \begin{aligned}
 \bar{P} &= P_\mu l_\mu^* \\
 P &= 1/4 \cdot \beta_\mu \bar{P} \beta_\mu
 \end{aligned} \right\}, \quad (7s)$$

$$\left. \begin{aligned}
 R &= R_\mu R_\mu^* \\
 \bar{R} &= 1/2 \cdot \beta_\mu R \beta_\mu
 \end{aligned} \right\}, \quad (7v)$$

then these satisfy the following relations:

$$\left. \begin{aligned}
 l &= \beta_\mu P l_\mu^* \\
 l^2 &= P, \quad l^2 = \bar{l} \\
 P \bar{P} &= \bar{P} P = 0 \\
 P + \bar{l} &= E^s \\
 P \beta_\mu + \beta_\mu P &= \beta_\mu^s \\
 \bar{l} \beta_\mu + \beta_\mu \bar{l} &= \beta_\mu^s \\
 P \beta_\mu &= \beta_\mu \bar{P}, \quad \bar{P} \beta_\mu = \beta_\mu P \\
 P \beta_\mu \dots \beta_\nu P &= \bar{P} \beta_\mu \dots \beta_\nu \bar{P} = 0 \\
 &\quad \text{odd number} \quad \text{odd number} \\
 P \beta_\mu \dots \beta_\lambda \bar{P} &= \bar{P} \beta_\mu \dots \beta_\lambda P = 0 \\
 &\quad \text{even number} \quad \text{even number}
 \end{aligned} \right\}, \quad (8s)$$

$$\left. \begin{aligned}
 R &= 1/3 \cdot \beta_\mu \bar{R} l_\mu^* \\
 R^2 &= R, \quad \bar{R}^2 = R \\
 R \bar{R} &= \bar{R} R = 0 \\
 R + \bar{R} &= E^v \\
 R \beta_\mu + \beta_\mu R &= \beta_\mu^v \\
 \bar{R} \beta_\mu + \beta_\mu \bar{R} &= \beta_\mu^v \\
 R \beta_\mu &= \beta_\mu \bar{R}, \quad \bar{R} \beta_\mu = \beta_\mu R \\
 R \beta_\mu \dots \beta_\nu R &= \bar{R} \beta_\mu \dots \beta_\nu \bar{R} = 0 \\
 &\quad \text{odd number} \quad \text{odd number} \\
 R \beta_\mu \dots \beta_\lambda \bar{R} &= \bar{R} \beta_\mu \dots \beta_\lambda R = 0 \\
 &\quad \text{even number} \quad \text{even number}
 \end{aligned} \right\}, \quad (8v)$$

Thus P , \bar{P} , R and \bar{R} correspond to the ones that are defined in I. Put

$$P_{\mu\nu} = P_\mu P_\nu^*, \quad (9s)$$

$$R_{\mu\nu} = R_\mu R_\nu^*. \quad (9v)$$

These satisfy the following relations[†]:

[†] P_μ and R_μ , defined in I, correspond to $P_{\mu\mu}$ and $R_{\mu\mu}$ (no summation) in this paper. $R_{\mu\nu}$ is just the same as the one defined in I.

$$\left. \begin{aligned} \bar{P} &= \delta_{\mu\nu} P_{\mu\nu} \\ P &= 1/4 \cdot \partial_{\lambda\rho} \partial_{\mu\nu} \partial_{\lambda\sigma} P_{\mu\nu} \beta_{\rho} \\ P \beta_{\mu} \beta_{\nu} &= \beta_{\mu} \beta_{\nu} P = \delta_{\mu\nu} P \\ \bar{P} \beta_{\mu} \beta_{\nu} &= \beta_{\mu} \beta_{\nu} \bar{P} = P_{\mu\nu} \end{aligned} \right\}, \quad (10s)$$

$$\left. \begin{aligned} R &= \delta_{\mu\nu} R_{\mu\nu} \\ \bar{R} &= 1/2 \cdot \partial_{\lambda\rho} \partial_{\mu\nu} \beta_{\lambda} R_{\mu\nu} \beta_{\rho} \\ R \beta_{\mu} \beta_{\nu} &= \beta_{\mu} \beta_{\nu} R = \delta_{\mu\nu} R - R_{\mu\nu} \\ \bar{R} \beta_{\mu} \beta_{\nu} &= \beta_{\mu} \beta_{\nu} R = \beta_{\rho} R_{\mu\nu} \beta_{\rho} \end{aligned} \right\}, \quad (10v)$$

$$\left. \begin{aligned} P_{\mu\nu} P_{\lambda\rho} &= \delta_{\nu\lambda} P_{\mu\rho} \\ P_{\mu\nu} \beta_{\lambda} \beta_{\rho} &= \delta_{\nu\lambda} P_{\mu\rho} \\ \beta_{\lambda} \beta_{\rho} P_{\mu\nu} &= \delta_{\mu\rho} P_{\lambda\nu} \\ P_{\mu\nu} \beta_{\tau} \beta_{\xi} P_{\lambda\rho} &= \delta_{\nu\tau} \delta_{\xi\lambda} P_{\mu\rho} \end{aligned} \right\}, \quad (11s)$$

$$\left. \begin{aligned} R_{\mu\nu} R_{\lambda\rho} &= \delta_{\nu\lambda} R_{\mu\rho} \\ R_{\mu\nu} \beta_{\lambda} \beta_{\rho} &= \delta_{\lambda\rho} R_{\mu\nu} - \delta_{\nu\rho} R_{\mu\lambda} \\ \beta_{\lambda} \beta_{\rho} R_{\mu\nu} &= \delta_{\lambda\rho} R_{\mu\nu} - \delta_{\mu\lambda} R_{\rho\nu} \\ R_{\mu\nu} \beta_{\tau} \beta_{\xi} R_{\lambda\rho} &= (\delta_{\tau\xi} \delta_{\nu\lambda} - \delta_{\tau\lambda} \delta_{\nu\xi}) R_{\mu\rho} \end{aligned} \right\}. \quad (11v)$$

The spurs of the matrices defined in (7) or (9) are as follows:

$$\left. \begin{aligned} S_P P &= 1 \\ S_P \bar{P} &= 4 \\ S_P P_{\mu\nu} &= \delta_{\mu\nu} \end{aligned} \right\}, \quad (12s)$$

$$\left. \begin{aligned} S_P R &= 4 \\ S_P \bar{R} &= 6 \\ S_P R_{\mu\nu} &= \delta_{\mu\nu} \end{aligned} \right\}. \quad (12v)$$

Next, we consider the connection between the wave functions of the Duffin-Kemmer field and that of Proca- or Klein-Gordon field. Put

$$\left. \begin{aligned} P_{\mu}^* \psi &= \frac{i}{\sqrt{x}} u_{\mu} \approx \frac{i}{\sqrt{x}} \partial_{\mu} u \\ \bar{\psi} P_{\mu} &= \frac{i}{\sqrt{x}} u_{\mu}^* \approx \frac{i}{\sqrt{x}} \partial_{\mu} u^* \\ P_{\mu}^* \beta_{\nu} \psi &= -i \sqrt{x} u \delta_{\mu\nu} \\ \bar{\psi} \beta_{\nu} P_{\mu} &= i \sqrt{x} u^* \delta_{\mu\nu} \end{aligned} \right\}, \quad (13s)$$

$$\left. \begin{aligned} R_{\mu}^* \psi &= \sqrt{x} v_{\mu} \\ \bar{\psi} R_{\mu} &= \sqrt{x} v_{\mu}^* \\ R_{\mu}^* \beta_{\nu} \psi &= \frac{1}{\sqrt{x}} h_{\mu\nu} \approx \frac{1}{\sqrt{x}} (\partial_{\mu} v_{\nu} - \partial_{\nu} v_{\mu}) \\ \bar{\psi} \beta_{\nu} R_{\mu} &= -\frac{1}{\sqrt{x}} h_{\mu\nu}^* \approx -\frac{1}{\sqrt{x}} (\partial_{\mu} v_{\nu}^* - \partial_{\nu} v_{\mu}^*) \end{aligned} \right\}, \quad (13v)$$

where u , u^* ; u_{μ} , u_{μ}^* ; v_{μ} , v_{μ}^* ; $h_{\mu\nu}$, $h_{\mu\nu}^*$ are certain scalar; vector; or tensor-quantities, respectively. If we regard u , u^* or v_{μ} , v_{μ}^* as the wave functions of Klein-Gordon- or Proca-equation, and require that the equations (1) and (5) are equivalent to the Klein-Gordon- or Proca-equation, u_{μ} , u_{μ}^* or $h_{\mu\nu}$, $h_{\mu\nu}^*$ are given by the right hand side of the equality. Hereafter, we adopt a sign \approx when the equality is established if we use the free equation of motion, and a sign \approx if we use the equation of motion with interaction. Also, we write as usual

$$f_{\mu\nu} \equiv \partial_{\mu} v_{\nu} - \partial_{\nu} v_{\mu}, \quad f_{\mu\nu}^* \equiv \partial_{\mu} v_{\nu}^* - \partial_{\nu} v_{\mu}^*, \quad (14)$$

then we have

$$h_{\mu\nu} \approx f_{\mu\nu}, \quad h_{\mu\nu}^* \approx f_{\mu\nu}^* \quad (13')$$

On account of the antisymmetry of $R_{\mu}^* \beta_{\nu}$ and $\beta_{\nu} R_{\mu}$ in μ , ν , $h_{\mu\nu}$ and $h_{\mu\nu}^*$ is evidently antisymmetric, and this fact is consistent with that $f_{\mu\nu}$ and $f_{\mu\nu}^*$ is antisymmetric in μ , ν .

The free-field ψ , ψ^* etc. can be rewritten as follows;

$$\left. \begin{aligned} \phi^s(x) &\approx -\frac{1}{x}(\beta\partial - x)P\psi(x) \\ \bar{\psi}(x) &\approx \frac{1}{x}\bar{\psi}(x)P(\beta\overleftarrow{\partial} + x) \end{aligned} \right\}, \quad (15s) \quad \left. \begin{aligned} \phi^v(x) &\approx -\frac{1}{x}(\beta\partial - x)R\psi(x) \\ \bar{\psi}^v(x) &\approx \frac{1}{x}\bar{\psi}(x)R(\beta\overleftarrow{\partial} + x) \end{aligned} \right\}. \quad (15v)$$

The commutation relations of the free fields are given by

$$[\phi_p(x), \bar{\psi}_\sigma(x')] = \frac{1}{i} \left\{ \beta\partial - \frac{1}{x}(\beta\partial)^2 \right\}_{\rho\sigma} \Delta(x-x'). \quad (16)$$

These can be rewritten for the scalar or vector meson respectively as follows;

$$[\phi_p^s(x), \bar{\psi}_\sigma^s(x')] = (i/x) (\beta\partial - x) P(\beta\partial - x)_{\rho\sigma} \Delta(x-x'), \quad (17s)$$

$$[\phi_p^v(x), \bar{\psi}_\sigma^v(x')] = (i/x) [(\beta\partial - x) R(\beta\partial - x) + \{(\beta\partial)^2 - x^2\} R]_{\rho\sigma} \Delta(x-x'). \quad (17v)$$

The Green function $1/2 T_F(x)$ for the meson is calculated easily by Umezawa-Takahashi's method and $T_F(x)$ is given by

$$T_F(x) = - \left\{ \beta\partial - \frac{1}{x}(\beta\partial)^2 + \frac{1}{x}(\square - x^2) \right\} \Delta_F(x). \quad (18)$$

This can also be rewritten for the scalar and vector meson as follows:

$$T_F^s(x) = \frac{1}{x} \{ (\beta\partial - x) P(\beta\partial - x) - \beta_\mu P \beta_\mu (\square - x^2) \} \Delta_F(x), \quad (19s)$$

$$T_F^v(x) = \frac{1}{x} \left\{ (\beta\partial - x) R(\beta\partial - x) - \frac{1}{2} \beta_\mu R \beta_\mu (\square - x^2) + [(\beta\partial)^2 - \square] R \right\} \Delta_F(x). \quad (19v)$$

§ 3. The interaction of the charged meson with the nucleon

The formulation, which is given for the free meson in the previous section, can be extended to the meson interacting with the other fields. We use, in this paper, heavy type for Heisenberg representation operators and light type for interaction representation operators. Especially, we use \mathcal{L} and \mathcal{H} for the Lagrangian and the Hamiltonian which consist of the Heisenberg representation operators, and use L and H for the ones which consist of the interaction representation operators.

We use the following source densities for the meson field

$$\begin{aligned} w &= \bar{\phi} \tau_- \phi, & j_\mu &= i \bar{\phi} \tau_- \gamma_\mu \phi, & m_{\mu\nu} &= \bar{\phi} \tau_- \sigma_{\mu\nu} \phi, \\ w^* &= \bar{\phi} \tau_+ \phi, & j_\mu^* &= i \bar{\phi} \tau_+ \gamma_\mu \phi, & m_{\mu\nu}^* &= \bar{\phi} \tau_+ \sigma_{\mu\nu} \phi. \end{aligned} \quad (20)$$

These are Hermitian except τ -spin, and τ 's itself have the character $\tau_+^* = \tau_-$ and $\tau_-^* = \tau_+$. ϕ and $\bar{\phi}$ represent the nucleon field, and $\sigma_{\mu\nu} = (1/2i) (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. At first, we consider the case of the wave formalism. We write f for an interaction which does not contain derivatives of the field variables and g for one which does. Then the total Lagrangian of the interacting system of the meson and the nucleon is given by

$$\mathcal{L}^{s(n)} = \mathcal{L}_N + \mathcal{L}_M^{s(n)} + \mathcal{L}_{\text{int}}^{s(n)}, \quad (21)$$

where \mathcal{L}_N is the Lagrangian of the free nucleon field and is given by

$$\mathcal{L}_N = -\bar{\phi}(\gamma\partial + M)\phi, \quad (22)$$

M is the mass of nucleon. \mathcal{L}_M is the Lagrangian of the free meson field and \mathcal{L}_{int} is that of the interaction of the both fields, and they are given by

$$\mathcal{L}_M^s = -(\partial_\mu u^* \partial_\mu u + \kappa^2 u^* u), \quad (23s)$$

$$\mathcal{L}_{\text{int}}^s = -f(u u^* + u^* u) - \frac{g}{\kappa} (j_\mu \partial_\mu u^* + j_\mu^* \partial_\mu u)$$

for the scalar meson, and

$$\mathcal{L}_M^v = -\left(\frac{1}{2} f_{\mu\nu}^* f_{\mu\nu} + \kappa^2 v_\mu^* v_\mu\right),$$

$$\mathcal{L}_{\text{int}}^v = -f(j_\mu v_\mu^* + j_\mu^* v_\mu) - \frac{g}{2\kappa} (m_{\mu\nu} f_{\mu\nu}^* + m_{\mu\nu}^* f_{\mu\nu}) \quad (23v)$$

for the vector meson.

In the β -formalism, considering the transformation properties, \mathcal{L}_M and \mathcal{L}_{int} are given by

$$\mathcal{L}_M = -\bar{\phi}(\beta\partial + \kappa)\phi, \quad (24)$$

$$\mathcal{L}_{\text{int}} = f_1 (M_{\mu\nu} \bar{\phi} \beta_\nu l'_\mu - M_{\mu\nu}^* l'^*_\mu \beta_\nu \phi) - f_2 (j_\mu \bar{\phi} l'_\mu + j_\mu^* l'^*_\mu \phi)$$

both for scalar and vector mesons. $M_{\mu\nu}$ are given by¹

$$M_{\mu\nu} = i\bar{\phi}\tau_{\mu\nu}\gamma_\nu\phi, \quad M_{\mu\nu}^* = i\bar{\phi}\tau_{\mu\nu}\gamma_\mu\phi, \quad (25)$$

and have the relations with the source densities (24) as follows:

$$M_{\mu\nu} = \begin{cases} -m_{\mu\nu} & \text{for } \mu \neq \nu \\ iw & \text{for } \mu = \nu \text{ (no summation),} \end{cases} \quad (26)$$

$$M_{\mu\nu}^* = \begin{cases} -m_{\mu\nu}^* & \text{for } \mu \neq \nu \\ iw^* & \text{for } \mu = \nu \text{ (no summation).} \end{cases}$$

Here we give some caution against (24). $l'^*_\mu \beta_\nu \phi$, $l'^*_\mu \phi$, etc., have the similar correspondences as (13) with the wave functions of Klein-Gordon or Proca equation with interaction; they are

$$\left. \begin{aligned} P_\mu^* \phi &= \frac{i}{\sqrt{\kappa}} u_\mu \\ \bar{\phi} P_\mu &= \frac{i}{\sqrt{\kappa}} u_\mu^* \\ P_\mu^* \beta_\nu \phi &= -i\sqrt{\kappa} u \delta_{\mu\nu} \\ \phi \beta_\nu P_\mu &= i\sqrt{\kappa} u^* \delta_{\mu\nu} \end{aligned} \right\}, \quad (27s)$$

$$\left. \begin{aligned} R_\mu^* \phi &= \sqrt{\kappa} v_\mu \\ \bar{\phi} R_\mu &= \sqrt{\kappa} v_\mu^* \\ R_\mu^* \beta_\nu \phi &= \frac{1}{\sqrt{\kappa}} h_{\mu\nu} \\ \bar{\phi} \beta_\nu R_\mu &= -\frac{1}{\sqrt{\kappa}} h_{\mu\nu}^* \end{aligned} \right\}. \quad (27v)$$

As it will soon be shown $u_\mu \neq \partial_\mu u$, $h_{\mu\nu} \neq f_{\mu\nu}$, etc., then \mathcal{L}_{int} (24) does not completely coincide with $\mathcal{L}_{\text{int}}^s$ (23s) or $\mathcal{L}_{\text{int}}^v$ (23v). Notwithstanding, following Harish-Chandra and Klein⁽⁶⁾ we take (24) as the Lagrangian. In order to have the same interaction terms, at least for the common ones in both formalisms, we must take

$$\left. \begin{aligned} f_1 &= -\frac{f}{4\sqrt{x}}, & f_2 &= \frac{g}{i\sqrt{x}} \\ M_{\mu\nu} &= i w \delta_{\mu\nu}, & M_{\mu\nu}^* &= i w^* \delta_{\mu\nu} \end{aligned} \right\} \quad (28s)$$

for the scalar meson, and

$$\left. \begin{aligned} f_1 &= \frac{g}{2\sqrt{x}}, & f_2 &= \frac{f}{\sqrt{x}} \\ M_{\mu\nu} &= -m_{\mu\nu}, & M_{\mu\nu}^* &= -m_{\mu\nu}^* \end{aligned} \right\} \quad (28v)$$

for the vector meson.

Summarizing the above, we may take as the total Lagrangian of the interacting system irrespective of the scalar or vector meson as follows:

$$\mathcal{L}_0 = -\bar{\phi}(\gamma\partial + M)\phi - \bar{\psi}(\beta\partial + x)\psi - f_1(M_{\mu\nu}\bar{\psi}\beta_\nu\Gamma_\mu - M_{\mu\nu}^*\Gamma_\mu^*\beta_\nu\psi) - f_2(j_\mu\bar{\psi}\Gamma_\mu + j_\mu^*\Gamma_\mu^*\psi). \quad (29)$$

The interaction Hamiltonian H_{int} of the interaction representation is deduced from (29) (cf. Appendix A):

$$\begin{aligned} H_{\text{int}} &= f_1(M_{\mu\nu}\bar{\psi}\beta_\nu\Gamma_\mu - M_{\mu\nu}^*\Gamma_\mu^*\beta_\nu\psi) + f_2(j_\mu\bar{\psi}\Gamma_\mu + j_\mu^*\Gamma_\mu^*\psi) \\ &+ \frac{f_1^2}{2x}(M_{\mu\nu}M_{\rho\sigma}^* + M_{\rho\sigma}^*M_{\mu\nu})\Gamma_\rho^*\beta_\sigma(1 + \beta_N^2)\beta_\nu\Gamma_\mu - \frac{f_2^2}{2x}(j_\mu j_\rho^* + j_\rho^*j_\mu)\Gamma_\rho^*(1 + \beta_N^2)\Gamma_\mu^*. \end{aligned} \quad (30)$$

Here $\beta_N = \beta_\mu n_\mu$, where n_μ is a time-like unit vector ($n_\mu^2 = -1$) normal to a space-like surface σ through the point x . Employing (6), we obtain

$$\left. \begin{aligned} P_\rho^*\beta_\sigma(1 + \beta_N^2)\beta_\nu P_\mu &= \delta_{\rho\sigma}\delta_{\mu\nu}(1 + n_\lambda^2) = 0 \\ P_\rho^*(1 + \beta_N^2)P_\mu &= \delta_{\rho\mu} + n_\mu n_\rho \end{aligned} \right\} \quad (31s)$$

for the scalar meson, and

$$\left. \begin{aligned} R_\rho^*\beta_\sigma(1 + \beta_N^2)\beta_\nu R_\mu &= \delta_{\rho\mu}\delta_{\sigma\nu} - \delta_{\rho\nu}\delta_{\sigma\mu} + n_\sigma n_\nu\delta_{\rho\mu} + n_\rho n_\mu\delta_{\sigma\nu} - n_\rho n_\nu\delta_{\sigma\mu} - n_\sigma n_\mu\delta_{\rho\nu} \\ R_\rho^*(1 + \beta_N^2)R_\mu &= \delta_{\rho\mu}(1 + n_\lambda^2) - n_\tau n_\tau\delta_{\rho\tau}\delta_{\mu\tau} = -n_\mu n_\rho \end{aligned} \right\} \quad (31v)$$

for the vector meson. Then, using (13), (30) is translated into the language of the wave formalism:

$$\begin{aligned} H_{\text{int}}^s &= f_1(M_{\mu\nu}\bar{\psi}\beta_\nu P_\mu - M_{\mu\nu}^*P_\mu^*\beta_\nu\psi) + f_2(j_\mu\bar{\psi}P_\mu + j_\mu^*P_\mu^*\psi) \\ &- \frac{f_2^2}{2x}(j_\mu j_\rho^* + j_\rho^*j_\mu)(\delta_{\rho\mu} + n_\mu n_\rho) \end{aligned}$$

$$\begin{aligned} \approx & f(\bar{\psi}\psi^* + \bar{\psi}\psi^*\psi) + \frac{g}{\kappa} (j_\mu \partial_\mu \psi^* + j_\mu^* \partial_\mu \psi) \\ & + \frac{g^2}{2\kappa^2} (j_\mu j_\mu^* + j_\mu^* j_\mu) + \frac{g^2}{2\kappa^2} (j_N j_N^* + j_N^* j_N) \end{aligned} \quad (32s)$$

for the scalar meson, and

$$\begin{aligned} H_{\text{int}} = & f_1 (M_{\mu\nu} \bar{\psi} \beta_\nu R_\mu - M_{\mu\nu}^* R_\mu^* \beta_\nu \psi) + f_2 (j_\mu \bar{\psi} R_\mu + j_\mu^* R_\mu^* \psi) \\ & + \frac{f_1^2}{2\kappa} (M_{\mu\nu} M_{\rho\sigma}^* + M_{\rho\sigma}^* M_{\mu\nu}) (\partial_{\rho\mu} \partial_{\sigma\nu} - \partial_{\rho\nu} \partial_{\sigma\mu} + n_\sigma n_\nu \partial_{\rho\mu} + n_\rho n_\mu \partial_{\sigma\nu} - n_\rho n_\nu \partial_{\sigma\mu} - n_\sigma n_\mu \partial_{\rho\nu}) \\ & + \frac{f_2^2}{2\kappa} (j_\mu j_\mu^* + j_\mu^* j_\mu) n_\mu n_\rho \\ \approx & \frac{g}{2\kappa} (m_{\mu\nu} f_{\mu\nu}^* + m_{\mu\nu}^* f_{\mu\nu}) + f (j_\mu \bar{\psi} \psi^* + j_\mu^* \bar{\psi} \psi) \\ & + \frac{g^2}{4\kappa^2} (m_{\mu\nu} m_{\mu\nu}^* + m_{\mu\nu}^* m_{\mu\nu}) + \frac{g^2}{2\kappa^2} (m_{\mu N} m_{\mu N}^* + m_{\mu N}^* m_{\mu N}) \\ & + \frac{f^2}{2\kappa^2} (j_N j_N^* + j_N^* j_N) \end{aligned} \quad (32v)$$

for the vector meson. Here

$$j_N = j_\mu n_\mu, \quad j_N^* = j_\mu^* n_\mu, \quad m_{\mu N} = m_{\mu\nu} n_\nu, \quad m_{\mu N}^* = m_{\mu\nu}^* n_\nu. \quad (33)$$

(32) differ from the ones which are deduced from (23) in the wave formalism in the normal independent direct couplings, $\frac{g^2}{2\kappa^2} (j_\mu j_\mu^* + j_\mu^* j_\mu)$ or $\frac{g^2}{4\kappa^2} (m_{\mu\nu} m_{\mu\nu}^* + m_{\mu\nu}^* m_{\mu\nu})$.

To avoid the troublesome normal dependent and independent direct terms, we follow Umezawa-Takahashi's rule. We calculate the S-matrix by taking

$$-L_{\text{int}}(x) = f_1 (M_{\mu\nu} \bar{\psi} \beta_\nu \Gamma_\mu - M_{\mu\nu}^* \Gamma_\mu^* \beta_\nu \psi) + f_2 (j_\mu \bar{\psi} \Gamma_\mu + j_\mu^* \Gamma_\mu^* \psi) \quad (34)$$

in place of $H_{\text{int}}(x)$, and the Green function $1/2 T_F(x-x')$ in place of the propagation function $\langle P(\bar{\psi}(x'), \psi(x)) \rangle$. The analysis as adopted in § 3 and § 4 of I is very simple in this case, since the elementary process which concerns with the meson fields is only nuclear force type (Fig. 1). We calculate this process for the various cases where the vertices A and B take scalar-, vector- or tensor-coupling in the β -formalism, and compare the results with that obtained in the wave formalism:

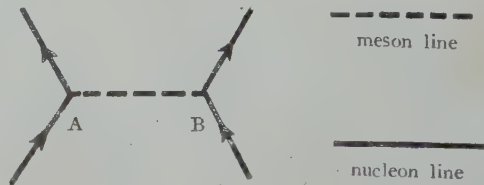


Fig. 1

(i) Scalar Meson

The S-matrix consists of the following line- and vertex-factors. (We may omit the nucleon factor for our purpose because they are same in both formalisms.)

	wave formalism	β -formalism
meson line	$\frac{1}{q^2 + x^2},$	$\frac{1}{x} \{ (i\beta q - x) P(i\beta q - x) + \bar{P}(q^2 + x^2) \} / (q^2 + x^2).$
S -vertex	$\left\{ \begin{array}{ll} S^* & f, \\ S & f, \end{array} \right.$	$\left\{ \begin{array}{ll} \frac{f}{4i\sqrt{x}} \beta_\mu P_\mu, \\ -\frac{f}{4i\sqrt{x}} P_\mu^* \beta_\mu. \end{array} \right.$
V -vertex	$\left\{ \begin{array}{ll} V^* & -i\frac{g}{2x} q_\mu, \\ V & i\frac{g}{2x} q_\mu, \end{array} \right.$	$\left\{ \begin{array}{ll} \frac{g}{i\sqrt{x}} P_\mu, \\ i\frac{g}{\sqrt{x}} P_\mu^*. \end{array} \right.$

(35s)

Here S^* means the scalar vertex with u^* (or $\bar{\psi}$), S with u (or ψ); V^* and V have similar meaning, P_μ^* means multiplication by P_μ^* from the left, and P_μ that by P_μ from the right, and so on. In the β -formalism, we calculate the nuclear force type elementary process for the various combinations of the vertices:

$$\begin{aligned}
 S-S^* &: \frac{1}{x} \left(\frac{f}{4i\sqrt{x}} \right) \left(\frac{-f}{4i\sqrt{x}} \right) P_\mu^* \beta_\mu \{ (i\beta q - x) P(i\beta q - x) + \bar{P}(q^2 + x^2) \} \beta_\nu P_\nu \\
 &= \frac{f^2}{16x^2} P_\mu^* \beta_\mu \left\{ (i\beta q - x) \frac{1}{4} \beta_\nu \bar{P} \beta_\nu (i\beta q - x) + \bar{P}(q^2 + x^2) \right\} \beta_\nu P_\nu \\
 &= \frac{f^2}{16x^2} \left\{ \frac{x^2}{4} \delta_{\mu\nu} P_\mu^* P_\nu P_\nu \delta_{\nu\nu} \right\} \\
 &= f^2,
 \end{aligned}
 \tag{36s}$$

$$S-V^*: \frac{1}{x} \left(\frac{-f}{4i\sqrt{x}} \right) \left(\frac{g}{i\sqrt{x}} \right) P_\mu^* \beta_\mu \{ (i\beta q - x) P(i\beta q - x) + \bar{P}(q^2 + x^2) \} P_\nu = -i \frac{fg}{x} q_\nu,$$

$$V-S^*: \frac{1}{x} \left(\frac{f}{4i\sqrt{x}} \right) \left(\frac{g}{i\sqrt{x}} \right) P_\mu^* \{ (i\beta q - x) P(i\beta q - x) + \bar{P}(q^2 + x^2) \} \beta_\nu P_\nu = i \frac{fg}{x} q_\mu,$$

$$V-V^*: \frac{1}{x} \left(\frac{g}{i\sqrt{x}} \right)^2 P_\mu^* \{ (i\beta q - x) P(i\beta q - x) + \bar{P}(q^2 + x^2) \} P_\nu = \frac{g^2}{x^2} [q_\mu q_\nu - \delta_{\mu\nu} (q^2 + x^2)],$$

For simplicity, the factor $(q^2 + x^2)^{-1}$ of the line factor is omitted in the above calculations. For $S-S^*$, $S-V^*$ and $V-S^*$ cases the results are the same as the ones obtained in the wave-formalism, but for $V-V^*$ case, the result differs from the one obtained in the wave formalism in the term with $(q^2 + x^2)$, the normal independent direct coupling term. This term is just the one that Klein pointed out.

(II) Vector Meson

In this case, the S -matrix consists of the following factors:

	wave formalism	β -formalism
meson line	$\frac{\delta_{\rho\sigma} + \kappa^{-2} q_\rho q_\sigma}{q^2 + \kappa^2}$	$\frac{1}{\kappa} \{ (i\beta q - \kappa) R (i\beta q - \kappa) + \bar{R} (q^2 + \kappa^2) + q_\rho q_\sigma R_{\rho\sigma} \} / (q^2 + \kappa^2)$
I -vertex	$\begin{cases} V^* & f, \\ V & f, \end{cases}$	$\begin{cases} \frac{f}{\sqrt{\kappa}} \cdot R_\mu, \\ \frac{f}{\sqrt{\kappa}} R_\mu^*. \end{cases}$
T -vertex	$\begin{cases} T^* & -i \frac{g}{2\kappa} (\delta_{\nu\rho} q_\mu - \delta_{\mu\rho} q_\nu), \\ T & i \frac{g}{2\kappa} (\delta_{\nu\rho} q_\mu - \delta_{\mu\rho} q_\nu), \end{cases}$	$\begin{cases} -\frac{g}{2\sqrt{\kappa}} \cdot \beta_\nu R_\mu, \\ \frac{g}{2\sqrt{\kappa}} R_\mu^* \beta_\nu. \end{cases}$

(35v)

The elementary processes give the following in the β -formalism :

$$\begin{aligned}
 V-V^* &: f^2 \left(\delta_{\mu\nu} + \frac{1}{\kappa^2} q_\mu q_\nu \right), \\
 V-T^* &: -i \frac{fg}{2\kappa} (q_\mu \delta_{\nu\rho} - q_\nu \delta_{\mu\rho}), \\
 T-V^* &: i \frac{fg}{2\kappa} (q_\mu \delta_{\nu\rho} - q_\nu \delta_{\mu\rho}), \\
 T-T^* &: \frac{g^2}{4\kappa^2} [(q_\nu q_\sigma \delta_{\mu\rho} + q_\mu q_\rho \delta_{\nu\sigma} - q_\mu q_\sigma \delta_{\nu\rho} - q_\nu q_\rho \delta_{\mu\sigma}) - (q^2 + \kappa^2) (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho})].
 \end{aligned}$$
(36v)

In this case, $V-V^*$, $V-T^*$ and $T-V^*$ give the same results as the ones in the wave formalism, but $T-T^*$ gives an additional normal independent direct coupling, too.

As shown by the above calculations, the S -matrix given in the β -formalism contains the redundant direct coupling. Klein concluded that this term canceled out by the same term of his interaction Hamiltonian [(A.10) Klein] and the final results were not affected by this term. But his argument has some questionable points, which we will discuss in the next section.

§ 4. Discussion on the normal independent direct coupling

As has been shown in the previous section, the S -matrices calculated by both formalisms differ in the normal independent direct coupling. This difference is caused by any of the following three reasons :

- (a) As Klein stated, the effects of this term are crossed out by considering the whole of the interaction Hamiltonian and then the results coincide in both formalisms.
- (b) In this problem, Umezawa-Takahashi's method can not be applied.
- (c) We treat the different systems from the beginning and it is a matter of course that they differ from each other.

Now we discuss these causes one by one. At first, we must know the interaction Hamiltonian. The system under consideration has the Lagrangian (29). To examine the case (b), we must obtain the interaction Hamiltonian H_{int} by Umezawa-Takahashi's method, but it is obtainable also by Matthew's method⁷⁾ and coincide with the above. Then we calculate by Matthews's method because it is more convenient for examining the case (a). Details of the calculation are given in Appendix A. According to Appendix A, \mathcal{H}_{int} the interaction part of the total Hamiltonian, is given by

$$\begin{aligned} \mathcal{H}_{\text{int}} = & f_1 (\mathbf{M}_{\mu\nu} \bar{\psi}_i \beta_\nu \Gamma_\mu - \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu \psi) + f_2 (\mathbf{j}_\mu \bar{\psi} \Gamma_\mu' + \mathbf{j}_\mu^* \Gamma_\mu'^* \psi) \\ & - \frac{f_1^2}{2\kappa} (\mathbf{M}_{\mu\nu} \mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^* \mathbf{M}_{\mu\nu}) \Gamma_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu \Gamma_\mu' \\ & + \frac{f_2^2}{2\kappa} (\mathbf{j}_\mu \mathbf{j}_\rho^* + \mathbf{j}_\rho^* \mathbf{j}_\mu) \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu. \end{aligned} \quad (37)$$

On the other hand, H_{int} , the interaction Hamiltonian of the interaction representation, is given by

$$\begin{aligned} H_{\text{int}} = & f_1 (\mathbf{M}_{\mu\nu} \bar{\psi}_i \beta_\nu \Gamma_\mu' - \mathbf{M}_{\mu\nu}^* \Gamma_\mu'^* \beta_\nu \psi) + f_2 (\mathbf{j}_\mu \bar{\psi} \Gamma_\mu' + \mathbf{j}_\mu^* \Gamma_\mu'^* \psi) \\ & + \frac{f_1^2}{2\kappa} (\mathbf{M}_{\mu\nu} \mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^* \mathbf{M}_{\mu\nu}) \Gamma_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu \Gamma_\mu' \\ & - \frac{f_2^2}{2\kappa} (\mathbf{j}_\mu \mathbf{j}_\rho^* + \mathbf{j}_\rho^* \mathbf{j}_\mu) \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu. \end{aligned} \quad (38)$$

We first argue the case (a). Klein pointed out that the normal independent direct coupling appears if $-\mathcal{L}_{\text{int}}^{\text{I}}$ alone is taken into account, but this term is crossed out if the whole \mathcal{H}_{int} is taken into account. (Although Klein calculated the case of the charge symmetrical meson in the Heisenberg representation, the essential point of our discussions are not affected by considering the charged meson.) Following his recipe, in the cases of B. Scalar Coupling and C. Pseudovector Coupling (following his notation, § VII) these direct coupling terms are actually crossed out, but in the case of A. Pseudoscalar Coupling the direct coupling appears anew. The argument that β_0 -term of his \mathcal{H}_{int} [(A.10) Klein] contributes only when $\beta_0(\beta_N$ for the flat surface σ) has the eigenvalue zero, hence we need not consider this term, is rather questionable. Therefore next we consider this term also. Following this recipe, in the cases A and B the terms in question disappear, but in the case C the direct coupling appears again. Thus following either recipe we can not wipe out the terms in question in all cases simultaneously. More importantly, he dropped a factor 1/2 in front of $\mathcal{H}_{\text{int}}^{\text{II}}$ [(A.10) Klein].^{††} If the correct \mathcal{H}_{int} is used, such cancellation does

[†] The first line of \mathcal{H}_{int} (37). Hereafter we write this part of \mathcal{H}_{int} as $\mathcal{H}_{\text{int}}^{\text{I}}$ and the remaining part as $\mathcal{H}_{\text{int}}^{\text{II}}$. Similar notations are used for H_{int} , too.

^{††} Compare with (74). The omission of a factor 1/2 is due to the omission of a factor 1/2 in \mathcal{L}_M , (70). In charge symmetrical theory, there is only one canonical variable ψ , and $\bar{\psi}$ is not canonically independent of ψ . So \mathcal{L}_M must be multiplied by a factor 1/2. Klein's misunderstanding is due to the omission of the factor 1/2 on the one hand, and to the sign difference of $\mathcal{H}_{\text{int}}^{\text{II}}$ from that of $H_{\text{int}}^{\text{II}}$ (cf. Appendix A) on the other hand.

not occur in any case following either recipe. As will soon be shown in the interaction representation, it is a correct procedure to take only $-\mathcal{L}_{int}$ of \mathcal{H}_{int} in our calculation, so the direct coupling is not wiped out by Klein's recipe.

Next, we examine the case (b). It is only necessary to repeat the Kinoshita's discussion (pp. 478-479). In the calculation of S-matrix by Dyson's method, we must use a vacuum expectation value of $P(\bar{\psi}(x'), \psi(x))$ as a propagation function of meson. By using the commutation relations (16), we get

$$\langle P(\bar{\psi}(x'), \psi(x)) \rangle_0 = -\frac{1}{2} \left\{ \beta \partial - \frac{1}{x} (\beta \partial)^2 \right\} \mathcal{A}_F(x-x') + \frac{i}{x} \beta_N^2 \delta(x-x'). \quad (39)$$

Employing

$$(\square - x^2) \mathcal{A}_F(x) = 2i\delta(x), \quad (40)$$

(39) can be rewritten as

$$\begin{aligned} \langle P(\bar{\psi}(x'), \psi(x)) \rangle_0 = & -\frac{1}{2} \left\{ \beta \partial - \frac{1}{x} (\beta \partial)^2 + \frac{1}{x} (\square - x^2) \right\} \mathcal{A}_F(x-x') \\ & + \frac{i}{x} (1 + \beta_N^2) \delta(x-x'). \end{aligned} \quad (41)$$

The first term is just the same as the Green function $T_F(x-x')/2$. The second term has the surface dependency with the factor $(1 + \beta_N^2)$, which is the same form as the ones of H_{int}'' . The contribution of each surface dependency just cancels out with each other in any order of expansion of the S-matrix. Thus, Umezawa-Takahashi's simplified rule is correct in our problem too.

Lastly, we examine the case (c). For this purpose, we only need to compare the Lagrangians (23) and (24). In order to translate (24) into the language of the wave formalism, we adopt the correspondence (27). In this case, all field variables are Heisenberg ones, hence they satisfy the field equations not the free ones (1) or (5) but (A.5) or (A.6). Multiplying (A.5) by P_μ^* (or P_μ) from the left (or right), we get

$$u_\mu \approx \partial_\mu u + \frac{g}{x} j_\mu, \quad u_\mu^* \approx \partial_\mu u^* + \frac{g}{x} j_\mu^*. \quad (42)$$

Similarly, multiplying (A.5) by $R_\mu^* \beta_\nu$ (or $\beta_\nu R_\mu$) from the left (or right), we get

$$h_{\mu\nu} \approx f_{\mu\nu} + \frac{g}{x} m_{\mu\nu}, \quad h_{\mu\nu}^* \approx f_{\mu\nu}^* + \frac{g}{x} m_{\mu\nu}^*. \quad (43)$$

Compare with the correspondence (13), the additional second term appears in each case. By the similar procedure as adopted in I (I.40), we get

$$\left. \begin{aligned} \phi^s & \approx -\frac{1}{x} (\beta \partial - x) P \phi - \frac{g}{i \sqrt{x^3}} j_\mu P_\mu \\ \bar{\psi} & \approx \frac{1}{x} P \psi (\beta \partial + x) - \frac{g}{i \sqrt{x^3}} j_\mu^* P_\mu^* \end{aligned} \right\}, \quad (44s)$$

$$\left. \begin{aligned} \phi^v &\approx -\frac{1}{x}(\beta\partial - x)R\phi + \frac{g}{2\sqrt{x^3}}m_{\mu\nu}\beta_\nu R_\mu \\ \bar{\phi}^v &\approx \frac{1}{x}\bar{\phi}R(\beta\partial + x) - \frac{g}{2\sqrt{x^3}}m_{\mu\nu}^*R_\mu^*\beta_\nu \end{aligned} \right\}. \quad (44v)$$

Using (42), (43) and (44), we can easily translate (24) into the language of the wave formalism :

$$\begin{aligned} \mathcal{L}^s &\approx -(\partial_\mu u^* \partial_\mu u + x^2 u^* u) - f(wu^* + u^* w) \\ &\quad - \frac{g}{x}(\mathbf{j}_\mu \partial_\mu u^* + \mathbf{j}_\mu^* \partial_\mu u) - \frac{g^2}{2x^2}(\mathbf{j}_\mu \mathbf{j}_\mu^* + \mathbf{j}_\mu^* \mathbf{j}_\mu), \end{aligned} \quad (45s)$$

$$\begin{aligned} \mathcal{L}^v &\approx -\left(\frac{1}{2}f_{\mu\nu}^* f_{\mu\nu} + x^2 v_\mu^* v_\mu\right) - f(\mathbf{j}_\mu v_\mu^* + \mathbf{j}_\mu^* v_\mu) - \frac{g}{2x}(m_{\mu\nu} f_{\mu\nu}^* + m_{\mu\nu}^* f_{\mu\nu}) \\ &\quad - \frac{g^2}{4x^2}(m_{\mu\nu} m_{\mu\nu}^* + m_{\mu\nu}^* m_{\mu\nu}). \end{aligned} \quad (45v)$$

The last terms of (45) are the normal independent direct couplings, which are not considered usually. If we start from the Lagrangian (45), and calculate throughout the wave language, we obtain the interaction Hamiltonian which contains the normal independent direct couplings (cf. Miyamoto⁸⁾, the case of vector meson), and coincides with the interaction Hamiltonian which is translated from (38). Therefore, in such a case the same normal independent direct couplings appear certainly in the S-matrix even if we use the wave formalism. That is, when we start from the Lagrangian (29), we treat the different system in the normal independent direct couplings from the usual one, and the difference appearing in the S-matrix is rather natural.

After all, we conclude as follows; we can also apply Umezawa-Takahashi's method in our system, and the direct couplings appear because our system differs from the usual one in such terms. If we want to do without the direct couplings, we must add from the outset the counter terms $\frac{g^2}{2x^2}(\mathbf{j}_\mu \mathbf{j}_\mu^* + \mathbf{j}_\mu^* \mathbf{j}_\mu)$ for the scalar case or $\frac{g^2}{4x^2}(m_{\mu\nu} m_{\mu\nu}^* + m_{\mu\nu}^* m_{\mu\nu})$ for the vector case. But these counter terms can not be introduced into the formalism as a unified term for both kinds of meson, and the unified description hitherto is destroyed.

§ 5. The case of the neutral meson and the charge symmetrical meson

Up to this place, the charged meson has been treated. In this section, we extend our theory to the neutral or charge symmetrical meson.

(i) Neutral Meson

In the case of a neutral meson, we must impose the additional conditions

$$I_\mu^* \phi = \bar{\phi} \Gamma_\mu, \quad I_\mu^* \beta_\nu \phi = -\bar{\phi} \beta_\nu \Gamma_\mu, \quad (46)$$

both in the scalar and vector mesons. From the correspondence (13), the meaning of

these conditions is easily understood. Also for this case both τ_+ and τ_- are to be replaced by $1/2$. In this case, there is only one canonical variable ϕ , accordingly $\bar{\psi}$ must be eliminated from the formalism. For this purpose, we first express $\bar{\psi}$ by ϕ^T , where the suffix T means the "transposed". As this is done by Harish-Chandra, we describe it briefly here. On taking the transposed of the commutation relations (2)

$$\beta_{\lambda l}^T \beta_{\mu}^T \beta_{\nu}^T + \beta_{\nu}^T \beta_{\mu}^T \beta_{\lambda}^T = \delta_{\mu\nu} \beta_{\lambda}^T + \delta_{\mu\lambda} \beta_{\nu}^T, \quad (47)$$

and reversing the sign, it is obvious that the matrices $-\beta_{\mu}^T$ satisfy the same commutation relations as β_{μ} , so that $-\beta_{\mu}^T$ are transformed from β_{μ} using the non-singular matrix θ as follows:

$$-\beta_{\mu}^T = \theta \beta_{\mu} \theta^{-1}. \quad (48)$$

From (48) and the relations satisfied by Γ_{μ} and Γ_{μ}^* , it is easy to prove that

$$\theta^T = \theta \quad (49)$$

and

$$\Gamma_{\mu}^* = \Gamma_{\mu}^T \theta, \quad \Gamma_{\mu} = \theta^{-1} \Gamma_{\mu}^{*T}. \quad (50)$$

From (46) and (50) we obtain the following relations between $\bar{\psi}$ and ϕ^T :

$$\bar{\psi} = \phi^T \theta. \quad (51)$$

The same is true for the Heisenberg variables, too. Thus in this case, there is the correspondence

$$\begin{aligned} \Gamma_{\mu}^* \phi &= \phi^T \Gamma_{\mu}^{*T} = \frac{i}{\sqrt{x}} u_{\mu} \quad (\text{for scalar}) \quad \text{or} \quad \sqrt{x} v_{\mu} \quad (\text{for vector}), \\ \Gamma_{\mu}^* \beta_{\nu} \phi &= \phi^T \beta_{\nu}^T \Gamma_{\mu}^{*T} = \frac{\sqrt{x}}{i} \delta_{\mu\nu} u \quad (\text{for scalar}) \quad \text{or} \quad \frac{1}{\sqrt{x}} h_{\mu\nu} \quad (\text{for vector}). \end{aligned} \quad (52)$$

As the source functions, we take in this case

$$w^0 = \bar{\phi} \phi, \quad j_{\mu}^0 = i \bar{\phi} \gamma_{\mu} \phi, \quad m_{\mu\nu}^0 = \bar{\phi} \sigma_{\mu\nu} \phi, \quad M_{\mu\nu}^0 = i \bar{\phi} \gamma_{\mu} \gamma_{\nu} \phi, \quad (53)$$

then we put as the Lagrangian

$$\mathcal{L} = -\frac{1}{2} \phi^T \theta (\beta \partial + x) \phi + f_1 M_{\mu\nu}^0 \Gamma_{\mu}^* \beta_{\nu} \phi - f_2 j_{\mu}^0 \Gamma_{\mu}^* \phi. \quad (54)$$

The factor $1/2$ is introduced because the first term is bilinear form of only one canonical variable ϕ . We must pay attention to the fact that the canonical variable is ϕ alone, in all the calculations. For example, the equations of motion are deduced as follows:[†]

[†] For convenience, the tensor suffices μ, ν are written properly at the upper-right. α, β, γ and δ designate the spinor suffices.

$$\mathcal{L} = -\frac{1}{2} (\phi_\alpha \theta_{\alpha\tau} \beta_{\tau\delta}^\mu \partial_\mu \phi_\delta + x \phi_\alpha \theta_{\alpha\delta} \phi_\delta) + f_1 \bar{M}_{\mu\nu}^0 l_\alpha^{*\mu} l_\delta^\nu \beta_{\alpha\delta} \phi_\delta - f_2 \bar{j}_\mu^0 l_\delta^{*\mu} \phi_\delta.$$

Then we have

$$\begin{aligned} -\frac{\partial}{\partial x_\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_\delta)} \right) + \frac{\partial \mathcal{L}}{\partial \phi_\delta} &= 0, \\ \frac{1}{2} \partial_\mu \phi_\alpha \theta_{\alpha\tau} \beta_{\tau\delta}^\mu - \frac{1}{2} \theta_{\delta\tau} \beta_{\tau\alpha}^\mu \partial_\mu \phi_\alpha - \frac{x}{2} (\phi_\alpha \theta_{\alpha\delta} + \theta_{\delta\alpha} \phi_\alpha) + f_1 \bar{M}_{\mu\nu}^0 l_\alpha^{*\mu} \beta_{\alpha\delta}^\nu - f_2 \bar{j}_\mu^0 l_\delta^{*\mu} &= 0, \\ \frac{1}{2} \beta_\mu^T \theta^T \partial_\mu \phi - \frac{1}{2} \theta \beta_\mu \partial_\mu \phi - \frac{x}{2} (\theta^T \phi + \theta \phi) &= -f_1 \bar{M}_{\mu\nu}^0 \beta_\nu^T l_\mu^{*T} + f_2 \bar{j}_\mu^0 l_\mu^{*T}, \\ -\theta (\beta \partial + x) \phi &= f_1 \bar{M}_{\mu\nu}^0 \theta l_\nu \Gamma_\mu + f_2 \bar{j}_\mu^0 \theta l_\mu, \\ (\beta \partial + x) \phi &= -f_1 \bar{M}_{\mu\nu}^0 \beta_\nu \Gamma_\mu - f_2 \bar{j}_\mu^0 \Gamma_\mu. \end{aligned} \quad (55)$$

Similarly we obtain

$$\phi^x \theta (\beta \partial - x) = -f_1 \bar{M}_{\mu\nu}^0 \Gamma_\mu^* \beta_\nu + f_2 \bar{j}_\mu^0 \Gamma_\mu^*. \quad (56)$$

The commutation relations and the Green function are given by†

$$[\phi_\rho(x), \phi'_\sigma(x')] = \frac{1}{i} \left[\left\{ \beta \partial - \frac{1}{x} (\beta \partial)^2 \right\} \theta^{-1} \right]_{\rho\sigma} \Delta(x-x'), \quad (57)$$

$$\frac{1}{2} T_F^0(x) = -\frac{1}{2} \left\{ \beta \partial - \frac{1}{x} (\beta \partial)^2 + \frac{1}{x} (\square - x^2) \right\} \theta^{-1} \Delta_F(x). \quad (58)$$

Also the interaction Hamiltonians are given by

$$\begin{aligned} \mathcal{H}_{\text{int}} &= -f_1 \bar{M}_{\mu\nu}^0 l_\mu^{*\nu} \beta_\nu \phi + f_2 \bar{j}_\mu^0 \Gamma_\mu^* \phi - \frac{f_1^2}{2x} \bar{M}_{\mu\nu}^0 \bar{M}_{\rho\sigma}^0 l_\rho^{*\nu} l_\sigma^\nu (1 + \beta_N^2) \beta_\nu l_\mu' \\ &\quad + \frac{f_2^2}{2x} \bar{j}_\mu^0 \bar{j}_\rho^0 \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu, \end{aligned} \quad (59)$$

$$\begin{aligned} H_{\text{int}} &= -f_1 \bar{M}_{\mu\nu}^0 \Gamma_\mu^* \beta_\nu \phi + f_2 \bar{j}_\mu^0 \Gamma_\mu^* \phi + \frac{f_1^2}{2x} \bar{M}_{\mu\nu}^0 \bar{M}_{\rho\sigma}^0 \Gamma_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu l_\mu' \\ &\quad - \frac{f_2^2}{2x} \bar{j}_\mu^0 \bar{j}_\rho^0 \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu. \end{aligned} \quad (60)$$

The circumstances about the S-matrix are similar for the case of charged meson.

(ii) Charge Symmetrical Meson

To extend our formalism to charge symmetrical form, we introduce a decomposition of ϕ and $\bar{\phi}$ into two neutral field ϕ_1 and ϕ_2 , which is the immediate generalization of the decomposition of a complex wave field into two real field. Here we call such field

† If we want to obtain (57) and (58) by Umezawa-Takahashi's method, we must notice that the field equation obtained from the free Lagrangian is not $(\beta \partial + x) \phi = 0$, but $-\theta(\beta \partial + x) \phi = 0$.

that satisfies (46) and (51) the "neutral" field. We tentatively decompose

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \quad \bar{\phi} = \frac{1}{\sqrt{2}}(\bar{\phi}_1 + i\bar{\phi}_2). \quad (61)$$

The inverse of (61) is

$$\phi_1 = \frac{1}{\sqrt{2}}(\phi + \theta^{-1}\bar{\phi}^T), \quad \phi_2 = \frac{i}{\sqrt{2}}(\phi - \theta^{-1}\bar{\phi}^T). \quad (62)$$

The isotopic-spin matrices τ_1 , τ_2 and τ_3 make a three-dimensional vector[†] $\underline{\tau} = (\tau_1, \tau_2, \tau_3)$ in charge space and τ_+ and τ_- are constructed from $\underline{\tau}$ as follows:

$$\tau_+ = \frac{1}{2}(\tau_1 + i\tau_2), \quad \tau_- = \frac{1}{2}(\tau_1 - i\tau_2). \quad (63)$$

Noticing that ϕ_1 and ϕ_2 satisfy (46) and (51), we can rewrite

$$\begin{aligned} & -f_1(\underline{M}_{\mu\nu}\bar{\phi}_\nu\Gamma_\mu - \underline{M}_{\mu\nu}^*\Gamma_\mu^*\beta_\nu\phi) - f_2(\underline{j}_\mu\bar{\phi}\Gamma_\mu + \underline{j}_\mu^*\Gamma_\mu^*\phi) \\ & = -\frac{f_1}{\sqrt{2}}(\underline{M}_{\mu\nu}^{(1)}\Gamma_\mu^*\beta_\nu\phi_1 + \underline{M}_{\mu\nu}^{(2)}\Gamma_\mu^*\beta_\nu\phi_2) - \frac{f_2}{\sqrt{2}}(\underline{j}_\mu^{(1)}\Gamma_\mu^*\phi_1 + \underline{j}_\mu^{(2)}\Gamma_\mu^*\phi_2), \end{aligned} \quad (64)$$

where

$$\underline{M}_{\mu\nu}^{(i)} = i\bar{\phi}\tau_i\gamma_\mu\gamma_\nu\phi, \quad \underline{j}_\mu^{(i)} = i\bar{\phi}\tau_i\gamma_\mu\phi \quad (i=1, 2). \quad (65)$$

Now, we introduce a third neutral field ϕ_3 and propose the interaction

$$f_3\underline{M}_{\mu\nu}^{(3)}\Gamma_\mu^*\beta_\nu\phi_3 - f_3\underline{j}_\mu^{(3)}\Gamma_\mu^*\phi_3 \quad (66)$$

where $\underline{M}_{\mu\nu}^{(3)}$ and $\underline{j}_\mu^{(3)}$ are given by putting $\tau_i \rightarrow \tau_3$ in (65). Multiplying (64) by $\sqrt{2}$ and adding (66), we obtain

$$\mathcal{L}_{\text{int}} = f_1\underline{M}_{\mu\nu}\Gamma_\mu^*\beta_\nu\phi - f_2\underline{j}_\mu\Gamma_\mu^*\phi, \quad (67)$$

as the interaction Lagrangian in the symmetrical theory. Here we consider that $\underline{M}_{\mu\nu}$, \underline{j}_μ and $\underline{\phi}$ are considered as vectors in charge space, that is

$$\underline{M}_{\mu\nu} = i\bar{\phi}\underline{\tau}\gamma_\mu\gamma_\nu\phi, \quad \underline{j}_\mu = i\bar{\phi}\underline{\tau}\gamma_\mu\phi \quad (68)$$

and

$$\underline{\phi} = (\phi_1, \phi_2, \phi_3). \quad (69)$$

The free meson part of the Lagrangian is given by

$$\mathcal{L}_M = -\frac{1}{2}\phi^T\theta(\beta\partial + \kappa)\phi. \quad (70)$$

Summarizing the above, we may take as the total Lagrangian of our system

[†] We denote by underlines a three dimensional vector in charge space in this paper.

$$\mathcal{L} = -\bar{\psi}(\gamma\partial + M)\psi - \frac{1}{2}\phi^T\theta(\beta\partial + \kappa)\phi + f_1\underline{M}_{\mu\nu}\Gamma_\mu^*\beta_\nu\phi - f_2\underline{j}_\mu\Gamma_\mu^*\phi. \quad (71)$$

The commutation relations, the Green function and the interaction Hamiltonians are given by

$$[\phi_{ip}(x), \phi_{j\sigma}(x')] = \frac{1}{i} \left[\left\{ \beta\partial - \frac{1}{\kappa}(\beta\partial)^2 \right\} \theta^{-1} \right]_{\rho\sigma} \delta_{ij} \Delta(x-x'), \quad (72)$$

$$\frac{1}{2}T_F^{\text{sym}}(x) = -\frac{1}{2} \left\{ \beta\partial - \frac{1}{\kappa}(\beta\partial)^2 + \frac{1}{\kappa}(\square - \kappa^2) \right\} \theta^{-1} \delta_{ij} \Delta_F(x), \quad (73)$$

$$\begin{aligned} \mathcal{H}_{\text{int}} = & f_1 \underline{M}_{\mu\nu} \Gamma_\mu^* \beta_\nu \phi + f_2 \underline{j}_\mu \Gamma_\mu^* \phi - \frac{f_1^2}{2\kappa} \underline{M}_{\mu\nu} \underline{M}_{\rho\sigma} \Gamma_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu \Gamma_\mu \\ & + \frac{f_2^2}{2\kappa} \underline{j}_\mu \underline{j}_\rho \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu, \end{aligned} \quad (74)$$

$$\begin{aligned} H_{\text{int}} = & -f_1 \underline{M}_{\mu\nu} \Gamma_\mu^* \beta_\nu \psi + f_2 \underline{j}_\mu \Gamma_\mu^* \psi + \frac{f_1^2}{2\kappa} \underline{M}_{\mu\nu} \underline{M}_{\rho\sigma} \Gamma_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu \Gamma_\mu \\ & - \frac{f_2^2}{2\kappa} \underline{j}_\mu \underline{j}_\rho \Gamma_\rho^* (1 + \beta_N^2) \Gamma_\mu, \end{aligned} \quad (75)$$

where suffices i and j designate the components of the vector in charge space. In (74), we must pay attention to the fact that the last two terms differ from the ones of Klein (A.10) in a factor $1/2$ (cf. Footnote on page 147).

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Appendix A. The calculation of \mathcal{H}_{int} and H_{int}

In this appendix we calculate \mathcal{H}_{int} (37) and H_{int} (30) from the total Lagrangian (29) by Matthew's method.⁷⁾ As the interaction Lagrangian contains no derivatives of the nucleon variables and the canonical conjugates of the nucleon variables are not altered by the interaction, we do not need to consider the free nucleon Lagrangian \mathcal{L}_N in order to evaluate \mathcal{H}_{int} and H_{int} . Thus in this case, we need merely to eliminate the dependent components of the meson variables ϕ and $\bar{\psi}$. We take as a Lagrangian

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_M + \mathcal{L}_{\text{int}} \\ = & -\bar{\psi}(\beta\partial + \kappa)\psi - f_1(\underline{M}_{\mu\nu}\bar{\psi}_i\beta_\nu\Gamma_\mu^* - \underline{M}_{\mu\nu}^*\Gamma_\mu^*\beta_\nu\psi) - f_2(\underline{j}_\mu\bar{\psi}\Gamma_\mu^* + \underline{j}_\mu^*\Gamma_\mu^*\psi). \end{aligned} \quad (\text{A.1})$$

The energy-momentum tensor $U_{\mu\nu}$ is defined in the usual way,

$$U_{\mu\nu} = \mathcal{L} \delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi. \quad (\text{A.2})$$

The canonical conjugate π of ϕ can be defined by

$$\pi = - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} n_\mu = \bar{\psi} \beta_N, \quad (\text{A.3})$$

where n_μ is a time-like unit vector ($n_\mu^2 = -1$) normal to a space-like surface σ through the point x . Then the Hamiltonian is defined by

$$\mathcal{H} = U_{\mu\nu} n_\mu n_\nu = -\mathcal{L} + \bar{\psi} \beta_N \partial_N \phi, \quad (\text{A.4})$$

where $\partial_N = \partial_\mu n_\mu$. The components of ϕ are not all independent. The independent components of ϕ are $-\beta_N^2 \phi$ and the dependent components $(1 + \beta_N^2) \phi$, and this separation depends on the orientation of the surface σ at the point x . The dependent components must be expressed by the independent components by use of the initial conditions. The initial conditions are obtained as follows: The equation of motion with interaction is given by the variation of the Lagrangian (A.1),

$$(\beta \partial + x) \phi = -f_1 \mathbf{M}_{\mu\nu} \beta_\nu \Gamma_\mu^* - f_2 \mathbf{j}_\mu \Gamma_\mu^*, \quad (\text{A.5})$$

$$\bar{\psi} (\beta \partial - x) = -f_1 \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu + f_2 \mathbf{j}_\mu^* \Gamma_\mu^*. \quad (\text{A.6})$$

Multiplying (A.5) by $(1 + \beta_N^2)$ from the left and changing terms properly, we obtain

$$\begin{aligned} (1 + \beta_N^2) \phi &= - \frac{1}{x} [(1 + \beta_N^2) (\beta \partial) \phi + f_1 \mathbf{M}_{\mu\nu} (1 + \beta_N^2) \beta_\nu \Gamma_\mu^* + f_2 \mathbf{j}_\mu (1 + \beta_N^2) \Gamma_\mu^*] \\ &= - \frac{1}{x} [\beta_N \partial_N \phi - (\beta \partial) \beta_N^2 \phi + f_1 \mathbf{M}_{\mu\nu} (1 + \beta_N^2) \beta_\nu \Gamma_\mu^* + f_2 \mathbf{j}_\mu (1 + \beta_N^2) \Gamma_\mu^*]. \end{aligned} \quad (\text{A.7})$$

Similarly, we obtain

$$\begin{aligned} \bar{\psi} (1 + \beta_N^2) &= \frac{1}{x} [\bar{\psi} (\beta \partial) (1 + \beta_N^2) + f_1 \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu (1 + \beta_N^2) - f_2 \mathbf{j}_\mu^* \Gamma_\mu^* (1 + \beta_N^2)] \\ &= \frac{1}{x} [\bar{\psi} \beta_N \partial - \bar{\psi} \beta_N^2 (\beta \partial) + f_1 \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu (1 + \beta_N^2) - f_2 \mathbf{j}_\mu^* \Gamma_\mu^* (1 + \beta_N^2)]. \end{aligned} \quad (\text{A.8})$$

These are the required initial conditions. Applying these initial conditions, we can eliminate the dependent components from the Hamiltonian (A.4).

$$\begin{aligned} \mathcal{H} &= \bar{\psi} (\beta \partial + x) \phi + f_1 (\mathbf{M}_{\mu\nu} \bar{\psi} \beta_\nu \Gamma_\mu^* - \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu \phi) + f_2 (\mathbf{j}_\mu \bar{\psi} \Gamma_\mu^* + \mathbf{j}_\mu^* \Gamma_\mu^* \phi) + \bar{\psi} \beta_N \partial_N \phi \\ &= -x \bar{\psi} \beta_N^2 \phi + \bar{\psi} (\beta \partial) \phi + f_1 (\dots) + f_2 (\dots) + \bar{\psi} \beta_N \partial_N \phi \\ &\quad + \frac{1}{2} [\bar{\psi} \beta_N \partial_N \phi - \bar{\psi} \beta_N^2 (\beta \partial) \phi + f_1 \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu (1 + \beta_N^2) \phi - f_2 \mathbf{j}_\mu^* \Gamma_\mu^* (1 + \beta_N^2) \phi] \\ &\quad - \frac{1}{2} [\bar{\psi} \beta_N \partial_N \phi - \bar{\psi} (\beta \partial) \beta_N^2 \phi + f_1 \mathbf{M}_{\mu\nu} \bar{\psi} (1 + \beta_N^2) \beta_\nu \Gamma_\mu^* + f_2 \mathbf{j}_\mu \bar{\psi} (1 + \beta_N^2) \Gamma_\mu^*] \end{aligned}$$

$$\begin{aligned}
&= -x\bar{\psi}\beta_N^2\phi + f_1(\dots) + f_2(\dots) \\
&+ \frac{1}{2x} [\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}^*I_\mu^*\beta_\nu(1+\beta_N^2)(\beta\partial)\phi - f_2j_\mu^*I_\mu^*(1+\beta_N^2)(\beta\partial)\phi] \\
&- \frac{1}{2x} [\bar{\psi}(\beta\partial)(1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}\bar{\mathbf{f}}(\beta\partial)(1+\beta_N^2)\beta_\nu I'_\mu + f_2j_\mu\bar{\mathbf{f}}(\beta\partial)(1+\beta_N^2)I'_\mu] \\
&- \frac{1}{2x} f_1\mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma[(1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}(1+\beta_N^2)\beta_\nu\Gamma_\mu + f_2j_\mu^*(1+\beta_N^2)\Gamma_\mu] \\
&+ \frac{1}{2x} f_2j_\rho^*\Gamma_\rho^*[(1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}(1+\beta_N^2)\beta_\nu\Gamma_\mu + f_2j_\mu(1+\beta_N^2)\Gamma_\mu] \\
&- \frac{1}{2x} f_1\mathbf{M}_{\mu\nu}[\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2) + f_1\mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma(1+\beta_N^2) - f_2j_\rho^*\Gamma_\rho^*(1+\beta_N^2)]\beta_\nu\Gamma_\mu \\
&- \frac{1}{2x} f_2j_\mu[\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2) + f_1\mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma(1+\beta_N^2) - f_2j_\rho^*\Gamma_\rho^*(1+\beta_N^2)]\Gamma_\mu \\
&= -x\bar{\psi}\beta_N^2\phi + \frac{1}{x}\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2)(\beta\partial)\phi \\
&- \frac{f_1^2}{2x} (\mathbf{M}_{\mu\nu}\mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^*\mathbf{M}_{\mu\nu})\Gamma_\rho^*\beta_\sigma(1+\beta_N^2)\beta_\nu I'_\mu + \frac{f_2^2}{2x} (j_\mu j_\rho^* + j_\rho^* j_\mu)\Gamma_\rho^*(1+\beta_N^2)\Gamma_\mu \\
&- f_1(\mathbf{M}_{\mu\nu}\bar{\mathbf{f}}\beta_N^2\beta_\nu\Gamma_\mu - \mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma\beta_N^2\phi) - f_2(j_\mu\bar{\mathbf{f}}\beta_N^2I'_\mu + j_\rho^*\Gamma_\rho^*\beta_N^2\phi) \\
&+ \frac{f_1}{x} \mathbf{M}_{\mu\nu}[\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2) + f_1\mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma(1+\beta_N^2) - f_2j_\rho^*I_\rho^*(1+\beta_N^2)]\beta_\nu\Gamma_\mu \\
&+ \frac{f_1}{x} \mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma[(1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}(1+\beta_N^2)\beta_\nu\Gamma_\mu + f_2j_\mu(1+\beta_N^2)I'_\mu] \\
&+ \frac{f_2}{x} j_\mu[\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2) + f_1\mathbf{M}_{\rho\sigma}^*I_\rho^*\beta_\sigma(1+\beta_N^2) - f_2j_\rho^*\Gamma_\rho^*(1+\beta_N^2)]\Gamma_\mu \\
&- \frac{f_2}{x} j_\rho^*\Gamma_\rho^*[(1+\beta_N^2)(\beta\partial)\phi + f_1\mathbf{M}_{\mu\nu}(1+\beta_N^2)\beta_\nu\Gamma_\mu + f_2j_\mu(1+\beta_N^2)\Gamma_\mu] \\
&= -x\bar{\psi}\beta_N^2\phi + \frac{1}{x}\bar{\psi}(\beta\partial)^\leftarrow (1+\beta_N^2)(\beta\partial)\phi \\
&+ \frac{f_1^2}{2x} (\mathbf{M}_{\mu\nu}\mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^*\mathbf{M}_{\mu\nu})\Gamma_\rho^*\beta_\sigma(1+\beta_N^2)\beta_\nu\Gamma_\mu - \frac{f_2^2}{2x} (j_\mu j_\rho^* + j_\rho^* j_\mu)\Gamma_\rho^*(1+\beta_N^2)I'_\mu \\
&+ \frac{f_1}{x} [\mathbf{M}_{\mu\nu}\bar{\mathbf{f}}(\beta\partial)^\leftarrow \beta_\nu\Gamma_\mu + \mathbf{M}_{\mu\nu}^*\Gamma_\mu^*\beta_\nu(\beta\partial)\phi] \\
&+ \frac{f_2}{x} [j_\mu\bar{\mathbf{f}}(\beta\partial)^\leftarrow \Gamma_\mu - j_\mu^*\Gamma_\mu^*(\beta\partial)\phi]
\end{aligned}$$

$$\begin{aligned}
& + \frac{f_1}{x} \{ \mathbf{M}_{\mu\nu} \bar{\psi} [(\beta \partial) - x] \beta_{N\mu}^2 \beta_\nu \Gamma_\mu + \mathbf{M}_{\mu\nu}^* I_\mu^* \beta_\nu \beta_{N\mu}^2 [(\beta \partial) + x] \psi \} \\
& + \frac{f_2}{x} \{ \mathbf{J}_\mu \bar{\psi} [(\beta \partial) - x] \beta_N^2 \Gamma_\mu - \mathbf{J}_\mu^* \Gamma_\mu^* \beta_N^2 [(\beta \partial) + x] \psi \} \\
& = \mathcal{H}_{\text{free}} + \mathcal{H}_{\text{int}}(\partial),
\end{aligned}$$

where $\mathcal{H}_{\text{free}}$ expresses the first line of the last expression and $\mathcal{H}_{\text{int}}(\partial)$ expresses the part below the second line. \mathcal{H}_{int} , the interaction part of the total Hamiltonian, can be obtained from $\mathcal{H}_{\text{int}}(\partial)$ by applying the equation of motion (A.5) or (A.6) because the variables contained in \mathcal{H}_{int} are the Heisenberg variables:

$$\begin{aligned}
\mathcal{H}_{\text{int}} = & f_1 (\mathbf{M}_{\mu\nu} \bar{\psi} \beta_\nu \Gamma_\mu - \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu \psi) + f_2 (\mathbf{J}_\mu \bar{\psi} \Gamma_\mu + \mathbf{J}_\mu^* I_\mu^* \psi) \\
& - \frac{f_1^2}{2x} (\mathbf{M}_{\mu\nu} \mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^* \mathbf{M}_{\mu\nu}) I_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu \Gamma_\mu + \frac{f_2^2}{2x} (\mathbf{J}_\mu \mathbf{J}_\rho^* + \mathbf{J}_\rho^* \mathbf{J}_\mu) I_\rho^* (1 + \beta_N^2) I_\mu^*.
\end{aligned} \tag{A.9}$$

On the other hand, H_{int} , the interaction Hamiltonian in the interaction representation, contains the free variables which satisfy the free equation of motion (1) or (5), then we have

$$\begin{aligned}
H_{\text{int}} = & f_1 (\mathbf{M}_{\mu\nu} \bar{\psi} \beta_\nu \Gamma_\mu - \mathbf{M}_{\mu\nu}^* \Gamma_\mu^* \beta_\nu \psi) + f_2 (\mathbf{J}_\mu \bar{\psi} \Gamma_\mu + \mathbf{J}_\mu^* \Gamma_\mu^* \psi) \\
& + \frac{f_1^2}{2x} (\mathbf{M}_{\mu\nu} \mathbf{M}_{\rho\sigma}^* + \mathbf{M}_{\rho\sigma}^* \mathbf{M}_{\mu\nu}) I_\rho^* \beta_\sigma (1 + \beta_N^2) \beta_\nu I_\mu^* - \frac{f_2^2}{2x} (\mathbf{J}_\mu \mathbf{J}_\rho^* + \mathbf{J}_\rho^* \mathbf{J}_\mu) I_\rho^* (1 + \beta_N^2) I_\mu^*.
\end{aligned} \tag{A.10}$$

In these form, it is easily seen that \mathcal{H}_{int} (or H_{int}) consists of the negative of the interaction Lagrangian $-\mathcal{L}_{\text{int}}$ (or $-I_{\text{int}}$), plus direct couplings, normal dependent and independent. The former corresponds to the first line of (A.9) or (A.10) and write $\mathcal{H}_{\text{int}}^I$ (or H_{int}^I) in this paper, and the latter corresponds to the second line of (A.9) or (A.10) and write $\mathcal{H}_{\text{int}}^{II}$ (or H_{int}^{II}). Comparing (A.9) and (A.10), it is clearly seen that $\mathcal{H}_{\text{int}}^I$ and H_{int}^I have the same sign but $\mathcal{H}_{\text{int}}^{II}$ and H_{int}^{II} have just the opposite sign. These circumstances exist quite generally in every interacting system usually considered.

Appendix B. The case of the interaction of the meson with the electromagnetic field

In the case of the interacting meson and electromagnetic fields, the β -formalism corresponds completely to the wave formalism as shown in I, and the difference such as the direct coupling does not appear. We discuss this point briefly.

In the β -formalism, the Lagrangian of the interacting system is given by

$$\mathcal{L} = -\bar{\psi}(\beta \partial^- + x)\psi \tag{A.11}$$

where $\partial_\mu^- = \partial_\mu - i\nu A_\mu$. Considering that the equation of motion for Heisenberg variables is

$$(\beta\partial^- + \kappa)\phi = 0, \quad (\text{A.12})$$

and there are the correspondences similar to (28), the Lagrangian (A.11) is translated into the following forms in the wave formalism :

$$\mathcal{L}^s \simeq -(\partial_\mu u^* \partial_\mu u + \kappa^2 u^* u) - ie(u^* \partial_\mu u - \partial_\mu u^* u) A_\mu - c^2 u^* u A_\mu A_\mu \quad (\text{A.13})$$

for the scalar meson, and

$$\mathcal{L}^v \simeq -\left(\frac{1}{2} F_{\mu\nu}^* F_{\mu\nu} + \kappa^2 v_\mu^* v_\mu\right) \quad (\text{A.14})$$

for the vector meson, where

$$\left. \begin{aligned} F_{\mu\nu}^* &\equiv f_{\mu\nu}^* + ie(A_\mu v_\nu^* - A_\nu v_\mu^*) \\ F_{\mu\nu} &\equiv f_{\mu\nu} - ie(A_\mu v_\nu - A_\nu v_\mu) \end{aligned} \right\} \quad (\text{A.15})$$

These are just the same Lagrangians as usually adopted.

As can be seen from the analysis done in I, the 4-vertex is produced by the δ -function term of the Green function. In the case of the interacting meson and nucleon system, this term produces the normal independent direct couplings. This correspondence is easily understood if we write the Feynman diagrams (Fig. 2).

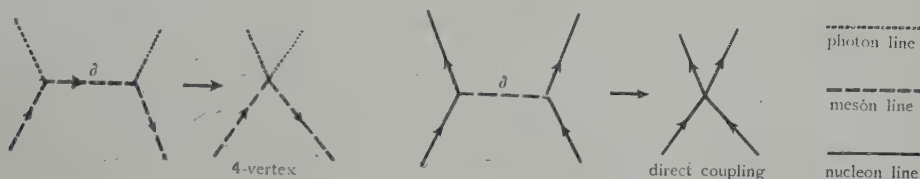


Fig. 2.

Then the only difference of the circumstance is that the 4-vertex is included also in the wave formalism while the normal independent direct couplings are not included usually in the wave formalism as such terms are meaningless. Indeed, when they introduced the interaction from the analogy of the Proca field to the Maxwell field in the earlier meson theory these direct couplings were contained in the interaction⁹⁾.

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Antiferromagnetism. The Kagomé Ising Net

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We can solve exactly the eigenvalue problem of the kagome Ising net with $z=4$. The transition temperature lies a little below than that of the square lattice. Its value is determined by $e^{4Lc} = 3 + 2\sqrt{3}$ and it teaches us that it is not determined only by the number of nearest neighbors. In the case of antiferromagnetism, especially, the kagomé lattice which does not fit to antiferromagnetic arrangement is disorderd at all temperature and possesses a finite zero point entropy just as in the case of the triangular lattice and the result runs as follows:

$$\frac{S_k(0)}{R} = \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \{ 21 - 4(\cos\omega_1 + \cos\omega_2 + \cos(\omega_1 + \omega_2)) \} d\omega_1 \cdot d\omega_2 \div 0.50183$$

§ 1. Introduction

After the problem of plane square lattice was solved exactly by Onsager¹⁾, the honeycomb and triangular lattices were treated by several authors²⁾. The exact values of transition temperature for these three typical lattices indicate an interesting rule which is given by the simple relation: $\text{ch } 2L_c = \sec \pi/z$ in terms of the nearest neighbors number: z . It is interesting to know whether it be determined solely by the nearest neighbors number or not. Recently, the same problem for the kagomé lattice was treated by I. Syozi³⁾. The kagomé lattice consists exclusively of equivalent lattice points and equivalent bonds and the number of nearest neighbors is four. Therefore it is equivalent to the square lattice in these points. The method of "star-triangle-transformation" that derives the partition function of the kagomé lattice from that of the honeycomb lattice was established by I. Syozi. The conclusion he drew out from his method reveals us clearly that the transition temperature is not determined by the nearest neighbors number only. On the other hand, the kagomé lattice leads us to another more interesting problem; "the case of antiferromagnetism." In the case of antiferromagnetism, the only one net triangular Ising net—has a peculiarity among three types of lattices that were already treated. Because the triangular Ising net does not fit to an antiferromagnetic arrangement and is disorderd at all temperature. Moreover, it was shown and evaluated by G. H. Wannier⁴⁾ that the zero point entropy remains finite. These circumstances lead us to treat the antiferromagnetic system of the kagomé Ising net. Because the kagomé lattice does not fit to an antiferromagnetic arrangement just as that of the triangular lattice.

We shall try to solve directly the eigenvalue problem of the kagomé lattice by a different method from Syozi's and discuss the case of antiferromagnetism.

§ 2. Eigenvalue problem of Kagomé-lattice and representation of its operator

Let us call "Kagomé" the lattice shown in Fig. 1. Separating the points with double suffix in Fig. 1, we are to deal with the lattice shown in Fig. 2, in order to be able to apply the matrix method of the crystal statistics. We can obtain the kagomé lattice from that of Fig. 2, when the interaction shown by the double-connected-line tends to infinite which causes the spins of atoms on both side in coincide. For such a lattice, we can write down the operator V of the eigenvalue problem as follows.

$$V = V_1 \cdot V_2, \quad (1)$$

$$V_1 = (2 \operatorname{sh} 2L)^{2n} e^{L(s_1 s_2 + s_2 s_3 + \dots)} e^{L^*(C_1 + C_2 + \dots C_{2n})} e^{L(s_1 s_2 + s_3 s_4 + \dots + s_{2n-1} s_{2n})} \\ \times e^{L^*(C_1 + C_2 + \dots C_{2n})}, \quad (2)$$

$$V_2 = (2 \operatorname{sh} 2L)^{2n} e^{L(s_1 s_2 + s_2 s_3 + \dots)} e^{L^*(C_1 + C_2 + \dots C_{2n})} e^{L(s_2 s_3 + s_4 s_5 + \dots s_{2n} s_1)} \\ \times e^{L^*(C_1 + C_2 + \dots C_{2n})}. \quad (3)$$

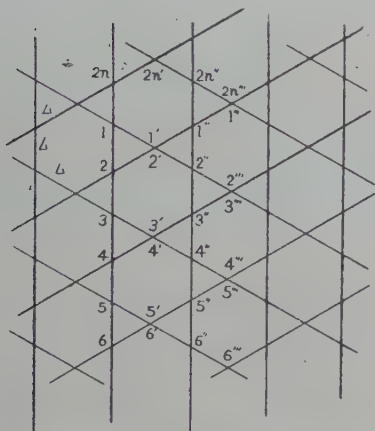


Fig. 1. Kagomé lattice: The interaction parameter of neighboring atoms is given by L . The number of one row's atoms is $2n$.

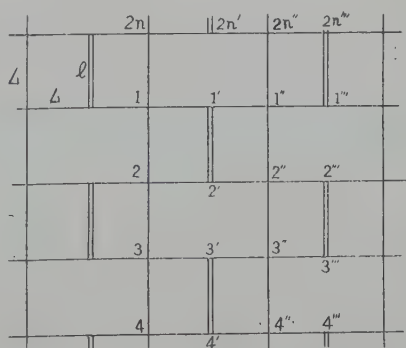


Fig. 2. The atoms with double suffix in Fig. 1 are separated into the two atoms which locate in the both ends of double connected line. The interaction parameter with double connected line is given by L .

The operator V_k of the kagomé lattice can be obtained as the limiting case of interaction: L tending to infinite.

$$V_k = \lim_{L \rightarrow \infty} 1 / (2 \operatorname{ch} 2L)^{2n} \cdot V_1 V_2. \quad (4)$$

In fact, the factor containing L in the operator V behaves as a δ -function that becomes either 1 or 0 according to the signs of the spins on the both ends of double-connected-interaction. That is:

$$\lim_{l \rightarrow \infty} \frac{1}{2 \operatorname{ch} l} \cdot e^{\frac{i s_1 s_2}{r_s}} = \lim_{l \rightarrow \infty} (\operatorname{ch} l + s_1 s_2 \operatorname{sh} l) / 2 \operatorname{ch} l = (1 + s_1 s_2) / 2.$$

Now let us introduce p_r, q_r , as follows :

$$\begin{aligned} p_r &= C_1 C_2 \cdots C_{r-1} s_r, \\ q_r &= C_1 C_2 \cdots C_{r-1} (i C_r s_r), \\ [p_i, p_k]_+ &= 2 \delta_{ik}, \quad [q_i, q_k]_+ = 2 \delta_{ik}, \quad [p_i, q_k]_+ = 0. \end{aligned}$$

Inversely $s_r s_{r+1}, C_r$ can be expressed by p_r, q_r .

$$\begin{aligned} s_r s_{r+1} &= -i p_{r+1} q_r, & s_{2n} s_1 &= i p_1 q_{2n} C, \\ C_r &= i p_r q_r, & C &= C_1 C_2 \cdots C_{2n}. \end{aligned}$$

The operator V of (1) is represented in terms of p_r, q_r as follows :

$$\begin{aligned} V_{1\pm} &= (2 \operatorname{sh} 2L)^{2n} e^{-iL(f_{2n} q_1 + f_{2n} q_2 + \cdots \mp f_{1n} q_{2n})} e^{iL^*(f_{1n} q_1 + f_{2n} q_2 + \cdots + f_{2n} q_{2n})} \\ &\quad \times e^{-iL(f_{2n} q_1 + f_{2n} q_2 + \cdots + f_{2n} q_{2n-1})} e^{iL^*(f_{1n} q_1 + f_{2n} q_2 + \cdots + f_{2n} q_{2n})}, \end{aligned} \quad (5)$$

$$\begin{aligned} V_{2\pm} &= (2 \operatorname{sh} 2L)^{2n} e^{-iL(f_{2n} q_1 + f_{2n} q_2 + \cdots \mp f_{1n} q_{2n})} e^{iL^*(f_{1n} q_1 + f_{2n} q_2 + \cdots + f_{2n} q_{2n})} \\ &\quad \times e^{-iL(f_{2n} q_2 + f_{2n} q_4 + \cdots \mp f_{1n} q_{2n})} e^{iL^*(f_{1n} q_1 + f_{2n} q_2 + \cdots + f_{2n} q_{2n})}, \end{aligned} \quad (6)$$

where V_{\pm} correspond to the space of even function with $C=1$ and that of odd function with $C=-1$, respectively.

We shall consider the "shift-operator" that shifts the indices of p_r, q_r . The shift operator: O is defined by

$$\begin{aligned} O_{\pm} p_j O_{\pm}^{-1} &= p_j, \\ O_{\pm} q_j O_{\pm}^{-1} &= q_{j+1} \quad (j \neq 2n), \quad O_{\pm} q_{2n} O_{\pm}^{-1} = \mp q_1 \quad (j=2n). \end{aligned} \quad (7)$$

Let us write for simplicity's sake.

$$\begin{aligned} \sum_{j=1}^{2n} p_j q_j &= A_0, \\ \sum_{j=1}^n p_{2j-1} q_{2j-1} &= A_1, \quad \sum_{j=1}^n p_{2j} q_{2j} = A_2. \end{aligned} \quad (8)$$

Then the operator V becomes by (5), (6) and (7) :
(the fore factors of V being neglected)

$$V_{1\pm} = O_{\pm}^{-1} e^{-iL\Lambda_0} O_{\pm} e^{iL^*\Lambda_0} O_{\pm}^{-1} e^{-iL\Lambda_2} O_{\pm} e^{iL^*\Lambda_0}, \quad (9)$$

$$V_{2\pm} = O_{\pm}^{-1} e^{-iL\Lambda_0} O_{\pm} e^{iL^*\Lambda_0} O_{\pm}^{-1} e^{-iL\Lambda_1} O_{\pm} e^{iL^*\Lambda_0}. \quad (10)$$

Here $V_{1\pm}$ and $V_{2\pm}$ are similar to each other except for the difference of the factors $e^{-iL\Lambda_2}$ and $e^{-iL\Lambda_1}$. In view of this point, we introduce a second shift operator: S by definitions as follows.

$$\begin{aligned} S_{\pm} p_j S_{\pm}^{-1} &= p_{j+1} \\ S_{\pm} q_j S_{\pm}^{-1} &= q_{j+1} \end{aligned} \quad (j \neq 2n), \quad \begin{aligned} S_{\pm} p_{2n} S_{\pm}^{-1} &= \mp p_1 \\ S_{\pm} q_{2n} S_{\pm}^{-1} &= \mp q_1. \end{aligned} \quad (11)$$

A_0 of (8) remains invariant under the transformation by the operator S and A_1 changes to A_2 and vice versa.

In fact

$$S_{\pm} A_0 S_{\pm}^{-1} = S_{\pm} \left(\sum_{j=1}^{2n} p_j q_j \right) S_{\pm}^{-1} = \sum_{j=1}^{2n} p_{j+1} q_{j+1} = \sum_{j=1}^{2n} p_j q_j = A_0. \quad (12)$$

Similarly

$$S_{\pm} A_1 S_{\pm}^{-1} = A_2, \quad S_{\pm} A_2 S_{\pm}^{-1} = A_1.$$

Thus (9), (10) become as follow.

$$\begin{aligned} V_{1\pm} &= O_{\pm}^{-1} e^{-iL\Lambda_0} O_{\pm} e^{iL^* \Lambda_0} O_{\pm}^{-1} S_{\pm}^{-1} e^{-iL\Lambda_1} S_{\pm} O_{\pm} e^{iL^* \Lambda_0}, \\ V_{2\pm} &= O_{\pm}^{-1} e^{-iL\Lambda_0} O_{\pm} e^{iL^* \Lambda_0} O_{\pm}^{-1} e^{-iL\Lambda_1} O_{\pm} e^{iL^* \Lambda_0} = S_{\pm}^{-1} V_{1\pm} S_{\pm}. \end{aligned} \quad (13)$$

By (12), S_{\pm} are commutable with the operators of $e^{\alpha\Lambda_0}$.

Now, we shall try to obtain the $4 \cdot n$ -dimensional representations due to Kaufman and Onsager⁵⁾. Introducing the $4 \cdot n$ -basis: η_r by the following definition:

$$\begin{aligned} p_r &= \eta_{2r-1} \\ q_r &= \eta_{2r} \end{aligned} \quad (r = 1, 2, \dots, 2n),$$

in general, we can get the representations of any operators M , N of the same kind as V on these $4 \cdot n$ -basis of η_r according to the theory of Nambu and Kaufman.

$$\begin{aligned} M &\longrightarrow R(M) = \{m_{ij}\}, \\ \text{operator} &\quad \text{representation} \\ N &\longrightarrow R(N) = \{n_{ij}\} \end{aligned} \quad (14)$$

in correspondence with

$$\begin{aligned} M \eta_i M^{-1} &= \sum_{j=1}^{2n} m_{ij} \eta_j, \\ N \eta_i N^{-1} &= \sum_{j=1}^{2n} n_{ij} \eta_j \end{aligned} \quad (15)$$

and for the product of M and N

$$MN \longrightarrow R(MN) = R(N) R(M) = \{n_{ij}\} \{m_{jk}\}$$

in correspondence with

$$\begin{aligned} MN \eta_i N^{-1} M^{-1} &= M \left(\sum_{j=1}^{2n} n_{ij} \eta_j \right) M^{-1} = \sum_{j=1}^{2n} n_{ij} \left(\sum_{k=1}^{2n} m_{jk} \eta_k \right) \\ &= \sum_{k=1}^{2n} \left(\sum_{j=1}^{2n} n_{ij} m_{jk} \right) \eta_k. \end{aligned}$$

Therefore the 4·n-dimensional representations of our operators become as follows, respectively.

$$R(e^{\pm iL\Lambda_0}) = \begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & \ddots \\ & & & & a \end{pmatrix}, \quad a = \begin{pmatrix} \text{ch } 2L & \mp i \text{ sh } 2L & 0 & 0 \\ \pm i \text{ sh } 2L & \text{ch } 2L & 0 & 0 \\ 0 & 0 & \text{ch } 2L & \mp i \text{ sh } 2L \\ 0 & 0 & \pm i \text{ sh } 2L & \text{ch } 2L \end{pmatrix}, \quad (16)$$

$$R(e^{-iL\Lambda_1}) = \begin{pmatrix} a_1 & & & \\ & a_1 & & \\ & & a_1 & \\ & & & \ddots \\ & & & & a_1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} \text{ch } 2l & i \text{ sh } 2l & 0 & 0 \\ -i \text{ sh } 2l & \text{ch } 2l & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (17)$$

$$R(e^{-iL\Lambda_2}) = \begin{pmatrix} a_2 & & & \\ & a_2 & & \\ & & a_2 & \\ & & & \ddots \\ & & & & a_2 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \text{ch } 2l & i \text{ sh } 2l \\ 0 & 0 & -i \text{ sh } 2l & \text{ch } 2l \end{pmatrix}. \quad (18)$$

Also the representation of our shift operators; O_{\pm} is

$$R(O_{\pm}) = \begin{pmatrix} A & B & 0 & 0 & \dots & 0 \\ 0 & A & B & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & A & B \\ \pm B & 0 & 0 & \dots & 0 & A \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (19)$$

and that of S_{\pm} is

$$R(S_{\pm}) = \begin{pmatrix} C & D & 0 & 0 & \dots & 0 \\ 0 & C & D & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & C & D \\ \pm D & 0 & 0 & \dots & 0 & C \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (20)$$

Thus $R(e^{\pm iL\Lambda_0})$, $R(e^{-iL\Lambda_1})$ and $R(e^{-iL\Lambda_2})$ are diagonal hyper matrices which have elements formed by the small 4-dimensional matrices. And also, $R(O_{\pm})$, $R(S_{\pm})$ are periodic hyper matrices which have elements formed by small 4-dimensional matrices, respectively. According to Kaufman, such a periodic hyper matrices can be transformed at once to a diagonal hyper matrices. Let us indicate by $\bar{R}(\bar{O}_{\pm})$, $\bar{R}(\bar{S}_{\pm})$ for the diagonal hyper matrices transformed from $R(O_{\pm})$, $R(S_{\pm})$.

That is

$$\bar{R}(\bar{O}_{\pm}) = \left\| \begin{array}{cccc} \bar{O}_{1\pm} & & & \\ & \bar{O}_{2\pm} & & \\ & & \ddots & \\ & & & \bar{O}_{r\pm} \\ & & & & \ddots \\ & & & & & \bar{O}_{n\pm} \end{array} \right\|, \quad \text{where} \quad \bar{O}_{r\pm} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & \xi_{r\pm} & 0 & 0 \end{array} \right\|, \quad \bar{O}_{r\pm}^{-1} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_{r\pm}^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right\|. \quad (21)$$

Similarly

$$\bar{R}(\bar{S}_{\pm}) = \left\| \begin{array}{cccc} S_{1\pm} & & & \\ & \bar{S}_{2\pm} & & \\ & & \ddots & \\ & & & S_{r\pm} \\ & & & & \ddots \\ & & & & & \bar{S}_{n\pm} \end{array} \right\|, \quad \bar{S}_{r\pm} = \left\| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \xi_{r\pm} & 0 & 0 & 0 \\ 0 & \xi_{r\pm} & 0 & 0 \end{array} \right\|, \quad \bar{S}_{r\pm}^{-1} = \left\| \begin{array}{cccc} 0 & 0 & \xi_{r\pm}^{-1} & 0 \\ 0 & 0 & 0 & \xi_{r\pm}^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right\| \quad (22)$$

where

$$\xi_{r+} = e^{(2r-1)\pi i/n}, \quad \xi_{r-} = e^{2i\pi i/n}.$$

Still, both $R(O_{\pm})$ and $R(S_{\pm})$, as we see in (19), (20) are periodic hyper matrices which have elements formed by the 4-dimensional matrices, respectively. Therefore, for the transformations:

$$R(O_{\pm}) \longrightarrow \bar{R}(\bar{O}_{\pm}),$$

$$R(S_{\pm}) \longrightarrow \bar{R}(\bar{S}_{\pm}),$$

there are similarity transformation-matrices as follows:

$$R(O_{\pm}) = \bar{R}(U_{\pm}) \bar{R}(\bar{O}_{\pm}) \bar{R}^{-1}(U_{\pm}),$$

$$R(S_{\pm}) = \bar{R}(U_{\pm}) \bar{R}(\bar{S}_{\pm}) \bar{R}^{-1}(U_{\pm}), \quad (23)$$

where $R(U_{\pm})$ are hyper matrices having elements formed by the constant multiple of 4-dimensional unit matrices. That is

$$R(U_{\pm}) = \left\| \begin{array}{cccc} U_{11} & U_{12} & U_{13} & \dots & U_{1n} \\ U_{21} & U_{22} & \dots & \dots & U_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ U_{n1} & \dots & \dots & \dots & U_{nn} \end{array} \right\|, \quad U_{ij} = u_{ij} \left\| \begin{array}{cccc} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{array} \right\|.$$

This $\bar{R}(U_{\pm})$ are evidently commutative with (16), (17) and (18), respectively. And the commutativity of $R(O_{\pm})$ and $R(S_{\pm})$ become evident from the commutativity of $O_{r\pm}$ and $\bar{S}_{r\pm}$ by (21), (22) and (23). Now let $\bar{S}_{r\pm}^0$ be the matrices $\bar{S}_{r\pm}$ and $\bar{S}_{r\pm}^{-1}$ multiplied by $\xi_{r\pm}^{-1/2}$ and $\xi_{r\pm}^{1/2}$, respectively.

That is

$$\bar{S}_{r\pm} \xi_{r\pm}^{-1/2} = \xi_{r\pm}^{1/2} \bar{S}_{r\pm}^{-1} = \bar{S}_{r\pm}^0 = \bar{S}_{r\pm}^{0-1} = \begin{vmatrix} 0 & 0 & \xi_{r\pm}^{1/2} & 0 \\ 0 & 0 & 0 & \xi_{r\pm}^{-1/2} \\ \xi_{r\pm}^{1/2} & 0 & 0 & 0 \\ 0 & \xi_{r\pm}^{1/2} & 0 & 0 \end{vmatrix}, \quad (24)$$

$$\bar{R}(\bar{S}_{\pm}^0) = \begin{vmatrix} \bar{S}_{1\pm}^0 & & & \\ & \bar{S}_{2\pm}^0 & & \\ & & \ddots & \\ & & & \bar{S}_{r\pm}^0 \\ & & & & \ddots \\ & & & & & \bar{S}_{n\pm}^0 \end{vmatrix}.$$

Thus, using $\bar{S}_{r\pm}^0$ instead of $\bar{S}_{r\pm}$ and $\bar{S}_{r\pm}^{-1}$, we may reproduce our relations in the next form: (25) of the same character as before. That is; (see (16), (17), (18), (21) and (24))

$$\begin{aligned} \bar{S}_{r\pm}^0 \bar{O}_{r\pm} \bar{S}_{r\pm}^{0-1} &= \bar{S}_{r\pm}^0 \bar{O}_{r\pm} \bar{S}_{r\pm}^0 = O_{r\pm}, \\ \bar{S}_{r\pm}^0 \bar{O}_{r\pm}^{-1} \bar{S}_{r\pm}^0 &= \bar{O}_{r\pm}^{-1}, \\ \bar{S}_{r\pm}^0 a \bar{S}_{r\pm}^0 &= a, \\ \bar{S}_{r\pm}^0 a_2 \bar{S}_{r\pm}^0 &= a_1, \\ \bar{S}_{r\pm}^0 a_1 \bar{S}_{r\pm}^0 &= a_2. \end{aligned} \quad (25)$$

After all, the matrices of the eigenvalue problem to be solve become as follows.

$$\bar{R}(\bar{V}_{\pm}) = \bar{R}(\bar{V}_{1\pm} \bar{V}_{r\pm}) = \bar{R}(\bar{V}_{2\pm}) \bar{R}(\bar{S}_{\pm}^0) \bar{R}(\bar{V}_{2\pm}) \bar{R}(\bar{S}_{\pm}^0).$$

This gives us

$$\bar{R}(\bar{V}_{\pm}) = \bar{R}(\bar{V}_{1\pm} \bar{V}_{2\pm}) = \{\bar{R}(\bar{S}_{\pm}^0 \bar{V}_{2\pm})\}^2.$$

Thus our eigenvalue problem has been simplified the whole operator being reduced to the product of the half numbers of operators. Similar considerations may be applied to the case of the honeycomb and triangular lattices. Now we may proceed to solve the eigenvalue problem of the reduced operator:

$$\begin{aligned} \bar{R}(\bar{S}_{\pm}^0 \bar{V}_{2\pm}) &= \bar{R}(e^{iL^* \Lambda_0}) \bar{R}(\bar{O}_{\pm}) \bar{R}(e^{-iL \Lambda_1}) \bar{R}(\bar{O}_{\pm}^{-1}) \\ &\times \bar{R}(e^{iL^* \Lambda_0}) \bar{R}(\bar{O}_{\pm}) \bar{R}(e^{-iL \Lambda_0}) \bar{R}(\bar{O}_{\pm}^{-1}) \bar{R}(\bar{S}_{\pm}^0). \end{aligned} \quad (26)$$

The products of the matrices of (26) can be calculated by (16), (17), (21) and (24). The results are following: (here, suffix: r is neglected. X stands for $\text{ch } 2l$, Y for $\text{sh } 2l$ and C for $\text{ch } 2L$, etc.)

$$R(S^0 V_2) = \begin{vmatrix} -C^* S^* S \xi^{1/2} & i S^{*2} S \xi^{-1/2} & S^{*2} C \xi^{-1/2} & -i C^* S^* C \xi^{-1/2} \\ -C^* S^* S X \xi^{1/2} & +i C^{*2} S X \xi^{-1/2} & +C^{*2} C X \xi^{-1/2} & -i C^* S^* C X \xi^{-1/2} \\ -C^* S^* C Y \xi^{-1/2} & +i C^{*2} C Y \xi^{-1/2} & +C^{*2} S Y \xi^{-1/2} & -i C^* S^* S Y \xi^{-1/2} \\ -i C^{*2} S \xi^{1/2} & -C^* S^* S \xi^{-1/2} & i C^* S^* C \xi^{-1/2} & C^{*2} C \xi^{-1/2} \\ -i S^{*2} S X \xi^{1/2} & -C^* S^* S X \xi^{-1/2} & +i C^* S^* C X \xi^{-1/2} & +S^{*2} C X \xi^{-1/2} \\ -i S^{*2} C Y \xi^{-1/2} & -C^* S^* C Y \xi^{-1/2} & +i C^* S^* S Y \xi^{-1/2} & +S^{*2} S Y \xi^{-1/2} \\ C^{*2} C \xi^{1/2} & -i C^* S^* C \xi^{1/2} & -C^* S^* S \xi^{1/2} & i C^{*2} S \xi^{-1/2} \\ +S^{*2} C X \xi^{1/2} & -i C^* S^* C X \xi^{1/2} & -C^* S^* S X \xi^{1/2} & +i S^{*2} S X \xi^{-1/2} \\ +S^{*2} S Y \xi^{3/2} & -i C^* S^* S Y \xi^{1/2} & -C^* S^* C Y \xi^{1/2} & +i S^{*2} C Y \xi^{1/2} \\ i C^* S^* C \xi^{1/2} & S^{*2} C \xi^{1/2} & -i S^{*2} S \xi^{1/2} & -C^* S^* S \xi^{-1/2} \\ +i C^* S^* C X \xi^{1/2} & +C^{*2} C X \xi^{1/2} & -i C^{*2} S X \xi^{1/2} & -C^* S^* S X \xi^{-1/2} \\ +i C^* S^* S Y \xi^{3/2} & +C^{*2} S Y \xi^{1/2} & -i C^{*2} C Y \xi^{1/2} & -C^* S^* C Y \xi^{1/2} \end{vmatrix}$$

§ 3. Solution of eigenvalue problem

The secular eq. derived from (27) becomes a reciprocal equation :

$$\lambda^4 + \Gamma_1 \lambda^3 + \Gamma_2 \lambda^2 + \Gamma_1 \lambda + 1 = 0$$

so that

$$\left(\lambda_{\pm} + \frac{1}{\lambda_{\pm}} \right) = 2 \operatorname{ch} \gamma_{\pm} = -\frac{\Gamma_1}{2} \pm \sqrt{\left(\frac{\Gamma_1}{2} \right)^2 - (\Gamma_2 - 2)} \equiv 2Z > 0. \quad (28)$$

The coefficients Γ_1 , Γ_2 calculated by (27) become

$$\Gamma_1 = 4C^* S^* S X \cos \omega + 4C^* S^* C Y \cos \omega + 4C^* S^* S \cos \omega \quad (29)$$

$$\Gamma_2 = 2 \{ -S^2 X - 2C^{*2} C S Y - C^{*4} C^2 X - S^{*4} C^2 X - 2C^{*2} S^{*2} X - 2C^* S^{*2} C^2 \\ - 2C^{*2} S^{*2} + 2S^{*2} C S Y \cos 2\omega + 2C^{*2} S^{*2} S^2 X \cos 2\omega + 2C^{*2} S^{*2} \cos 2\omega \}, \quad (30)$$

where

$$(\xi^{1/2} + \xi^{-1/2}) = 2 \cos \omega.$$

The transition temperature is defined by $z=1$ in (28). That is

$$2\Gamma_1 + \Gamma_2 + 2 = 0. \quad (31)$$

We must treat the limiting case of the interaction energy tending to infinity. Therefore the equation determining the transition temperature is

$$4C^* S^* S \cos \omega + 4C^* S^* C \cos \omega - S^2 - 2C^{*2} C S - C^{*4} C^2 - S^{*4} C^2 - 2C^{*2} S^{*2} \\ + 2S^{*2} C S \cos 2\omega + 2C^{*2} S^{*2} S^2 \cos 2\omega = 0.$$

Thus we obtain the transition temperature with $n \rightarrow \infty$ ($\omega \rightarrow 0$) as follows :

$$e^{A_{Lo}} = 3 + 2\sqrt{3}. \quad (32)$$

This figure is to be compared with $3 + 2\sqrt{2}$ for the square lattice with the same number 4 of nearest neighbours.

Now let us evaluate the partition function. Our secular equation is of a reciprocal form and the roots may be written as λ_+ , $1/\lambda_+$, λ_- , $1/\lambda_-$, respectively. Using the formula :

$$\int_0^{2\pi} \log(2 \operatorname{ch} x - 2 \cos \omega') d\omega' = 2\pi x, \quad (33)$$

$(\gamma_{r+} + \gamma_{r-})$ is written as follows.

$$\gamma_{r+}(\omega) + \gamma_{r-}(\omega) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=+,-} \log\left(\lambda_i + \frac{1}{\lambda_i} - 2 \cos \omega'\right) d\omega',$$

where γ_{r+} and γ_{r-} correspond to plus and minus in (28), respectively. Thus

$$\begin{aligned} (\gamma_{r+} + \gamma_{r-}) &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{i=+,-} \log\left(\lambda_i + \frac{1}{\lambda_i} - 2 \cos \omega'\right) d\omega' \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log(-I_2' - 2I_1' \cos \omega' - 4 \cos^2 \omega' + 2) d\omega' \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log(-I_2' - 2I_1' \cos \omega') d\omega', \end{aligned} \quad (34)$$

(where, $-4 \cos^2 \omega'$ and 2 are neglected in limiting case of $I \rightarrow \infty$) Therefore the partition function : λ_k per atom becomes as follows from (34), restoring the fore factors of (2), (3), (4),

$$6n \log \lambda_k = \lim_{I \rightarrow \infty} \left\{ 8n \frac{1}{2} \log(2 \operatorname{sh} 2L) - 8n \frac{1}{4} \log(2 \operatorname{ch} I) + \frac{1}{2\pi} \sum_{r=1}^n \int_0^{2\pi} \log(-I_2' - 2I_1' \cos \omega') d\omega' \right\}. \quad (35)$$

Substituting I_1' , I_2' of (29), (30) in (35), we can obtain the partition function of the kagomé lattice. Our result is

$$\begin{aligned} \log(\lambda_k/2) &= \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \frac{1}{4} [\operatorname{ch}^6 2L + 2 \operatorname{ch}^3 2L \operatorname{sh}^3 2L + \operatorname{sh}^6 2L + 3 \operatorname{ch}^2 2L \\ &\quad - (2 \operatorname{ch} 2L \operatorname{sh}^3 2L + 2 \operatorname{ch}^2 2L \operatorname{sh}^2 2L) (\cos \omega_1 + \cos \omega_2 + \cos(\omega_1 + \omega_2))] d\omega_1 d\omega_2, \end{aligned} \quad (36)$$

where

$$\omega' = \frac{\omega_1 - \omega_2}{2}, \quad \omega = \frac{\omega_1 + \omega_2}{2}.$$

§ 4. Discussion of antiferromagnetism

There still remains for us to analyze the antiferromagnetism of our kagomé lattice which does not fit to an antiferromagnetic arrangement as we indicated in our introduction. The antiferromagnetic system of the triangular lattice was discussed by G. H. Wannier in

1950. According to his result, the system is disordered at all temperature and has no Curie point. Especially the system has a finite value of zero point entropy;

$$S_t(0)/R = \frac{2}{\pi} \int_0^{\pi/3} \log(2 \cos \omega) d\omega \doteq 0.3383.$$

In this section, we shall try to discuss the system of the antiferromagnetic kagomé lattice in comparison with that of the triangular lattice of G. H. Wannier. Firstly let us make qualitative discussions. Later we shall give the exact computation.

Qualitative Discussion

In the case of the so-called antiferromagnetism, any conceivable arrangements of spins in the kagomé lattice possess still a great many bonds of unequal pairs in the ground state as in that of the triangular lattice. One of examples for such arrangements in the ground state of the triangular lattice was shown by G. H. Wannier. (see Fig. 3) It consists of rows of positive spins alternating with rows of negative spins as shown by Fig. 3. In our kagomé lattice, an example of arrangements is shown by Fig. 4. It also consists

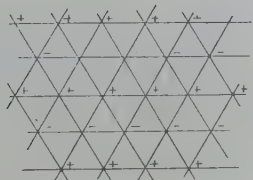


Fig. 3 Triangular lattice: One example of the arrangements of minimum energy after G. H. Wannier.

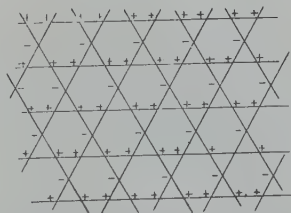


Fig. 4 A simple arrangement of minimum energy in the kagomé lattice. This arrangement is in correspondence with that of Fig. 3.

of rows of positive spins alternating with rows of negative spins. In these arrangements of the ground state, one of three directions of the lattice consists of rows of equal pair bonds and the other two directions consist of rows of unequal pair bonds. Now let us put $-J$ the interaction of equal pair bond and J that of unequal pair bond. Then, in the ground state, the binding energies of the triangular and kagomé lattices become as follows, respectively. (where, $J > 0$ in ferromagnetic case, $J < 0$ in antiferromagnetic case. $ZN/2$ is the total number of bonds.)

$$\begin{aligned} E_t(0) &= 1/2 \cdot ZN(2/3 \cdot J) + 1/2 \cdot ZN(-1/3 \cdot J) = 1/6 \cdot ZNJ = NJ \\ E_k(0) &= 1/6 \cdot ZNJ = 2/3 NJ \end{aligned} \quad (J < 0)$$

In the case of the ferromagnetism, these are equal to

$$\begin{aligned} E_t(0) &= 1/2 \cdot ZN(-J) = -3NJ \\ E_k(0) &= 1/2 \cdot ZN(-J) = -2NJ \end{aligned} \quad (J > 0)$$

In the antiferromagnetic case, therefore, the binding energy of the kagomé lattice is just one-third of the binding energy of its ferromagnetic case in the ground state as in that of the triangular-lattice. In spite of such a coincidence in the binding energies of triangular and kagomé lattices, the entropy of the kagomé lattice at absolute zero is larger than that of the triangular lattice. Nextly let us consider about it. Such an arrangement as shown by Fig. 3 and Fig. 4 is not likely to be ever realized because there are arrangements of much higher weight. In the system of the triangular lattice, one of examples that have much higher weight in the ground state was given by G. H. Wannier. (see Fig. 5) It consists of rows of alternate spins and each row may be laid independently. But the entropy of the arrangement is still zero because the weight is proportional to $N^{1/2}$ only. In the system of our kagomé lattice, however, such an arrangements as

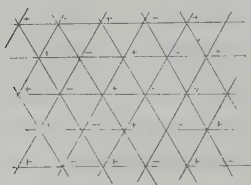


Fig. 5 An arrangement of minimum energy having medium high weight shown by G. H. Wannier. Each rows of alternating spins may be laid independently.

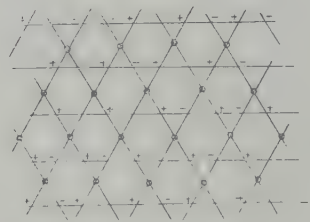


Fig. 6 An arrangement of minimum energy having medium high weight in the kagomé lattice. Each rows of alternating spins may be laid independently. Moreover there are the free spins of $N/3$.

consists of rows of alternate spins which may be laid independently each other is not similar as that of the triangular-lattice. (see Fig. 6) This arrangement has much higher weight because each one of encircled spins which locate among the two rows of alternate spins can reverse independently the sign without changing the energy. Thus we obtain a weight of $2^{N/3}$. But this is not yet the full weight because there is a large amount of "contingent freedom" which was called so by G. H. Wannier. Since the encircled spins can be varied independently, it will occur quite often that the two neighboring encircled spins have equal sign. Thus the non-encircled central-spin among the two neighboring encircled spins may be chosen at-random at one out of four positions without changing the binding energy. Hence there are another $N/6$ free spins added to the original number of $N/3$. Thus we have

$$S_k(0)/R > (1/3 + 1/6) \log 2 = 1/2 \cdot \log 2 \doteq 0.3466$$

from the consideration of an arrangement of Fig. 6 for the zero point entropy.

Now the arrangement of Fig. 6 shows that it corresponds to the division into the three sublattices of diamond as shown by Fig. 7. (these three sublattices are distinguished by \bullet , \circ , \triangle , respectively.) An arbitrary lattice point which belongs to one sublattice is encircled by four lattice points which belong to the other two sublattices. Thus the

arrangement of Fig. 6 shows that all lattice points which belong to one sublattice may be free independently each other if the other two sublattices have an arrangement of minimum energy. Now, as the triangular lattice is divided into three triangular sublattices, the kagomé lattice is also divided into three kagomé sublattices. (see Fig. 8). It forms the sublattices which is different from the three diamond sublattices as shown by Fig. 7. Thus all lattice points which belong to one kagomé sublattice may be free independently each other if the other two sublattices have an arrangement of minimum energy as shown by Fig. 9. Therefore there are the free spins of $N/3$ which belong to one sublattice. In this arrangement too, there are the ordered hexagons

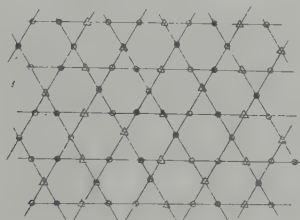


Fig. 8 Kagomé lattice is divided into the three kagomé sublattices by the different method from Fig. 7.

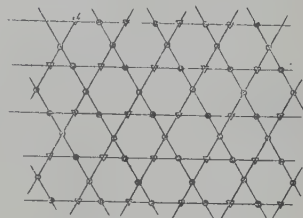


Fig. 7 Kagomé lattice is divided into the three diamond sublattices.

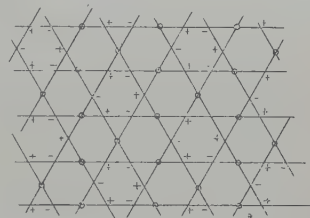


Fig. 9 An arrangement of minimum energy having finite entropy. There are the free spins of $N/3$ which belong to one kagomé sublattice. The lattice points of $2N/3$ which belong to the other two sublattices form the ordered hexagons of $1/9 \cdot N$. These hexagons individually have cyclic freedom.

of which have independent cyclic freedom. Thus we obtain a weight of $2^{N/3 + N/9}$. Moreover there are the contingent freedom of $6/N$ just as discussed in the arrangement of Fig. 6. Therefore the zero point entropy becomes as follows,

$$S_k(0)/R > \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{6} \right) \log 2 = \frac{11}{18} \log 2 \doteq 0.4236 .$$

Thus the qualitative discussion shows us that the zero point entropy of the kagomé lattice has larger value than that of the triangular lattice. The exact value of this entropy will be derived below.

Calculation of the Thermal Properties

Now let us put for brevity,

$$X = e^{\varepsilon L}, \quad [\omega] = \cos \omega_1 + \cos \omega_2 + \cos(\omega_1 + \omega_2).$$

Then the partition function; (36) becomes

$$\log \lambda_k = \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log 4 \left[\frac{1}{4} (X^6 + 18X^2 + 21X^{-2} + 24) - (X^4 - X^2 + X^{-2} - 1)[\omega] \right] \times d\omega_1 d\omega_2. \quad (37)$$

The energy per atom is given by the formula

$$E_k = -J \partial \log \lambda_k / \partial L, \quad L = J/kT. \quad (38)$$

Applying the formula (38) to (37), we find for the binding energy E_k

$$E_k / -J = \frac{1}{6} \left(\frac{b'}{b} \right) + \left(\frac{ba' - b'a}{6\pi \cdot b} \right) \int_{-1}^1 \frac{dx}{\sqrt{(a^2 - 2b^2) - 2b(a+b)x + b^2x^2} \sqrt{1-x^2}}.$$

The integral is a complete elliptic integral of the first kind. One can find the next formula in handbooks.

$$\int_{-1}^1 \frac{dx}{\sqrt{(a-x)(\beta-x)(\gamma-x)(x-\delta)}} = \frac{2K\left(\sqrt{\frac{(a-\beta)(\gamma-\delta)}{(a-\gamma)(\beta-\delta)}}\right)}{\sqrt{(a-\gamma)(\beta-\delta)}}. \quad (39)$$

This gives in our case

$$E_k / -J = \frac{1}{6} \left(\frac{b'}{b} \right) + \frac{1}{6\pi} \left(\frac{ba' - b'a}{b^2} \right) \frac{2K\left(\sqrt{\frac{4\sqrt{3+2(a/b)}}{(a/b)^2 + 2\sqrt{3+2(a/b)} - 3}}\right)}{\sqrt{(a/b)^2 + 2\sqrt{3+2(a/b)} - 3}}, \quad (40)$$

where

$$a = [X^6 + 18X^2 + 21X^{-1} + 24], \quad b = 4[X^4 - X^2 + X^{-2} - 1]$$

and

$$a' = \frac{\partial a}{\partial L}, \quad b' = \frac{\partial b}{\partial L}.$$

The curves which illustrate formula (40) are plotted in Fig. 10.

To obtain the entropy at absolute zero, we return to (37). The binding energy $E_k(0)$ in the ground state is

$$\begin{aligned} E_k(0) &= \lim_{L \rightarrow -\infty} (-J \partial \log \lambda_k / \partial L) \\ &= \lim_{L \rightarrow -\infty} \frac{-J}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \\ &\quad \times \frac{1/4(-84X^{-2}) - (-4X^{-2})[\omega]}{1/4(21X^{-2}) - (X^{-2})[\omega]} d\omega_1 d\omega_2 \\ &= (2/3)J. \quad (41) \end{aligned}$$

Therefore

$$-\frac{E_k(0)}{kT} = -\frac{2}{3}J/kT = -\frac{2}{3}L. \quad (42)$$

Remembering that

$$\log \lambda_k = -\frac{F_k}{kT} = -\frac{E_k}{kT} + \frac{S_k}{k}, \quad (43)$$

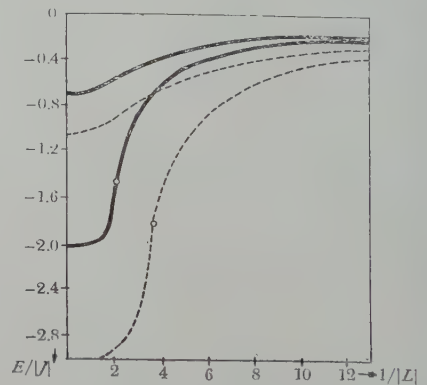


Fig. 10 Energy- $2S$ -temperature plot for kagomé Ising net. Ferromagnetic and anti-ferromagnetic coupling. The dotted curves are that of triangular Ising net.

we obtain the next expression from (42) and (43) at absolute zero.

$$\lim_{L \rightarrow -\infty} \log \lambda_k = -\frac{E_k(0)}{kT} + \frac{S_k(0)}{k} = -\frac{2}{3}L + \frac{S_k(0)}{k} \quad (44)$$

Drawing out $\frac{1}{6} \log X^{-2}$ from the integral of (37), we get

$$\begin{aligned} \log \lambda_k = & \frac{1}{6} \log X^{-2} + \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log 4 \left[\frac{1}{4} (X^8 + 18X^4 + 21 + 24X^2) \right. \\ & \left. - (X^6 - X^4 + 1 - X^2) [\omega] \right] d\omega_1 d\omega_2 \quad (45) \end{aligned}$$

Remembering that $X^{-1/3}$ equals $e^{-2/3L}$, we obtain at absolute zero

$$\lim_{L \rightarrow -\infty} \log \lambda_k = -\frac{2}{3}L + \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log(21 - 4[\omega]) d\omega_1 d\omega_2 \quad (46)$$

Comparing (44) with (46), we thus get the exact value of zero point entropy.

$$S_k(0)/R = \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log[21 - 4(\cos \omega_1 + \cos \omega_2 + \cos(\omega_1 + \omega_2))] d\omega_1 d\omega_2 \quad (47)$$

$$\doteq 0.50183.$$

In conclusion, we should like to express our sincerest thanks to Prof. Kōdi Husimi, Mr. Itirō Syōzi for their kind interest in this work.

Appendix I

The complete partition function of the anisotropic kagomé lattice which have the different interaction parameters: L_1 , L_2 and L_3 for the three directions of the lattice is obtained readily by generalizing our treatments in § 2, § 3. The result becomes as follows.

$$\begin{aligned} \log(\lambda_k/2) = & \frac{1}{24\pi^2} \int_0^{2\pi} \int_0^{2\pi} \log \frac{1}{4} \{ \text{ch}^2 2L_1 \text{ch}^2 2L_2 \text{ch}^2 2L_3 + \text{sh}^2 2L_1 \text{sh}^2 2L_2 \text{sh}^2 2L_3 \\ & + 2\text{ch} 2L_1 \text{ch} 2L_2 \text{ch} 2L_3 \text{sh} 2L_1 \text{sh} 2L_2 \text{sh} 2L_3 + \text{ch}^2 2L_1 + \text{ch}^2 2L_2 + \text{ch}^2 2L_3 \\ & - 2\text{ch} 2L_1 \text{sh} 2L_1 \text{sh} 2L_2 \text{sh} 2L_3 \cos \omega_1 - 2\text{ch} 2L_2 \text{sh} 2L_1 \text{sh} 2L_2 \text{sh} 2L_3 \cos \omega_2 \\ & - 2\text{ch} 2L_3 \text{sh} 2L_1 \text{sh} 2L_2 \text{sh} 2L_3 \cos(\omega_1 + \omega_2) - 2\text{ch} 2L_2 \text{ch} 2L_3 \text{sh}^2 2L_1 \cos \omega_1 \\ & - 2\text{ch} 2L_1 \text{ch} 2L_3 \text{sh}^2 2L_2 \cos \omega_2 - 2\text{ch} 2L_1 \text{ch} 2L_2 \text{sh}^2 2L_3 \cos(\omega_1 + \omega_2) \} d\omega_1 d\omega_2. \end{aligned}$$

Appendix II

The partition function of the kagomé lattice can be transformed from that of the triangular lattice which have been already treated.

Now let us consider the diced lattice as shown in Fig. 11. Let its interaction parameter be L^* . By summing at first over the spin variables with respect to the vertices

having three nearest neighbor atoms, we arrive at the partition function of the triangular lattice (Star triangle transformation), with an interaction parameter T ; in fine

$$\sum_{\mu_0=\pm 1} e^{L^* \mu_0 (\mu_1 + \mu_2 + \mu_3)} = 2 (\text{ch}^3 L^* \cdot \text{ch} 3 L^*)^{1/4} \cdot e^{\frac{T}{2} (\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1)}$$

$$e^{2T} = 2 \text{ch} 2L^* - 1.$$

Then there holds the relation between the partition function of the diced lattice $f_d(L^*)$ and that of the triangular lattice $f_t(T)$.

$$f_d(L^*) = 2^{2N/3} (\text{ch} 3L^* \cdot \text{ch}^3 L^*)^{N/6} \cdot f_t(T).$$

On the other hand, the diced lattice is evident the dual net of our kagomé lattice. (see Fig. 12).

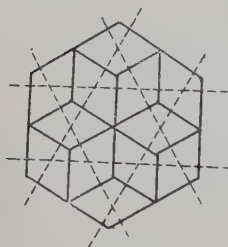


Fig. 12

where

Then the partition function of the kagomé lattice is connected to that of the diced lattice by

$$f_k(L) = \frac{1}{2} (\text{sh} 2L)^N f_d(L^*),$$

where

$$\text{sh} 2L \cdot \text{sh} 2L^* = 1.$$

Thus we have

$$f_k(L) = 2^{2N/3} (e^T \text{ch}^5 L \cdot \text{sh} L)^{N/3} \cdot f_t(T),$$

$$e^{2T} = 2 \coth 2L - 1.$$

The transition temperature of the kagomé lattice is given by $e^{4T_c} = 3 + 2\sqrt{3}$, corresponding to $e^{4T_c} = 3$ of the triangular lattice, which is coincide with (32).

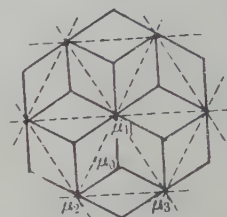


Fig. 11 Diced-lattice

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Miscellanea in Elementary Quantum Mechanics, III

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III. Uncertainty relations and ground states

On the ground that there exists an intimate correlation between the uncertainty relation and the ground state of a quantum-mechanical system, we present here variants of the usual uncertainty relation and a method of determining the ground states for a number of systems. Strangely enough, the arguments may be applied also to excited states.

§ 1. The harmonic oscillator

The Heisenberg uncertainty relation in its usual form, in natural units

$$\int_{-\infty}^{+\infty} x^2 |\psi|^2 dx \cdot \int_{-\infty}^{+\infty} \left| \frac{d\psi}{dx} \right|^2 dx \geq \frac{1}{4} \quad (1.1)$$

lies in a deep connection with the eigenvalue problem of a harmonic oscillator. For, if we put the Hamiltonian of the oscillator with its angular frequency 1 in the form

$$H = - (1/2m) d^2/dx^2 + (m/2) x^2,$$

then we know that the expectation value of H for any state function $\psi(x)$ is greater than the lowest eigenvalue $1/2$, that is the zero point energy;

$$\frac{1}{2m} \int_{-\infty}^{+\infty} \left| \frac{d\psi}{dx} \right|^2 dx + \frac{m}{2} \int_{-\infty}^{+\infty} x^2 |\psi|^2 dx \geq \frac{1}{2} \int_{-\infty}^{+\infty} |\psi|^2 dx. \quad (1.2)$$

The condition for this inequality to hold irrespective of (positive) values of m is just the standard uncertainty relation (1.1).

As for the proof of the inequality (1.2) one writes down, after H. Weyl⁽¹⁾, the trivial inequality

$$\int_{-\infty}^{+\infty} \left| \frac{d\psi}{dx} + mx\psi \right|^2 dx \geq 0, \quad (1.3)$$

then expands the square

$$\int \left| \frac{d\psi}{dx} \right|^2 dx + m \int x \frac{d|\psi|^2}{dx} dx + m^2 \int x^2 |\psi|^2 dx \geq 0,$$

and finally resorts to a partial integration in the middle integral. Once the inequality

(1.2) has been proved, then it establishes not only a lower bound $1/2$ for the possible eigenvalues of the oscillator, but also confirms that the lower bound $1/2$ is actually an eigenvalue. For the corresponding equality holds for a function ϕ_0 obeying the first order differential equation

$$d\phi_0/dx + mx\phi_0 = 0, \quad (1.4)$$

which immediately integrates itself to

$$\phi_0 \propto \exp(-mx^2/2), \quad (1.5)$$

the well-known ground state function.

It is a notable fact that there exists an inequality, analogous to (1.3), which leads to the excited states. Consider the inequality

$$\int_{-\infty}^{+\infty} \left| \frac{d\phi}{dx} + \left(-\frac{1}{x} + mx \right) \phi \right|^2 dx \geq 0, \quad (1.6)$$

where it is necessary for ϕ to vanish as x at the origin, because of the singular character of the integrand. This becomes

$$\begin{aligned} \int \left| \frac{d\phi}{dx} \right|^2 dx + \int \left(-\frac{1}{x} + mx \right) \frac{d|\phi|^2}{dx} dx + \int \left(-\frac{1}{x} + mx \right)^2 |\phi|^2 dx &\geq 0, \\ \int \left| \frac{d\phi}{dx} \right|^2 dx - \int \left(\frac{1}{x^2} + m \right) |\phi|^2 dx + \int \left(\frac{1}{x^2} + m^2 x^2 - 2m \right) |\phi|^2 dx &\geq 0, \\ \frac{1}{2m} \int \left| \frac{d\phi}{dx} \right|^2 dx + \frac{m}{2} \int x^2 |\phi|^2 dx &\geq \frac{3}{2} \int |\phi|^2 dx. \end{aligned} \quad (1.7)$$

Computed for any function with a node at the origin, the energy expectation value of the harmonic oscillator is always greater than $3/2$. The corresponding uncertainty relation runs

$$\int \left| \frac{d\phi}{dx} \right|^2 dx \cdot \int x^2 |\phi|^2 dx \geq \left(\frac{3}{2} \right)^2. \quad (1.8)$$

The inequality becomes an equality for the eigenfunction ϕ_1 defined by

$$d\phi_1/dx + (-1/x + mx)\phi_1 = 0, \quad (1.9)$$

that is

$$\phi_1 \propto x \exp(-mx^2/2). \quad (1.10)$$

In the same way we can derive the second excited state of the oscillator. We consider the inequality

$$\int_{-\infty}^{+\infty} \left| \frac{d\phi}{dx} + \left(-\frac{1}{x-a} - \frac{1}{x+a} + mx \right) \phi \right|^2 dx \geq 0,$$

and adjust the parameter a in such a way that in the resulting potential energy the singular part $\propto (x^2 - a^2)^{-1}$ evanesces out. We get in this way $a = (2m)^{-1/2}$. For any

function ψ with two nodes at $x = \pm (2m)^{-1/2}$, we have

$$\frac{1}{2m} \int \left| \frac{d\psi}{dx} \right|^2 dx + \frac{m}{2} \int x^2 |\psi|^2 dx \geq \frac{5}{2} \int |\psi|^2 dx.$$

The corresponding uncertainty relation cannot be obtained in the manner described above, since ψ should depend on the parameter m .

It is fairly obvious how to invent inequalities which lead higher excited states.

One might object against the inequality (1.7) for an arbitrary wave function ψ with a sole condition of vanishing at the origin. For, under that condition, the function ψ may have as a component the ground state wave function ψ_0 and the energy expectation value may well lie below the first excited level. Nevertheless the inequality is valid, as can be seen by an example: $\psi \propto x^2 \exp(-mx^2/2)$ is a linear combination of ψ_0 and ψ_2 , the energy expectation value for this ψ is easily found to be $11/6$, which is larger than $3/2$. In fact we find that the mixing weights of ψ_0 and ψ_2 in this function is $1:2$, so that the energy is $(1/3)(1/2) + (2/3)(5/2) = 11/6$. A node at the origin does not exclude the intervention of the ground state, but makes the excited states appear with sufficiently large weights.

§ 2. The hydrogen atom

We now attempt to formulate the corresponding theory for the hydrogen atom problem. Let us start from the inequality:

$$\int |\text{grad } \psi + Z \text{ grad } r \cdot \psi|^2 dv \geq 0 \quad (2.1)$$

that is

$$\int |\text{grad } \psi|^2 dv + Z \int (\text{grad } r \cdot \text{grad } |\psi|^2) dv + Z^2 \int (\text{grad } r)^2 |\psi|^2 dv \geq 0.$$

Since we have

$$(\text{grad } r)^2 = 1, \quad \Delta r = 2/r, \quad (2.2)$$

the inequality becomes on a partial integration

$$\frac{1}{2} \int |\text{grad } \psi|^2 dv - \int \frac{Z}{r} |\psi|^2 dv \geq -\frac{Z^2}{2} \int |\psi|^2 dv. \quad (2.3)$$

On the left we find the Hamiltonian of the hydrogen atom with a nuclear charge Z , all in atomic units. The lower bound $-Z^2/2$ is actually reached by the ground state wave function

$$\psi_0 \propto \exp(-Zr), \quad (2.4)$$

being the unique solution of the system of first order differential equations

$$\text{grad } \psi + Z \text{ grad } r \cdot \psi = 0. \quad (2.5)$$

Since the parameter Z is arbitrary in the inequality (2.3), it entails a sort of uncertainty relation

$$\int |\text{grad } \psi|^2 dv \cdot \int |\psi|^2 dv \geq \left(\int \frac{1}{r} |\psi|^2 dv \right)^2. \quad (2.6)$$

Let us write T and U for the expectation values of the kinetic energy and the repulsive Coulomb energy for a unit charge. Then (2.6) may be expressed as

$$2T \geq U^2. \quad (2.7)$$

When this relation is known a priori, it is easy to derive from it the ground state energy. For the energy

$$E = T - ZU \geq (1/2)U^2 - ZU = (1/2)(U - Z)^2 - Z^2/2 \quad (2.8)$$

will attain its minimum value

$$E_0 = -Z^2/2$$

for

$$U_0 = Z \quad (-ZU_0 = -Z^2, T_0 = Z^2/2).$$

One sees that the virial theorem is verified for the ground state.

The lowest p -states can be obtained in the following manner: let us envisage the inequality

$$\int |\text{grad } \psi - \frac{\mathbf{i}}{x} \psi + \frac{Z}{2} \text{grad } r \cdot \psi|^2 dv \geq 0, \quad (2.9)$$

for a wave function ψ with a nodal plane $x=0$. \mathbf{i} is the unit vector directed along the positive x -axis, the coefficients -1 and $Z/2$ are so chosen that the resulting inequality becomes to be of the desired type. (2.9) can be transformed by a partial integration into

$$\frac{1}{2} \int |\text{grad } \psi|^2 dv - \int \frac{Z}{r} |\psi|^2 dv \geq -\frac{Z^2}{8} \int |\psi|^2 dv. \quad (2.10)$$

Thus the lowest energy of the states with a nodal plane is given by

$$-Z^2/8 = -(Z/2)^2/2, \quad (2.11)$$

and the corresponding eigenfunction is of the type

$$\psi \propto x \exp(-Zr/2) \quad (2.12)$$

obtained as the solution of the system of differential equations

$$\text{grad } \psi - (\mathbf{i}/x) \psi + (Z/2) \text{grad } r \cdot \psi = 0. \quad (2.13)$$

The nodal plane may be any plane through the origin and among the infinite variety of such functions three may be chosen as a linear basis.

From (2.10) owing to the arbitrariness of Z we get the uncertainty relation

$$\int |\text{grad } \phi|^2 dv \cdot \int |\phi|^2 dv \geq 4 \left(\int \frac{1}{r} |\phi|^2 dv \right)^2, \quad (2.14)$$

and (2.7) is replaced by

$$T \geq 2U^2$$

and again the virial theorem is seen to hold :

$$U_0 = Z/4, \quad -ZU_0 = -Z^2/4, \quad T_0 = Z^2/8, \quad E_0 = T_0 - ZU_0 = -Z^2/8.$$

We now proceed to get the lowest d -states. It is not difficult to see that an appropriate inequality is given by

$$\int \left| \text{grad } \phi - \frac{i}{x} \phi - \frac{j}{y} \phi + \frac{Z}{3} \text{grad } r \cdot \phi \right|^2 dv \geq 0 \quad (2.15)$$

to be computed for a wave function with two mutually orthogonal nodal planes $x=0$, $y=0$. For such functions we have

$$\frac{1}{2} \int |\text{grad } \phi|^2 dv - \int \frac{Z}{r} |\phi|^2 dv \geq -\frac{1}{2} \left(\frac{Z}{3} \right)^2 \int |\phi|^2 dv. \quad (2.16)$$

The lowest energy

$$-(Z/3)^2/2 \quad (2.17)$$

is attained by functions of the type

$$\phi \propto xy \exp(-Zr/3), \quad (2.18)$$

from which the complete set of $3d$ -functions can be obtained by rotations of the coordinate system.

In general the lowest l -states can be determined from the initial inequality

$$\int \left| \text{grad } \phi - \sum_{\alpha=1}^l \frac{i_{\alpha}}{x_{\alpha}} \phi + \frac{Z}{l+1} \text{grad } r \cdot \phi \right|^2 dv \geq 0, \quad (2.19)$$

where i_{α} 's are unit vectors on the xy -plane around the origin, each making an angle $(\alpha-1)\pi/l$ with the x -axis and $x_{\alpha} = (r \cdot i_{\alpha})$ is the orthogonal projection of the radius vector r on to the unit vector i_{α} . It gives rise to the following inequality

$$\frac{1}{2} \int |\text{grad } \phi|^2 dv - \int \frac{Z}{r} |\phi|^2 dv \geq -\frac{1}{2} \left(\frac{Z}{l+1} \right)^2 \int |\phi|^2 dv, \quad (2.20)$$

provided that we have the equality

$$\sum_{(\alpha, \beta)} \frac{(i_{\alpha} \cdot i_{\beta})}{x_{\alpha} x_{\beta}} = 0, \quad \text{or} \quad \sum (i_{\alpha} \cdot i_{\beta}) (i_{\gamma} \cdot r) (i_{\delta} \cdot r) \dots (i_{\lambda} \cdot r) = 0. \quad (2.21)$$

This equation is equivalent to

$$\frac{1}{2} \sum_{\alpha=1}^l i_{\alpha} (i_{\alpha} \cdot r) = 0, \quad (2.22)$$

so that we have to prove the product $\prod_{\alpha=1}^l (\mathbf{i}_\alpha \cdot \mathbf{r})$ is a harmonic function, particularly a harmonic function in the $x'y$ -plane. Introducing the polar coordinates ρ, φ in the plane, the product becomes

$$\rho^l \prod_{\alpha=1}^l \cos(\varphi - (a-1)\pi/l).$$

The cosine product is a periodic function of φ with period $2\pi/l$ and is odd or even according as l is even or odd, so that it is fairly obvious that the cosine product is proportional to $\sin l\varphi$, if l is even, and to $\cos l\varphi$, if l is odd. Exact formulae are*

$$\prod_{\alpha=1}^l \cos\left(\varphi - \frac{(a-1)\pi}{l}\right) = \begin{cases} (-1)^{(l+1)/2} 2^{1-l} \cos l\varphi, & l: \text{odd.} \\ (-1)^{l/2} 2^{1-l} \sin l\varphi, & l: \text{even.} \end{cases} \quad (2.23)$$

That $\rho^l \cos l\varphi$ or $\rho^l \sin l\varphi$ is harmonic needs no comments.

In this way we have found a lowest l -state in the form

$$\psi \propto \prod_{\alpha=1}^l (\mathbf{i}_\alpha \cdot \mathbf{r}) \exp\{-Zr/(l+1)\} \quad (2.24)$$

with the energy

$$- \{Z/(l+1)\}^2/2, \quad (2.25)$$

the lower bound of the energy expectation value for wave functions with l symmetrically arranged nodal planes.

The $2s$ state wave function emerges from the consideration of inequalities of the type :

$$\int |\text{grad } \psi + \left(-\frac{1}{r-a} + \frac{Z}{2}\right) \text{grad } r \cdot \psi|^2 dv \geq 0.$$

One finds immediately that the parameter a should be chosen to be equal to $2/Z$, if one wants to have a pure Coulomb field. $-(1/2)(Z/2)^2$ is the lower bound of the energy

* A simple proof of these formulae has kindly been afforded us by Mr. Masamoto Ôtuka: Let z be a complex variable and consider the equation

$$\begin{aligned} z^l - e^{2il\varphi} &= \prod_{k=0}^{l-1} (z - e^{2i(\varphi-k\pi/l)}) \\ &= \prod_{k=0}^{l-1} e^{i(\varphi-k\pi/l)} \cdot \prod_{k=0}^{l-1} (ze^{-i(\varphi-k\pi/l)} - e^{i(\varphi-k\pi/l)}) \\ &= e^{il\varphi - (l-1)\pi/2} \cdot \prod_{k=0}^{l-1} (ze^{-i(\varphi-k\pi/l)} - e^{i(\varphi-k\pi/l)}). \end{aligned}$$

Then, if $z = -1$, we have

$$\begin{aligned} (-1)^l - e^{2il\varphi} &= e^{il\varphi} e^{-i(l-1)\pi/2} (-1)^l 2^l \prod_{k=0}^{l-1} \cos(\varphi - k\pi/l), \\ e^{il\varphi} - (-1)^l e^{-2il\varphi} &= e^{-i(l-1)\pi/2} (-1)^{l-1} 2^l \prod_{k=0}^{l-1} \cos(\varphi - k\pi/l). \end{aligned}$$

expectation value for functions with the spherical nodal surface $r=2/Z$ and is realized for the function

$$\psi_{2s} \propto (r-2/Z)\exp(-Zr/2).$$

In this way any eigenstate of the hydrogen-like problem can be obtained by invention of suitably chosen inequalities.

§ 3. The rotating oscillator

We can slightly generalize the inequality of the foregoing section. Let us consider with $\lambda > 0$

$$\int |\text{grad } \phi + \text{grad}(-\lambda \log r + Cr) \cdot \phi|^2 d\tau \geq 0. \quad (3.1)$$

This becomes

$$\frac{1}{2} \int |\text{grad } \phi|^2 d\tau + \int \left\{ \frac{\lambda(\lambda+1)}{2r^2} - \frac{(\lambda+1)C}{r} \right\} |\phi|^2 d\tau \geq -\frac{C^2}{2} \int |\phi|^2 d\tau. \quad (3.2)$$

Thus the energy expectation value of an electron within a central field: the Coulomb attraction

$$-Z/r$$

plus the inverse cube force repulsive potential

$$\lambda(\lambda+1)/2r^2, \quad (3.3)$$

is always greater than

$$-Z^2/2(\lambda+1)^2, \quad (3.4)$$

this minimum value being attained by the ground state wave function

$$\phi_0 \propto r^\lambda \exp\{-Zr/(\lambda+1)\}. \quad (3.5)$$

Such a system is designated by A. Sommerfeld⁽²⁾ as rotating oscillator.

It is by no means necessary to impose the condition $\lambda > 0$ so far as it concerns with the inequality (3.2) for any function ϕ which makes the left-hand side of (3.2) convergent. In the critical case $\lambda = -1/2$ the ground state wave function ϕ_0 makes the energy expectation value indefinite. Too strong an attractive inverse cube force potential ($\lambda(\lambda+1) < -1/4$) cannot be included in our scheme.

Let us write the inequality (3.2) in the following form with obvious abbreviations:

$$2T + \lambda(\lambda+1)U_{-2} - 2(\lambda+1)CU_{-1} + C^2 \geq 0. \quad (3.2a)$$

The left-hand side is a quadratic expression in C and λ and we can derive the corresponding uncertainty relation:

$$2T \geq U_{-2}^2/4(U_{-2} - U_{-1}^2). \quad (3.6)$$

This implies the less strict inequality

$$2T \geq U_{-2}/4. \quad (3.6a)$$

and also previous inequality (2.7), since (3.6) may be written

$$2T - U_{-1}^2 \geq (4T - U_{-2})^2/8T. \quad (3.6b)$$

Hence for the ground state of hydrogen-like atom we have $U_{-2} = 2Z^2$ as much as $U_{-1} = Z$.

We may regard the repulsive potential (3.3) with $\lambda = l$ as a centrifugal potential for an l -state. The expectation value of the energy of the hydrogen atom for an l -state $\psi = R(r)Y_l(\theta, \varphi)$ is

$$\int \psi^* \left(-\frac{1}{2} \Delta - \frac{Z}{r} \right) \psi d\tau = \int R^* \left\{ -\frac{1}{2} \frac{1}{r} \frac{\partial^2}{\partial r^2} r + \frac{l(l+1)}{2r^2} - \frac{Z}{r} \right\} R d\tau,$$

if Y_l is normalized according to $\int |Y_l|^2 d\omega = 4\pi$. Thus the left-hand side of (3.2) evaluated for a spherically symmetric wave function $\psi = R(r)$ is equal to the above expectation value and we are led to conclude that the lowest l -state of the hydrogen atom is

$$R_{l0}(r)Y_l \propto r^l \exp\{-Zr/(l+1)\} \cdot Y_l, \quad (3.7)$$

where R_{l0} is obtained as solution of the first order differential equation

$$dR/dr + (-l/r + Z/(l+1))R = 0, \quad (3.8)$$

and the corresponding eigenvalue is

$$-Z^2/(l+1)^2. \quad (3.9)$$

If we start from the inequality ($D > 0$)

$$\int |\text{grad } \psi + \text{grad} \left(-\lambda \log r + \frac{1}{2} D r^2 \right) \cdot \psi|^2 d\tau \geq 0, \quad (3.10)$$

we are led to

$$\frac{1}{2} \int |\text{grad } \psi|^2 d\tau + \int \left\{ \frac{\lambda(\lambda+1)}{2r^2} + \frac{D^2}{2} r^2 \right\} |\psi|^2 d\tau \geq \left(\lambda + \frac{3}{2} \right) D \int |\psi|^2 d\tau. \quad (3.11)$$

This shows that the ground state energy of an electron in the central field

$$\lambda(\lambda+1)/2r^2 + (1/2)D^2r^2 \quad (3.12)$$

is

$$E_0 = (2\lambda + 3)D/2 \quad (3.13)$$

(in units where $m=1$, $\hbar=1$) and the corresponding wave function is

$$\psi_0 \propto r^\lambda \exp(-Dr^2/2). \quad (3.14)$$

Or, alternatively, the lowest l -state of an isotropic harmonic oscillator in three dimensions is given by

$$E_0 = (l + 3/2)D, \quad \phi_0 \propto r^l \exp(-Dr^2/2) \cdot Y_l. \quad (3.15)$$

The formula may also be applied to the problem of an electron within a uniform magnetic field.

The uncertainty relation derived from (3.11) runs

$$2T \geq \frac{U_{-2}(U_2 U_{-2} - 3/4)}{4(U_2 U_{-2} - 1)} = \frac{1}{4} U_{-2} + \frac{1}{16} \frac{U_{-2}}{U_2 U_{-2} - 1}. \quad (3.16)$$

§ 4. Applications of the uncertainty relations

We first establish a theorem on the many center problem. Let the energy of the electron in the field of many point charges $Z_i (i=1, 2, \dots)$ be

$$E = T - \sum_i Z_i U_i, \quad (4.1)$$

where

$$U_i = \int \frac{1}{|\mathbf{r} - \mathbf{r}_i|} |\phi|^2 dv$$

and \mathbf{r}_i stands for the position of the i -th charge Z_i . The uncertainty relation (2.7) applies to each and every charge:

$$2T \geq U_i^2 \quad (i=1, 2, \dots).$$

We have then, for a suitably chosen system of weight factors p_i ($\sum p_i = 1$, $p_i > 0$),

$$\begin{aligned} E &= \sum_i (p_i T - Z_i U_i) \\ &\geq \sum (1/2 \cdot p_i U_i^2 - Z_i U_i) \\ &\geq \sum 1/2 \cdot p_i (U_i - Z_i/p_i)^2 - \sum Z_i^2/2p_i \\ &\geq -\sum Z_i^2/2p_i. \end{aligned}$$

The lower bound thus obtained will be maximum for

$$p_i = Z_i / \sum_k Z_k.$$

Thus we have established

$$E \geq -(\sum_i Z_i)^2/2, \quad (4.2)$$

or, in words, the energy of the many center problem is always greater than the lowest energy of one center problem, where all the charges Z_i coalesce to a single charge ($\sum_i Z_i$).

As a second application of our uncertainty relation we consider the helium-like atom. The energy expectation value is of the form

$$E = T_1 + T_2 - ZU_1 - ZU_2 + U_{12}, \quad (4.3)$$

where the subscripts refer to the first and the second electron, and

$$U_{12} = \iint \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} |\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 d\mathbf{r}_1 d\mathbf{r}_2$$

is the interaction energy of the two electrons. Our uncertainty relation (2.7) may also be applied to U_{12} , if the position of the second electron is regarded as a parameter and a due account is paid for the "linearity" of our original inequality (2.3). Thus either

$$2T_1 \geq U_{12}^2 \quad \text{or} \quad 2T_2 \geq U_{12}^2. \quad (4.4)$$

Now we take for $\psi(\mathbf{r}_1, \mathbf{r}_2)$ a product of hydrogenic wave functions of an effective charge ζ , as a trial function in the sense of the variation principle. Then

$$2T_1 = 2T_2 = \zeta^2, \quad U_1 = U_2 = \zeta, \quad U_{12} \leq \zeta,$$

and we get an *upper* bound

$$E < \zeta^2 - 2Z\zeta + \zeta = (\zeta - Z + 1/2)^2 - (Z - 1/2)^2,$$

the optimum value being

$$E < - (Z - 1/2)^2 \quad (4.5)$$

with the effective charge

$$\zeta = Z - 1/2. \quad (4.6)$$

Our screening constant $1/2$ comes out too large, as compared with the usually accepted value, because of our admittedly rough estimation of U_{12} . Our uncertainty relation is not precise enough to be able to predict the electron affinity of the hydrogen ($Z=1$), although this still makes the effective charge ζ positive.

Since our inequality is established for an attractive Coulomb field, it is natural that it is far from an equality when applied to the repulsive case. It is highly desirable to improve our inequality in this respect, just as we have done for wave functions with nodal surfaces.

§ 5. The general observation

By good fortune we have been able in §§ 1, 2 and 3 to devise appropriate inequalities, which lead to the physically interesting potentials. Apart from the chance successes, the essential points of our foregoing considerations will be the following:

"Let

$$\psi_0 \propto \exp(-\varphi_0) \quad (5.1)$$

be a essentially positive and bounded function satisfying appropriate boundary conditions. Then it represents the *ground state* of a particle immersed within the potential given by

$$V \equiv (1/2) \{(\text{grad } \varphi_0)^2 - \Delta \varphi_0\} \quad (= \Delta \psi_0 / 2\psi_0), \quad (5.2)$$

and the corresponding ground state energy is zero."

For the proof we consider the inequality

$$\int |\text{grad } \psi + \text{grad } \varphi_0 \cdot \psi|^2 dv \geq 0, \quad (5.3)$$

which, on a partial integration, reduces to

$$\frac{1}{2} \int |\text{grad } \psi|^2 dv + \int V \cdot |\psi|^2 dv \geq 0. \quad (5.4)$$

The left-hand side is the energy expectation value of the particle in the potential V , for which we have established the lower limit zero. This limit is reached by the wave function ψ_0 , from which we have started.

It will be a non-sense, if one tries to obtain a φ_0 (or ψ_0) satisfying the eq. (5.2) for a potential V , a priori given. It amounts to solve the Schrödinger equation in the ordinary manner. It is here proposed to reverse the whole attitude; the solution is first given and the corresponding problem is established a fortiori. Practical applications of this idea will be given in later sections.

Mathematically the above theorem, though simple and rather trivial, has a remarkable consequence. The ground state wave function ψ_0 , which can be given arbitrary except the general conditions imposed above, determines the whole set of the excited states uniquely.

It is often stated that the complete set of orthogonal functions and the associated eigenvalues determines a Hamiltonian operator. But this is so only when one deals with Hermite operators in their abstract and widest sense. If the Hamiltonian should be a second order differential operator of the standard form, then our theorem asserts it is completely determined by its ground state function only. Likewise an arbitrarily given wave function ψ_1 with a nodal surface represents an excited state (but has the lowest energy among functions with the same nodal surface) and the corresponding potential is uniquely determined.

Thus it is not in general possible to prescribe the ground and excited states independently. Nevertheless examples will be given, in which the ground state and the first excited state with a common potential are given a priori. This is done by introducing a sufficient number of undetermined parameters into the functions.

Now confining ourselves to the one-dimensional case, we ascertain the following theorem:

"If there exists the eigenstate wave function ψ_k with k nodes ($k=0, 1, 2, \dots$) for an one-particle Schrödinger equation with a potential of finite depth, then its energy eigenvalue E_k is always higher (or lower) than that of the eigenfunction with less (or more) number of nodes. Based on the lemma that, on account of the above general consideration, the energy eigenvalue is lower than the energy expectation value for any wave function ψ satisfying appropriate boundary conditions and vanishing at the k points $\xi_1^{(k)}, \xi_2^{(k)}, \dots, \xi_k^{(k)}$ which represent the k nodes of ψ_k respectively, a proof will be given below.

- i) Case $k=0$. It is obvious from the lemma that E_0 is the lowest of all.
- ii) Case $k=1$. Linearly combining ψ_0 with ψ_j ($j=2$ or $3, 4, \dots$) we consider the wave function ψ vanishing at $\xi^{(1)}$, i.e. the node of ψ_1 :

$$\psi = a\psi_0 + b\psi_j, \quad (5.5)$$

$$\psi(\xi^{(1)}) = 0 = a\psi_0(\xi^{(1)}) + b\psi_j(\xi^{(1)}), \quad (5.6)$$

and the normalization condition is

$$|a|^2 + |b|^2 = 1. \quad (5.7)$$

(a) If $a=0$ ($b=1$), ψ becomes identical with ψ_j . So $E_1 < E_j$ from the lemma. (This is the case when the potential is symmetrical and j is odd.)

(b) If $0 < |b| < 1$, then the energy expectation value for ψ is

$$E = |a|^2 E_0 + |b|^2 E_j < E_j,$$

where the last inequality holds from (i) and (5.7). On the other hand $E_1 < E$ from the lemma, thus $E_1 < E_j$.

(c) The case $b=0$ is forbidden, since ψ_0 vanishes nowhere.

iii) Case $k=2$. In a similar way, we consider

$$\psi = a\psi_0 + b\psi_1 + c\psi_j \quad (j=3 \text{ or } 4, 5, \dots) \quad (5.8)$$

under the conditions

$$\psi(\xi_1^{(2)}) = 0 = a\psi_0(\xi_1^{(2)}) + b\psi_1(\xi_1^{(2)}) + c\psi_j(\xi_1^{(2)}), \quad (5.9)$$

$$\psi(\xi_2^{(2)}) = 0 = a\psi_0(\xi_2^{(2)}) + b\psi_1(\xi_2^{(2)}) + c\psi_j(\xi_2^{(2)}), \quad (5.10)$$

and

$$|a|^2 + |b|^2 + |c|^2 = 1. \quad (5.11)$$

(a) If $a=b=0$ ($c=1$), the proof is done in the same way as in the case (ii a).

(b) If $0 < |c| < 1$, the energy expectation value is

$$E = |a|^2 E_0 + |b|^2 E_1 + |c|^2 E_j < E_j,$$

with the help of (i), (ii) and (5.11). It follows that $E_2 < E_j$, since the lemma assures $E_2 < E$.

(c) The case $c=0$ never occurs. If it did, the wave function ψ would have the same nodes as the eigenfunction ψ_2 (but the former is not identical with the latter because of linear independence of the eigenfunctions). Then, from the lemma, its energy expectation value E would be by no means lower than the eigenvalue E_2 , while the direct calculation shows

$$E = |a|^2 E_0 + |b|^2 E_1$$

and

$$E < E_1 < E_2$$

from (ii).

iv) The proof can be easily extended for any value of k .

§ 6. One-dimensional examples: Double minima. Band edges.

First we want to invent a potential with symmetrical double minima. The corresponding ground state wave function will, we may suppose, have symmetrical double maxima.

Accordingly we put

$$\psi_0 \propto (a^2 + x^2)^k \exp(-x^2/2). \quad (6.1)$$

The formula (5.2) leads to the following potential

$$V_0(x) = \frac{x^2}{2} + \frac{k(2k-1+2a^2)}{a^2+x^2} - \frac{2k(k-1)a^2}{(a^2+x^2)^2}, \quad (6.2)$$

and the ground state energy is

$$E_0 = 1/2 + 2k. \quad (6.3)$$

The calculations we have experienced in deriving the above potential suggest that a potential of the same analytical form results from the following odd function

$$\psi_1 \propto x(a^2 + x^2)^l \exp(-x^2/2), \quad (6.4)$$

which in fact leads to the potential

$$V_1(x) = \frac{x^2}{2} + \frac{l(2l+1+2a^2)}{a^2+x^2} - \frac{2l(l-1)a^2}{(a^2+x^2)^2}, \quad (6.5)$$

and the energy

$$E_1 = 3/2 + 2l. \quad (6.6)$$

Now we adjust our parameters k and l in such a way that the two potentials V_0 and V_1 coincide with each other:

$$k(2k-1+2a^2) = l(2l+1+2a^2),$$

$$k(k-1) = l(l-1).$$

Apart from the trivial case $k=l=0$, we get $k+l=1$, that is

$$k = (1+\beta)/2, \quad l = (1-\beta)/2, \quad (6.7)$$

where

$$\beta = \{2(1+a^2)\}^{-1}. \quad (6.8)$$

In this we have a potential with double minima

$$V(x) = \frac{x^2}{2} + \frac{(1-\beta)(1-\beta^2)}{2\beta(a^2+x^2)} + \frac{(1-\beta^2)a^2}{2(a^2+x^2)^2}, \quad (6.9)$$

for which the *ground* and the *first* excited states are given by the simple expressions:

$$\left. \begin{aligned} E_0 &= 3/2 + \beta, & \psi_0 &\propto (a^2 + x^2)^{(1+\beta)/2} \exp(-x^2/2), \\ E_1 &= 5/2 - \beta, & \psi_1 &\propto x(a^2 + x^2)^{(1-\beta)/2} \exp(-x^2/2). \end{aligned} \right\} \quad (6.10)$$

In the limit $a \rightarrow \infty$ ($\beta \rightarrow 0$) we get simply the harmonic oscillator. In the other extreme case $a \rightarrow 0$ ($\beta \rightarrow 1/2$) we have again the analytically soluble case:

$$V = x^2/2 + 3/8x^2$$

with

$$E_0 = 2, \quad \psi_0 \propto |x|^{3/2} \exp(-x^2/2),$$

$$E_1 = 2, \quad \psi_1 \propto \operatorname{sgn}(x) \cdot |x|^{1/2} \exp(-x^2/2).$$

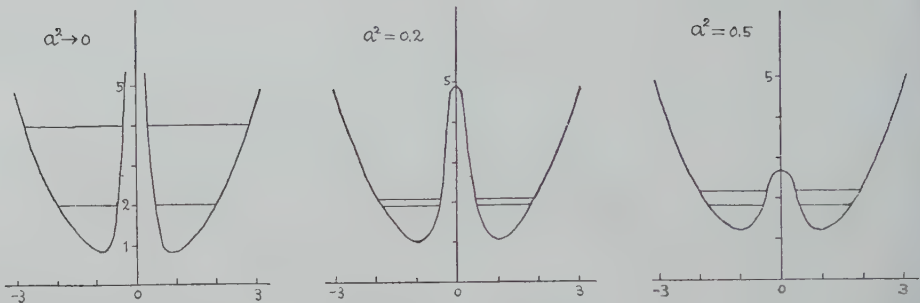


Fig. 1. Double minimum potential (6.9) and its ground

In this case there occurs a degeneration. The positive and negative domains are analytically disconnected and our eigenfunctions are quite artificial. Our potential (6.9) for different values of the parameter α (or β) and the corresponding levels are illustrated in Fig. 1. Further we have checked the validity of the first order perturbation calculation starting from the harmonic case ($\beta=0$), the trend of the ground state energy being compared with the exact one (6.10) in Fig. 2.

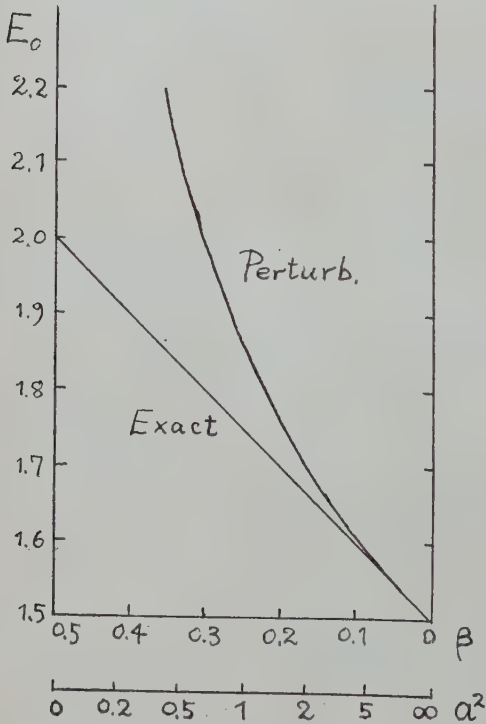


Fig. 2. Ground state energy as function of parameter (6.10): Exact and calculated values as first order perturbation.

Secondly we have constructed an example of the band theory. As well known, the lower edge of a band is represented by a periodic wave function with the same period as that of the potential and the upper edge by a periodic wave function with the double period, alternating its sign. With this circumstance in mind we tentatively put

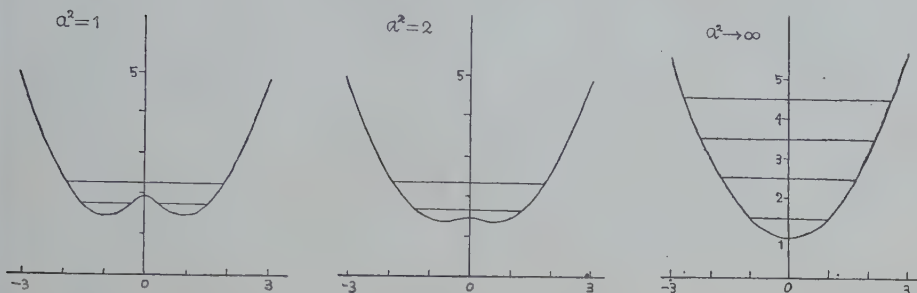
$$\left. \begin{aligned} \psi_0 &\propto (A + \cos x)^{\lambda} \exp(\mu \cos x), \\ \psi_1 &\propto \cos(x/2) \cdot (A + \cos x)^{\mu} \exp(\lambda \cos x). \end{aligned} \right\} \quad (6.11)$$

These wave functions give rise to the potentials of the following form :

$$2V = \frac{a}{(A + \cos x)^2} + \frac{b}{A + \cos x} + c \cos x - a^2 \cos^2 x. \quad (6.12)$$

The conditions for the equality of the coefficients a , b and c for the two potentials uniquely determine our parameters A , λ , μ :

$$\left. \begin{aligned} \lambda &= 3/4, & \mu &= 1/4, \\ A &= 1/4a - 1, \end{aligned} \right\} \quad (6.13)$$



and first excited levels for different values of parameter a .

(the other possible case $A=1$ is abandoned because of the infinity appearing in the potential.) The only remaining parameter a is restricted by the condition $A > 1$, i.e.

$$0 < a < 1/8. \quad (6.14)$$

Our conclusion is this : in the periodic potential given by

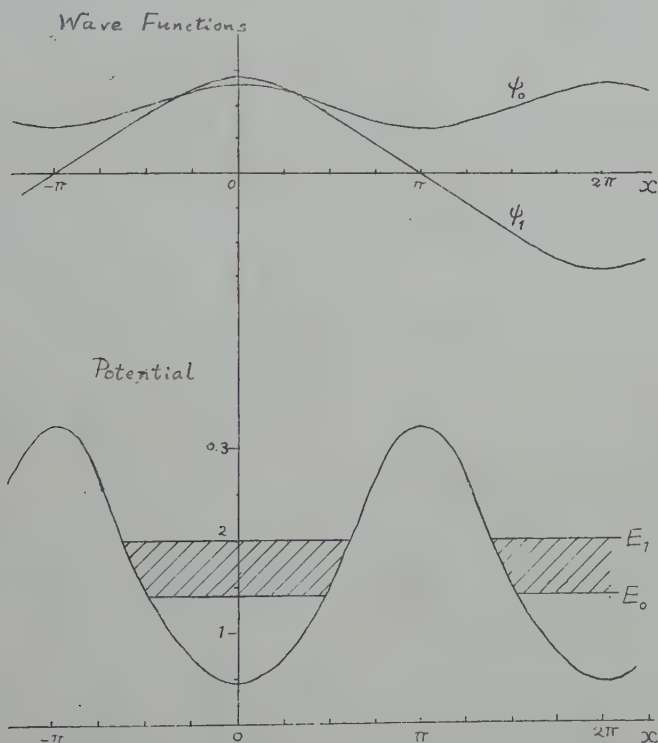


Fig. 3. Periodic potential (6.15) with $a=1/16$, its band edges and corresponding wave functions.

$$V = \frac{3}{64a} \left(\frac{1}{8a} - 1 \right) \left\{ \frac{1}{4a} - 1 + \cos x \right\}^{-2} + \frac{3}{16} \left\{ \frac{1}{4a} - 1 + \cos x \right\}^{-1} - \frac{3}{4} a \cos x - \frac{1}{2} a^2 \cos^2 x, \quad (6.15)$$

the edges of the first band are represented as follows:

$$\left. \begin{aligned} E_0 &= \frac{1}{2} \left\{ \frac{3}{4} - \left(a - \frac{3}{4} \right)^2 \right\}, \\ \psi_0 &\propto \left\{ \frac{1}{4a} - 1 + \cos x \right\}^{3/4} \exp(a \cos x), \\ E_1 &= \frac{1}{2} \left\{ \frac{1}{2} - \left(a + \frac{1}{4} \right)^2 \right\}, \\ \psi_1 &\propto \cos \frac{x}{2} \cdot \left\{ \frac{1}{4a} - 1 + \cos x \right\}^{1/4} \exp(a \cos x), \end{aligned} \right\} \quad (6.16)$$

and the band width is

$$E_1 - E_0 = 1/8 - a.$$

The potential, the band edges and the wave functions are illustrated in Fig. 3, for $a = 1/16$.

§ 7. The Coulomb repulsion + the harmonic attraction in three dimensions

The only problem of two bodies which can be reduced to the one body problem is the case of two electrons immersed in a common harmonic potential with an interaction depending only on their distance r_{12} . Suggested from an approximate solution of the helium problem we put as our starting wave function (with $c > 0$)

$$\psi_0 \propto \exp \{ -\omega(r_1^2 + r_2^2)/2 \} \cdot (1 + cr_{12}). \quad (7.1)$$

Then the corresponding potential derived from the formula (5.2) extended for the six-dimensional space is

$$V = -4\omega + \omega^2/2 \cdot (r_1^2 + r_2^2) + 2c/r_{12} - (2c^2 - \omega)/(1 + cr_{12}). \quad (7.2)$$

If the parameter c is put equal to $\sqrt{\omega/2}$, then the interaction turns out to be purely Coulombian, but the charge will be rigidly connected with the strength of the harmonic binding. If we start, instead of (7.1), from a closely akin wave function of the form

$$\psi_0 \propto \exp \{ -\omega(r_1^2 + r_2^2)/2 \} \cdot (1 + cr_{12})(1 + c'r_{12}),$$

the corresponding potential is again of the quite similar analytical form as (7.2) and the interaction becomes purely Coulombian by suitable choice of the parameters c and c' . The charge constant is again rigidly connected with ω , but the relation is different from the above. By augmenting linear factors in the wave function we can invent a large class

of potentials which consist of the harmonic binding and the Coulomb repulsion. The curious fact is that the charge constant can take only discrete values if ω is given a priori. (A sort of quantization?)

§ 8. The shift operators in the hydrogen atom problem

We have seen in § 2 that the ground state ψ_0 of the hydrogen atom is characterized by the property of being annihilated under the operation

$$\nabla_i \equiv \partial/\partial x_i + Z \partial r/\partial x_i \quad (i=1, 2, 3). \quad (8.1)$$

Along with this we also have to deal with its adjoint

$$\nabla_i^\dagger \equiv -\partial/\partial x_i + Z \partial r/\partial x_i. \quad (8.2)$$

The calculations involved in transforming from (2.1) to (2.3) show the validity of the operator equation

$$\sum_i \nabla_i^\dagger \nabla_i = -\Delta - 2Z/r + Z^2 = 2H + Z^2, \quad (8.3)$$

where H is the Hamiltonian of the hydrogen atom. We now develop an operator calculus for ∇_i and ∇_i^\dagger .

The components of the vector operator ∇ are mutually commutative; so are also the components of ∇^\dagger . This is a well known fact of differential geometry. But we have

$$[\nabla_j, \nabla_k] = -2Z \partial^2 r / \partial x_j \partial x_k = [\nabla_k^\dagger, \nabla_j^\dagger]. \quad (8.4)$$

By virtue of this commutation relation we can compute the commutator with the Hamiltonian:

$$[H, \nabla_k] = (1/2) \sum_j [\nabla_j^\dagger, \nabla_k] \nabla_j = -Z \sum_j \partial^2 r / \partial x_k \partial x_j \cdot \nabla_j, \quad (8.5)$$

$$[H, \nabla_k^\dagger] = (1/2) \sum_j \nabla_j^\dagger [\nabla_j, \nabla_k^\dagger] = +Z \sum_j \nabla_j^\dagger \partial^2 r / \partial x_j \partial x_k. \quad (8.6)$$

Now we have the relation (2.2), so that

$$\sum_j \partial^2 r / \partial x_k \partial x_j \cdot \partial r / \partial x_j = 0. \quad (8.7)$$

On account of this, we see from (8.5) that H and ∇_k are effectively commutative for any spherically symmetric function $S = S(r)$:

Thus, if the function S represents an s -eigenfunction, we can derive from it the complete set of three p -states with the same energy upon application of the vector operator ∇ :

$$S \rightarrow \nabla_k S \propto P_k. \quad (8.8)$$

For the ground $1s$ -state this operation leads of course to the null result. Inversely we can conclude from (8.6) that the s -state is obtained from the p -states by the scalar product $\sum_k \nabla_k^\dagger P_k$:

$$P_k \rightarrow \sum_k \nabla_k P_k \propto S. \quad (8.9)$$

We can in quite an analogous way work out shift operators which engender d -states from p -states. But this vectorial generalization of Schrödinger-Infeld ladder method⁽³⁾ will be given elsewhere in collaboration with Mr. H. Arita.

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On Gauge Invariance in Electrodynamics and the Self-energy Problem of the Photon

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The physical meaning implied in the gauge invariance of electrodynamics is analyzed using the Hamiltonian formalism (§ 2). The self-energy of the photon is calculated as a limit of diminishing the rest mass to zero of the self-energy of a neutral vector quantum with small rest mass (§ 3).

The inevitability of introducing the method of renormalizing the light velocity as proposed in our previous paper is concluded from these results (§ 4).

§ 1. Introduction and summary

Although the gauge invariance in electrodynamics and the self-energy problem of the photon have been discussed by many authors, the situation has not been fully cleared. For example, satisfactory answer has not been given to questions such as what is the consequence of the gauge invariance, why the interaction of the electromagnetic field must be introduced in a gauge invariant way, or how this requirement is related to the self-energy problem of the photon.

The first aim of the present paper is to try to throw light on these problems, and the physical meaning implied in the gauge invariance of electrodynamics is analyzed in § 2 using the Hamiltonian formalism.

The second aim is to supplement our previous discussion on the self-energy problem of the photon. In the paper "*A New Attempt on the Self-energy Problem of the Photon*"¹⁾, one of the authors proposed the method of renormalizing the light velocity for the self-energy problem of the photon. The inevitability of introducing such an unfamiliar method however was not elucidated there, and room of wondering whether it is not possible, instead of attempting such a drastic approach, to amalgamate the self-energy into the presumed rest mass of the photon in such a way that the resulting rest mass just vanishes seems not to be ruled out. Such a weak point of the theory came mainly from an insufficiency of the discussion on the gauge invariance of electrodynamics. Indeed the gauge invariance is a very characteristic feature of the electromagnetic field²⁾, and its investigation affords a powerful measure also in the self-energy problem of the photon, for which purpose the result of § 2 can be utilized. In § 3, the self-energy of the photon is calculated as a limit of diminishing the rest mass to zero of the self-energy of a neu-

*) We restrict our attention to fields with spin less than $3/2$.

tral vector quantum with small rest mass. Contrary to general believing, the result is obtained in a non-zero and still gauge invariant form. This result, when combined with those of § 2, enables us to conclude the inevitability of introducing our method.

§ 2. Gauge invariance in electrodynamics

The physical meaning implied in the gauge invariance of electrodynamics is discussed in this section. In order to simplify the discussion, we restrict our attention to the case of classical theory, since the situation is not altered essentially in quantum electrodynamics.

Electrodynamics is invariant under the gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu - \partial_\mu \lambda, \\ [\square] \lambda &= 0. \end{aligned} \quad (2.1)$$

As will be seen easily by expanding λ into Fourier series, (2.1) is a transformation which, conserving transverse waves, alters the homogeneous parts of the longitudinal and scalar components of the electromagnetic field. Therefore, the fact that electrodynamics is invariant under gauge transformations is another expression of the requirement that all physically significant results must not depend upon these parts of the electromagnetic field.

It is instructive to write the system in the Hamiltonian form to understand why such a situation occurs. The interacting system of charged particles and the electromagnetic field can be described by equations of motion^{*)};

$$\begin{aligned} \partial H / \partial q_k &= -\dot{p}_k, \quad \partial H / \partial p_k = \dot{q}_k && \text{(particles),} \\ \partial H / \partial q_\lambda &= -\dot{p}_\lambda, \quad \partial H / \partial p_\lambda = \dot{q}_\lambda && \text{(transverse waves),} \\ \left. \begin{aligned} \partial H / \partial q_\sigma &= -\dot{p}_\sigma, \quad \partial H / \partial p_\sigma = \dot{q}_\sigma \\ \partial H / \partial \alpha_\sigma &= -\dot{b}_\sigma, \quad \partial H / \partial b_\sigma = \dot{\alpha}_\sigma \end{aligned} \right\} && \text{(longitudinal waves),} \end{aligned} \quad (2.2)$$

where H is the Hamiltonian of the total system, and (p_k, q_k) , (p_λ, q_λ) , (p_σ, q_σ) and $(b_\sigma, \alpha_\sigma)$ are canonical variables describing charged particles and the transverse, longitudinal, and scalar components of the electromagnetic field respectively. Performing a canonical transformation generated by the modulus function

$$\begin{aligned} S = & -\sum i k_\sigma \lambda(\mathbf{k}_\sigma; t) P_\sigma - \sum i k_\sigma \lambda(\mathbf{k}_\sigma; t) B_\sigma \\ & + \sum q_k P_k + \sum q_\lambda P_\lambda + \sum q_\sigma P_\sigma + \sum a_\sigma B_\sigma \end{aligned} \quad (2.3)$$

with $\lambda(\mathbf{k}_\sigma; t)$ satisfying

$$\ddot{\lambda}(\mathbf{k}_\sigma; t) + k_\sigma^2 \lambda(\mathbf{k}_\sigma; t) = 0,$$

^{*)} Notations in this section are the same with those in Heitler's book (*Quantum Theory of Radiation*), and natural unit is used throughout.

these variables are transformed into

$$\begin{aligned} p_k &\rightarrow \dot{p}_k, & q_k &\rightarrow q_k, \\ p_\lambda &\rightarrow \dot{p}_\lambda, & q_\lambda &\rightarrow q_\lambda, \\ p_\sigma &\rightarrow \dot{p}_\sigma, & q_\sigma &\rightarrow q_\sigma - ik_\sigma \lambda(\mathbf{k}_\sigma; t), \\ b_\sigma &\rightarrow b_\sigma, & a_\sigma &\rightarrow a_\sigma - ik_\sigma \lambda(\mathbf{k}_\sigma; t). \end{aligned} \quad (2.4)$$

In terms of A_μ , (2.4) is equivalent to

$$A_\mu \rightarrow A_\mu - \partial_\mu \lambda, \quad \square \lambda = 0.$$

This is nothing but the gauge transformation (2.1). Therefore, the homogeneous parts of the longitudinal and scalar components of the electromagnetic field lie within the arbitrariness of describing the electromagnetic field which is allowed by the canonical transformation. This is even the case when the electromagnetic field is treated as external, where a suitable canonical transformation applied to the particles system just reproduces the gauge transformation. To see this, taking again the classical case, the motion of a charged particle in an external electromagnetic field is described by the Hamiltonian

$$H = c\phi + \sqrt{m^2 + (\mathbf{p} - e\mathbf{A})^2}. \quad (2.5)$$

Performing a canonical transformation generated by the modulus function

$$W = -X_i p_i - e\lambda(X_i, t), \quad (2.6)$$

these variables are transformed into

$$\begin{aligned} H &\longrightarrow H - e \partial \lambda / \partial t, \\ p_i &\longrightarrow p_i + e \partial \lambda / \partial x_i, \end{aligned} \quad (2.7)$$

and

$$x_i \longrightarrow x_i.$$

Written in terms of new variables, therefore, (2.5) takes the form

$$H = e(\phi - \partial \lambda / \partial t) + \sqrt{m^2 + (\mathbf{p} - e\mathbf{A} + e \partial \lambda / \partial \mathbf{x})^2},$$

which is completely identical with the result of applying a gauge transformation to the external electromagnetic field A_μ .

The above discussion can be brought into a form more suitable to practical use, if we decompose the electromagnetic field into the transverse, longitudinal and scalar components;

$$A_\mu = n_\mu (n_\nu \partial / \partial x_\nu) A - (\partial / \partial x_\mu + n_\mu n_\nu \partial / \partial x_\nu) A' + \mathfrak{A}_\mu,$$

where n_μ is an arbitrary time-like unit vector and A , A' and \mathfrak{A}_μ satisfy*)

*) We consider the case of free field only, since the interaction representation is used mainly in actual calculations.

$$\square A = 0,$$

$$\square A' = 0,$$

and

$$\square \mathfrak{A}_\mu = 0.$$

Under the gauge transformation (2.1) they transform as

$$A \longrightarrow A - \lambda,$$

$$A' \longrightarrow A' - \lambda,$$

and

$$\mathfrak{A}_\mu \longrightarrow \mathfrak{A}_\mu.$$

Therefore, $A - A'$ and \mathfrak{A}_μ remain gauge invariant, and the gauge invariance of the theory is made explicit if we can express all physical quantities with $A - A'$ and \mathfrak{A}_μ only. (Actually, terms containing $A - A'$ do not appear, since in this decomposed form Lorentz condition takes the form $A - A' = 0$.) Written in this form, therefore, the requirement of gauge invariance is equivalent to saying that terms containing $\partial A' / \partial x_\mu$ should vanish from all physically significant results.

§ 3. The self-energy of the photon

Next we calculate the self-energy of the photon. This quantity has been a typical example of the mathematical ambiguity of quantum electrodynamics. The ambiguity, however, can be reduced by attributing small rest mass to the photon, which enable us to use its rest system in the actual calculation. The continuous approach of such field to the electromagnetic field in the limit of diminishing the rest mass to zero has already been shown by Glauber²⁾. According to him, the neutral vector field with rest mass μ is described by a vector $A_\mu(x)$ and a scalar $B(x)$ satisfying the equations of motion

$$\left. \begin{aligned} (\square - \mu^2) A_\mu(x) &= 0 \\ (\square - \mu^2) B(x) &= 0 \end{aligned} \right\}, \quad (3.1)$$

and the commutation relations

$$\begin{aligned} [A_\mu(x), A_\nu(x')] &= i\delta_{\mu\nu} \mathcal{D}(x-x') \\ [B(x), B(x')] &= i\mathcal{D}(x-x'). \end{aligned} \quad (3.2)$$

The Schrödinger equation and the Lorentz condition are given by

$$i \partial \Psi[\sigma] / \partial \sigma(x) = -j_\mu(x) A_\mu(x) \Psi[\sigma], \quad (3.3)$$

and

$$\left\{ \frac{\partial A_\mu(x)}{\partial x_\mu} + \mu B(x) - \int \mathcal{D}(x-x') j_\mu(x') d\sigma'_\mu \right\} \Psi[\sigma] = 0 \quad (3.4)$$

respectively.

(The notations are the same as those of Schwinger³⁾ unless otherwise noted.) (3.1) and (3.2) approach to the usual equations of motion and the commutation relations of quantum electrodynamics in the limit of $\mu \rightarrow 0$, while $B(x)$ disappears from the interaction Hamiltonian density and from the supplementary condition. Therefore, $B(x)$ field becomes entirely irrelevant to the interaction between our field and the electron field, and in this sense we can say that $A_\mu(x)$ is reduced to the ordinary electromagnetic field in the limit of $\mu \rightarrow 0$.

It is convenient first to calculate the induced current $\partial j_\mu(x)$. $\partial j_\mu(x)$ is obtained by taking the expectation value $j_\mu(x)$ of the current vector operator in the Heisenberg representation at the vacuum state of the electron. Since the interaction Hamiltonian density is $-j_\nu(x)A_\mu(x)$, the transformation function transforming variables in the Heisenberg representation into those in the interaction representation is given by

$$S[\sigma] = 1 + (-i) \int_{-\infty}^{\sigma} j_\mu(x') A_\mu(x') (dx') \\ + (i)^2 \int_{-\infty}^{\sigma} \int_{-\infty}^{\sigma'} j_\mu(x') A_\mu(x') j_\mu(x'') A_\mu(x'') (dx') (dx'') + \dots \quad (3.5)$$

Therefore, remembering that the vacuum expectation value of $j_\mu(x)$ vanishes, it is found that $\partial j_\mu(x)$ is given by

$$\partial j_\mu(x) = i \int_{-\infty}^{\sigma} \langle [j_\mu(x), j_\nu(x')] \rangle_0 A_\nu(x') (dx') \quad (3.6)$$

in the ϵ^2 approximation, where $\langle \rangle_0$ means an expectation value at the vacuum state of the electron. The explicit form of $\langle [j_\mu(x), j_\nu(x')] \rangle_0$ was calculated by Schwinger to be

$$\langle [j_\mu(x), j_\nu(x')] \rangle_0 = (ie^2/2) S \not{p} [S^{(1)}(x' - x) \gamma_\mu S(x - x') \gamma_\nu \\ - S^{(1)}(x - x') \gamma_\nu S(x' - x) \gamma_\mu], \quad (3.7)$$

where

$$S(x) = (\gamma \partial / \partial x - x) \Delta(x), \quad S^{(1)}(x) = (\gamma \partial / \partial x - x) \Delta^{(1)}(x), \quad (3.8)$$

$$\Delta(x) = \frac{-i}{(2\pi)^3} \int e^{ipx} \delta(p^2 + x^2) \varepsilon(p) (dp), \quad (3.9)$$

and

$$\Delta^{(1)}(x) = \frac{1}{(2\pi)^3} \int e^{ipx} \delta(p^2 + x^2) (dp).$$

Substituting (3.8) and (3.9) into (3.7), (3.6) is written in the form

$$\partial j_\mu(x) = \frac{ie^2}{(2\pi)^6} \int G_{\mu\nu}(k) A_\nu(k) e^{ikx} \cdot (dk), \quad (3.10)$$

where

$$G_{\mu\nu}(k) = \frac{1}{2 \cdot (2\pi)^4} \int_{-\infty}^{\sigma} \int_{-\infty}^{\sigma'} S \not{p} \{ (i\gamma \not{p}_1 - x) \gamma_\mu (i\gamma \not{p}_2 - x) \gamma_\nu \} (\varepsilon(p_2) - \varepsilon(p_1)) \\ \times \delta(p_1^2 + x^2) \delta(p_2^2 + x^2) e^{i(-p_1 + p_2 - k)(x - x')} (dp_1) (dp_2) (dx'). \quad (3.11)$$

Rewriting (3.11) using the relation

$$\partial(\mathbf{p}^2 + \mathbf{x}^2) = \frac{1}{2} \left\{ \frac{\partial(\sqrt{\mathbf{p}^2 + \mathbf{x}^2} - p_0)}{\sqrt{\mathbf{p}^2 + \mathbf{x}^2}} + \frac{\partial(\sqrt{\mathbf{p}^2 + \mathbf{x}^2} + p_0)}{\sqrt{\mathbf{p}^2 + \mathbf{x}^2}} \right\}$$

where \mathbf{p} means space components of p_μ ,

$$\begin{aligned} G_{\mu\nu}(k) = & \frac{1}{4 \cdot (2\pi)^4} \int_{-\infty}^{\infty} \int [Sp\{(i\gamma p_1^* - \mathbf{x})\gamma_\mu(i\gamma p_2 - \mathbf{x})\gamma_\nu\} e^{i(-p_1^* + p_2 - k)(x - x')} \\ & - Sp\{(i\gamma p_1 - \mathbf{x})\gamma_\mu(i\gamma p_2^* - \mathbf{x})\gamma_\nu\} e^{i(-p_1 + p_2^* - k)(x - x')}] \\ & \times \frac{d\mathbf{p}_1}{\sqrt{\mathbf{p}_1^2 + \mathbf{x}^2}} \cdot \frac{d\mathbf{p}_2}{\sqrt{\mathbf{p}_2^2 + \mathbf{x}^2}}(dx'), \end{aligned} \quad (3.12)$$

with $(p_\mu = \mathbf{p}, i\sqrt{\mathbf{p}^2 + \mathbf{x}^2})$, and $p_\mu^* = (\mathbf{p}, -i\sqrt{\mathbf{p}^2 + \mathbf{x}^2})$.

Referring to the rest system hereafter, we take k_μ as $(0, 0, 0, i\mu)$. Thus, performing integrations over space components of x' , (3.12) becomes

$$\begin{aligned} G_{\mu\nu}(k) = & \frac{1}{4(2\pi)} \int_{-\infty}^{\infty} \int [Sp\{(i\gamma p^* - \mathbf{x})\gamma_\mu(i\gamma p - \mathbf{x})\gamma_\nu\} e^{-i(-p_0^* + p_0 - k_0)(x_0 - x_0')} \\ & - Sp\{(i\gamma p - \mathbf{x})\gamma_\mu(i\gamma p^* - \mathbf{x})\gamma_\nu\} e^{-i(-p_0 + p_0^* - k_0)(x_0 - x_0')}] \\ & \times \frac{d\mathbf{p}}{\mathbf{p}^2 + \mathbf{x}^2}(dx'). \end{aligned} \quad (3.13)$$

Since the spur calculation yields

$$1/4 Sp\{(i\gamma p_1 - \mathbf{x})\gamma_\mu(i\gamma p_2 - \mathbf{x})\gamma_\nu\} = -(\mathbf{p}_1)_\mu(\mathbf{p}_2)_\nu + (\mathbf{p}_1)_\nu(\mathbf{p}_2)_\mu + (\mathbf{p}_1\mathbf{p}_2 + \mathbf{x}^2)\delta_{\mu\nu},$$

it is readily seen that (3.13) vanishes for $\mu \neq \nu$. For $\mu = \nu$, the calculation must be done separately for each value of μ and ν . After a simple calculation we find

$$G_{\mu\nu}(k) = \begin{cases} \frac{-2i}{2\pi} \int_0^\infty \frac{(-\mathbf{p}^2 + \mathbf{p}^2 + \mathbf{x}^2)}{\mathbf{p}^2 + \mathbf{x}^2 - \mu^2/4} \frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2 + \mathbf{x}^2}} & (\mu = \nu = 1, 2, 3) \\ 0 & (\mu = \nu = 4), \end{cases}$$

and $G_{\mu\nu}(k)$ takes the form

$$(G_{\mu\nu})_0 = -iA \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (3.14)$$

in this rest system, where A is a divergent coefficient given by

$$A = 4 \int_0^\infty \left(\frac{2\mathbf{p}^2}{3} + \mathbf{x}^2 \right) \frac{1}{\mathbf{p}^2 + \mathbf{x}^2 - \mu^2/4} \frac{d\mathbf{p}}{\sqrt{\mathbf{p}^2 + \mathbf{x}^2}} d\mathbf{p}. \quad (3.15)$$

The transformation to the original coordinate system can be performed at once according to

$$G_{\mu\nu}(k) = \sum a_{\mu\nu\alpha\beta}(G_{\alpha\beta})_0,$$

where we wrote the coefficients of Lorentz-transformation as $a_{\mu\nu}$. The explicit form of $a_{\mu\nu}$ is easily found to be

$$a_{\mu\nu} = \begin{pmatrix} -1 + (k_1/K)^2 & k_1 k_2 / K^2 & k_1 k_3 / K^2 & -i k_1 / \mu \\ k_1 k_2 / K^2 & -1 + (k_2/K)^2 & k_2 k_3 / K^2 & -i k_2 / \mu \\ k_1 k_3 / K^2 & k_2 k_3 / K^2 & -1 + (k_3/K)^2 & -i k_3 / \mu \\ i k_1 / \mu & i k_2 / \mu & i k_3 / \mu & -i k_4 / \mu \end{pmatrix}, \quad (3.16)$$

with

$$K = \sqrt{x(k_0 - x)}.$$

Performing this transformation,

$$G_{\mu\nu}(k) = -iA \begin{pmatrix} 1 + k_1^2 / \mu^2 & k_1 k_2 / \mu^2 & k_1 k_3 / \mu^2 & k_1 k_4 / \mu^2 \\ k_1 k_2 / \mu^2 & 1 + k_2^2 / \mu^2 & k_2 k_3 / \mu^2 & k_2 k_4 / \mu^2 \\ k_1 k_3 / \mu^2 & k_2 k_3 / \mu^2 & 1 + k_3^2 / \mu^2 & k_3 k_4 / \mu^2 \\ k_1 k_4 / \mu^2 & k_2 k_4 / \mu^2 & k_3 k_4 / \mu^2 & 1 + k_4^2 / \mu^2 \end{pmatrix}, \quad (3.17)$$

Substituting (3.17) into (3.10), we finally find that $\partial j_\mu(x)$ is given by

$$\begin{aligned} \partial j_\mu &= \frac{e^2 A}{(2\pi)^2} \int \left(\delta_{\mu\nu} + \frac{1}{\mu^2} k_\mu k_\nu \right) A_\nu(k) e^{ikx} (dk) \\ &= e^2 A / (2\pi)^2 \cdot (A_\mu(x) - \mu^{-2} \partial_\mu \partial_\nu A_\nu(x)). \end{aligned} \quad (3.18)$$

It is satisfactory that this result is just of the form required from the discussion of § 1 (note that the equation of motion for $A'(x)$ is given by $(\square - x^2)A'(x) = 0$ instead of $\square A'(x) = 0$), and further satisfies the conservation law of charge

$$\partial \partial j_\mu(x) / \partial x_\mu = 0. \quad (3.19)$$

(The second term of (3.18) which plays an important role to secure the gauge invariance and the conservation equation of charge of $\partial j_\mu(x)$ is missed in ordinary calculations. This is due to a careless dropping of some odd functions appearing in the course of the integrations, but such a procedure would not be safe unless sufficient ground exist for the symmetry of integrating variables.)

We are now in place to discuss the self-energy caused by the interaction between $\partial j_\mu(x)$ and $A_\mu(x)$. To such divergence problem the method of renormalization has gained brilliant success. This method has its base in that, when we distinguish fictitious bare quanta and interacting physical ones, the infinite self-energy can be condensed into the effect of altering the structure constants of quanta such as mass or charge. Therefore, in order that the renormalization method is successful, it must be shown firstly that variables describing physical quanta should satisfy the same type of equations of motion and commutation relations as those of bare ones which differ from them only in the numerical values of the structure constants, and secondly that divergent terms just vanish when calculation is made referring to the Hamiltonian written in terms of field variables describing physical quanta.

Returning to our problem, field variables describing physical quanta in free space are given by taking the expectation value of $A_\mu(x)^{*)}$ at the vacuum state of the electron. Therefore, its equation of motion is given by

$$(\square - \mu^2) \langle A_\mu(x) \rangle_0 = -\partial j_\mu(x). \quad (3 \cdot 20)$$

The equation (3·20) shows that the induced current can never be interpreted as giving an alteration to the rest mass of the bare quanta, so long as it is of the form (3·18). In order that such an interpretation is possible, $\partial j_\mu(x)$ must be of the form (const.) $\times A_\mu(x)$, which is in violent contradiction to the requirement of the gauge invariance of $\partial j_\mu(x)$. Therefore, the condition of the gauge invariance imposes a severe restriction to the physical interpretation of the self energy, and so long as the whole scheme is formulated in a gauge invariant form, the ensuing self-energy can never be interpreted as giving an alteration to the rest mass of the quanta. Thus, we are faced with a serious difficulty in confronting with energy which is hard to interpret. The most natural way out of this situation seems to renormalize the light velocity as proposed in our previous paper. Indeed, as was shown there, the propagation velocity of the physical photon is altered from that of the bare one if $\partial j_\mu(x) \neq 0$, and difficulties concerning with the self-energy problem of the photon can be solved very naturally if we reformulate Lorentz transformation using this altered value of the light velocity.

§ 4. Conclusion

Summarizing the results of § 2 and § 3, we may conclude that

- i) the condition of the gauge invariance does not require that the induced current itself should vanish. It rather requires that terms containing $\partial l / \partial x_\mu$ should vanish in the induced current,
- ii) indeed a natural calculation yields a non-vanishing induced current with the required form,
- iii) the self-energy due to the induced current of this form can never be interpreted as given rest mass to the photon,
- iv) the situation can be understood consistently if we adopt the idea of renormalizing the light velocity.

Thus, it seems to us an inevitable course, so long as the induced current survives non-zero, to have to resort to the method of renormalizing the light velocity, although it seems strange at first sight.

In conclusion, the authors wish to express their sincere gratitudes to Prof. S. Sakata for his interest taken in this work.

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*) $A_\mu(x)$ mean the Heisenberg representation of $A_\mu(x)$; $A_\mu(x) = S^{-1}[\sigma] A_\mu(x) S[\sigma]$.

On the Effects of Excited States of Nucleons upon Static Nuclear Potential in Symmetrical Pseudoscalar Meson Theory, I

—Derivation of Nuclear Potential and Qualitative Conclusions—

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The effects of excited states of nucleons upon static nuclear potential in symmetrical pseudoscalar meson theory are investigated. An additional interaction Hamiltonian is introduced, according to which a nucleon can make a transition from its ground states to the excited one of both spin and isotopic spin $3/2$, or vice versa. The fourth order perturbation contribution of this Hamiltonian is calculated in the non relativistic and static approximations for nucleons. The effects of excited states prove to be very large. The derived nuclear potential contains generally a very strong attractive central force and a very small tensor force of various signs, both of which have very high singularities at the origin. The most interesting feature is that the effects are much larger for charge triplet states than for charge singlet states. The derived nuclear potentials are summarized in the final section with the interesting five qualitative conclusions. The quantitative investigations of the potentials here derived will be given in detail in the subsequent paper (part II).

§ 1. Introduction and the method for treatment of excited states

As the result of various investigations, the type of meson field seems to have been determined as pseudoscalar. As to the coupling of meson field with the nucleon, however, it is not decided whether the pseudoscalar type interaction or the pseudovector one should be preferred. Although the pseudoscalar interaction is favorable for renormalization technique, the nuclear potential derived from pseudoscalar interaction only seems to be inconsistent with empirical facts. It is to be noted, however, that these results are only obtained from ordinary perturbation calculation and the existence of excited states of nucleon is not taken into account at all. Indeed, we wish to show in this paper that the existence of excited states of nucleons seems to modify the situations generally for both types of interactions.

How to estimate the effects of excited states upon static nuclear potential is the essential point of this paper. An exact treatment of excited states would be a very difficult problem and the conventional perturbation method will certainly be inadequate for this purpose. In this paper, we do not enter into these basic problems further, but adopt the following approximate treatment of the excited states of nucleons rather phenomenologically. According to the present meson theory a nucleon can make transition between various states accompanied by the emission or absorption of one meson, as the result of which, for example, the state of a nucleon interacting with its self-field is described as the superposition of various states composed of bare nucleons and several mesons. The convergency of these asymptotic expansions is, however, very bad in meson theory because

of the possible existence of the nucleon excited states as well as the largeness of the interaction constant. We consider in this paper that the perturbation expansion will be significant sufficiently if we include in the above expansion such states that are composed of the nucleons in their excited states and several mesons emitted. Thus a nucleon interacting with its self-field can make the virtual transition to excited states accompanied by one meson emission. Indeed, one of the authors (M.S.)¹⁾ has showed under the same idea that the anomalous magnetic moment of nucleons can be explained by the inclusion of the virtual transition to the excited state with spin and isotopic spin $3/2$. The recent experiments on pion-nucleon scattering can readily be understood in the same line; a nucleon can make virtual transition to excited states accompanied by the absorption of the incident pion and therefore resonance phenomenon will appear. In this case also, the excited state is required to have spin and isotopic spin $3/2$. Thus a nucleon interacting with its self-field can make virtual transition to its excited states with the emission of a virtual pion or with the absorption of a pion from the external field. As a summary we believe after these considerations that at least a good approximation can be attained by perturbation calculations if we include the above virtual transitions in the perturbation expansions in addition to the usual ones. Thus the effect of nucleon excited states on the interaction of two nucleons can be calculated as the fourth order effect in perturbation calculations that comes from the processes of two-meson exchange accompanied by the transitions of nucleons through excited states.

It must be noted here that, in the p.s. coupling of nucleon with the p.s. meson field, because of the inclusion of γ_5 in the interaction Hamiltonian, the contribution from two-meson exchange processes to the nuclear potential (fourth order potential) is rather dominant compared with the one from one-meson exchange processes (second order potential). That is in the (p.s., p.s.) theory, the two-meson exchange interaction is large compared with the one-meson exchange one. In the (p.s., p.v.) theory the circumstances are found to be quite similar. According to the considerations of nucleon excited states mentioned above the leading effect of excited states arises from two meson exchange interactions. It is expected, therefore, that in pseudoscalar meson theory nucleon excited states may contribute very much to nucleon interactions, which will be shown to be the case in the subsequent paragraphs.

In order to take account of the above new virtual states we have only to add a new interaction Hamiltonian, which causes the virtual transitions to excited states accompanied by the emission or absorption of one meson. A nucleon in its excited state is considered to be described to obey the same equation of motion as an elementary particle of spin $3/2$ and, for the convenience of calculations, Rarita-Schwinger formalism²⁾ is adopted. The possible four charge states of excited nucleon are all taken into account and only the charge independence hypothesis is imposed. Thus it is assumed that in the free Hamiltonian, besides ordinary terms, an additional free Hamiltonian of spin and isotopic spin $3/2$ field is inserted. In the interaction Hamiltonian, in addition to the usual coupling term, those which describe the transition of ordinary nucleon to the excited states of spin and isotopic spin $3/2$ accompanied by the emission or absorption of a meson are added with some

coupling constant. These interaction terms are treated as the small perturbation in the calculations of various processes. The examination of pion-nucleon scattering and γ -pion production processes was carried out by Minami et al.⁽³⁾ in the same line. According to their calculations the agreement with the experimental data is improved compared with other approaches carried out previously on the same problem. As to γ -pion production process, because it seems to be necessary to take into account the effect of anomalous magnetic moment of nucleon more adequately, the results of S. Minami et al. is considered to be satisfactory from the approximate nature of our investigations.

Basing upon these considerations the static nuclear potential is analyzed by the perturbation calculation to the fourth order in nonrelativistic approximation. Nucleons are considered to be infinitely heavy and at rest neglecting nucleon recoils entirely. This static approximation is, however, inadequate in p.s. coupling theory. In order to avoid this difficulty the p.s. interaction Hamiltonian is transformed into the equivalent sum of scalar pair coupling term and p.v. coupling one by means of the canonical transformation used by Dyson.^{(4),(5)} After this transformation the static approximation becomes also possible in p.s. coupling case. In p.v. coupling case, the static approximation is adequate. On the other hand, in the interaction Hamiltonian term that describes the transition of a nucleon from the ground state to the excited one (or the inverse process), one must insert γ_5 or 1 according to whether the excited state has the same or different parity as the ground state of a nucleon. According to the calculations by Minami et al.⁽³⁾ the latter gives better agreement with the experiment than the former. Further the phenomenological treatment of anomalous magnetic moments of nucleons⁽¹⁾ also favours the latter case, since, if γ_5 is inserted, it would be improper to have the image that a nucleon makes transition from the ground state to the excited one with the emission of a pion, because the coupling with the negative energy state would become dominant. From these reasons we assume in this paper that the intrinsic parity of the excited state is different from that of the ground state and hence the interaction Hamiltonian contains no γ_5 factor. Nucleons can thus be in a good approximation treated as at rest also for this term, simplifying practical calculations exceedingly.

By means of these formulations one can immediately calculate the interaction energy between two nucleons at rest. In our calculations three important techniques are employed. First of them is the method to treat the four charge states of the excited nucleon and is explained in Appendix I. Secondly, because it is impossible to use the so called sum rule with respect to the spin 3/2 field, in obtaining various matrix elements corresponding to the processes through excited states, we adopt the procedure explained in Appendix II. It is shown there that the matrix element, calculated practically in each case and summed over four intermediate spin states, can be replaced by the equivalent quantity containing Pauli spin matrix only. Finally, because of the appearance of mass difference between excited and ground state, it is generally impossible to carry out integrations exactly. Therefore we employ the approximate method in the integration with respect to virtual meson momenta. Here it is to be remarked that the above relatively small mass difference (about two times the pion mass) must not be neglected even if we employ the assumption

of infinitely heavy nucleons, because otherwise the important virtual processes characteristic of the effects of excited states cannot be taken into account.

The present analysis is based on the presumption that the method of calculations adopted here is a good approximation for the forthcoming exact theoretical treatment. The nearest way to judge the validity of our picture is to compare the theoretical results with empirical facts. We will thereafter investigate various other processes in order to clarify and justify our picture.

In the following sections our calculations are given in detail. The fundamental formulation for the analysis will be given in § 2. In § 3 and § 4 the derivation of nuclear potential will be performed for p.s. and p.v. couplings respectively. In the final section only the qualitative conclusions are given. The quantitative results are given in detail in the subsequent paper (Part II).

§ 2. General formulation

The unperturbed (free) Hamiltonian density H_0 is the sum of the energy density of nucleon field, pion field, and spin 3/2 field which describes the excited states of nucleon :

$$\begin{aligned} H_0 = & \hbar c [\bar{\psi}(\gamma_i \partial_i + \kappa)\psi] + (1/2) \sum_{\alpha=1}^3 [c^2 V_\alpha V_\alpha + \partial_i U_\alpha \partial_i U_\alpha + \mu^2 U_\alpha U_\alpha] \\ & + \hbar c [\bar{\Psi}_\mu(\gamma_i \partial_i + \lambda)\Psi_\mu - (1/3)\bar{\Psi}_\mu \gamma_\mu \partial_i \Psi_\mu - (1/3)\bar{\Psi}_i \partial_i \gamma_\mu \Psi_\mu \\ & + (1/3)\bar{\Psi}_\mu \gamma_\mu (\gamma_i \partial_i - \lambda)\gamma_\nu \Psi_\nu], \end{aligned} \quad (2.1)$$

where ψ , Ψ_μ and U_α , V_α represent the spin 1/2-, spin 3/2-, and pion-field quantities, respectively. $\kappa \equiv M_0 c/\hbar$, $\lambda \equiv Mc/\hbar$, and $\mu \equiv m_\pi c/\hbar$, where M_0 , M , and m_π are the rest masses of a ground state nucleon, an excited state one, and a pion, respectively.

The interaction energy density of the pion and the ground state nucleon is

$$if\bar{\psi}\gamma_5\tau_\alpha\psi U_\alpha \quad \text{or} \quad (g/\mu)\bar{\psi}i\gamma_5\gamma_\mu\tau_\alpha\psi(\partial U_\alpha/\partial x_\mu), \quad (2.2)$$

according to whether the pion is coupled to the nucleon through p.s. or p.v. coupling, respectively. For p.s. coupling, in order to carry out the calculations in the static approximation for nucleons, we make so called Dyson transformation which transforms p.s. coupling term into, up to the second order in the coupling constant, scalar pair coupling term and equivalent p.v. coupling term :

$$(f^2/2\kappa\hbar c)\bar{\psi}\psi U_\alpha^2 + if/(2\kappa)\bar{\psi}i\gamma_5\gamma_\mu\tau_\alpha\psi(\partial U_\alpha/\partial x_\mu), \quad (2.3)$$

which are sufficient for our calculation up to the fourth order. The first term in (2.3) permits the simultaneous emission or absorption of two pions, the physical meaning of which will be discussed later in § 3, while the second one is the well known equivalent p.v. coupling term.

As for the interaction term which describes the transitions of a nucleon to or from its excited states, we assume the following form with the phenomenological coupling constant G having the dimension of the charge :

$$(G/\mu)\bar{\Psi}_\mu T_\alpha \psi (\partial U_\alpha / \partial x_\mu) + (G/\mu)\bar{\psi} T_\alpha^* \Psi_\mu (\partial U_\alpha / \partial x_\mu), \quad (2.4)$$

in which T_α and its hermite conjugate T_α^* are the 4-2 matrices determined from the requirement that the nuclear interactions caused by (2.4) satisfy the requirement of charge independence and play the role of τ -spin in the conventional formulation. Thus the pion field can be treated symmetrically, simplifying the analysis exceedingly. The practical representations of matrix T_α 's are as follows:

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1/\sqrt{3} \\ -1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -i & 0 \\ 0 & i/\sqrt{3} \\ -i/\sqrt{3} & 0 \\ 0 & i \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & 0 \\ 2/\sqrt{3} & 0 \\ 0 & -2/\sqrt{3} \\ 0 & 0 \end{bmatrix}. \quad (2.5)$$

Ψ_μ and ψ in (2.4) must accordingly be considered to have the following four and two components:

$$\Psi_\mu = \begin{bmatrix} \Psi_{\mu}^{++} \\ \Psi_{\mu}^{+} \\ \Psi_{\mu}^{0} \\ \Psi_{\mu}^{-} \end{bmatrix}, \quad \psi = \begin{bmatrix} \psi_P \\ \psi_N \end{bmatrix}, \quad (2.6)$$

where the superscripts in Ψ_μ refer to the charge values while P and N in ψ do to the proton and neutron, respectively. The form of the interaction Hamiltonian, when the meson field is expressed as the sum of charged and neutral fields, is given in Appendix I, together with the proof that the charge independence of nucleon-nucleon interaction can be attained by choice of the coupling term (2.4) with (2.5) and (2.6).

In order to evaluate the matrix elements, free field solutions of Rarita-Schwinger field will be needed. The equations for free Rarita-Schwinger field are

$$(\gamma_\mu \partial_\mu + \lambda) \Psi_\nu = 0, \quad \gamma_\mu \Psi_\mu = 0, \quad \partial_\mu \Psi_\mu = 0. \quad (2.7)$$

Four independent solutions of these equations belonging to the positive energy eigenvalue E and momentum \mathbf{p} and corresponding to four different spin orientations are given as follows:*

$$\begin{aligned} \{\Psi_\mu^{(1)}\} &= \{e_1 \phi_+, 0\}, \quad \{\Psi_\mu^{(2)}\} = \{e_2 \phi_-, 0\}, \\ \{\Psi_\mu^{(3)}\} &= \left\{ \frac{1}{\sqrt{3}} e_2 \phi_+ + \sqrt{\frac{2}{3}} \frac{E}{Mc^2} e_3 \phi_-, i \sqrt{\frac{2}{3}} \frac{\mathbf{p}}{Mc} \phi_- \right\}, \\ \{\Psi_\mu^{(4)}\} &= \left\{ \frac{1}{\sqrt{3}} e_1 \phi_- - \sqrt{\frac{2}{3}} \frac{E}{Mc^2} e_3 \phi_+, -i \sqrt{\frac{2}{3}} \frac{\mathbf{p}}{Mc} \phi_+ \right\}, \end{aligned} \quad (2.8)$$

where

* S. Kusaka, Phys. Rev. 63 (1940), 61. There are certain misprints in Kusaka's paper. Our eq. (2.8) is correct.

$$E = \sqrt{(Mc^2)^2 + (pc)^2},$$

$$e_1 = \left\{ \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0 \right\}, e_2 = \left\{ \frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, 0 \right\}, e_3 = \frac{\mathbf{p}}{p} = \{0, 0, 1\},$$

$$\phi_+ = \frac{1}{\sqrt{(Mc^2 + E)^2 + (pc)^2}} \begin{bmatrix} Mc^2 + E \\ 0 \\ pc \\ 0 \end{bmatrix} \frac{1}{\sqrt{V}} e^{i(p\mathbf{x} - Et)/\hbar},$$

$$\phi_- = \frac{1}{\sqrt{(Mc^2 + E)^2 + (pc)^2}} \begin{bmatrix} 0 \\ Mc^2 + E \\ 0 \\ -pc \end{bmatrix} \frac{1}{\sqrt{V}} e^{i(p\mathbf{x} - Et)/\hbar},$$

in which the normalization $\sum_{\mu} \int \Psi_{\mu}^{(i)*} \Psi_{\mu}^{(j)} dV = \delta_{ij}$ is used.

If one omits such terms that vanish in the approximation for a nucleon to be at rest, the interaction Hamiltonian density is given as

$$H' = H_1 + H_2 + H_3,$$

$$H_1 = (f/2\kappa) \bar{\psi} \sigma_j \tau_a \psi (\partial U_a / \partial x_j), \quad \text{or} \quad (g/\mu) \bar{\psi} \sigma_j \tau_a \psi (\partial U_a / \partial x_j),$$

$$H_2 = (f^2/2\kappa\hbar c) \bar{\psi} \psi U_a^2,$$

$$H_3 = (G/\mu) (\Psi_j T_a \psi) \partial U_a / \partial x_j + (G/\mu) (\bar{\psi} T_a^* \Psi_j) \partial U_a / \partial x_j. \quad (2.9)$$

The static nuclear potential $V(r)$ can be calculated as the expectation value of the above interaction for the state of two nucleons at rest which are apart from each other by the distance r in the usual fourth order perturbation calculation as follows:

$$\begin{aligned} V(r) = & \sum_I \frac{\langle i | \bar{H}' | I \rangle \langle I | \bar{H}' | i \rangle}{E_i E_I} + \sum_{II} \frac{\langle i | \bar{H}' | II \rangle \langle II | \bar{H}' | I \rangle \langle I | \bar{H}' | i \rangle}{(E_i - E_I)(E_i - E_{II})} \\ & + \sum_{I II III} \frac{\langle i | \bar{H}' | III \rangle \langle III | \bar{H}' | II \rangle \langle II | \bar{H}' | I \rangle \langle I | \bar{H}' | i \rangle}{(E_i - E_I)(E_i - E_{II})(E_i - E_{III})}, \end{aligned} \quad (2.10)$$

where

$$\bar{H}' = \int \bar{H}' d^3x.$$

In order to evaluate the above matrix element, meson field quantities are expanded as usual,

$$U_a(\mathbf{x}) = (1/\sqrt{V}) \sum_{\mathbf{k}} c \sqrt{\hbar/2\omega_{\mathbf{k}}} (a_{\mathbf{k}}^{(\alpha)} + a_{-\mathbf{k}}^{(\alpha)*}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (2.11)$$

$$\omega_{\mathbf{k}} = c \sqrt{\mu^2 + \mathbf{k}^2},$$

in which $a_{\mathbf{k}}^{(\alpha)}$ or $a_{\mathbf{k}}^{(\alpha)*}$ are the operators that absorb or emit a meson with wave number vector \mathbf{k} respectively. On the other hand, the nucleon field quantities become, according to our assumption of nucleon at rest,

$$\begin{aligned}
\bar{\psi}(x)\sigma_j\psi(x) &= \sum_s \sigma_j^{(s)} \delta(x-x^{(s)}), \\
\bar{\psi}(x)\psi(x) &= \sum_s \delta(x-x^{(s)}), \\
(\bar{\Psi}_j(x)\psi(x)) &= \sum_s H_j \delta(x-x^{(s)}).
\end{aligned} \tag{2.12}$$

Concerning the treatment of matrix element H_j , some techniques are necessary as was mentioned in § 1. In Appendix II they are given in details and the necessary quantities are all calculated. After these preparations we perform the practical calculation of nuclear potential in the subsequent section.

§ 3. Calculation of nuclear potential for p.s. coupling

Each of the interaction terms in (2.9) gives the characteristic contribution to the nuclear potential. If calculated up to the fourth order in the coupling constant, keeping in mind the fact that H_2 itself is of the second order, one obtains contributions to nuclear potential from three terms in (2.10) as follows:

By the second order perturbation calculation (the first term in (2.10)), (a) the second order perturbation contributions from H_1 —This gives the well known second order nuclear potential (Graph a).

(b) the second order perturbation contributions from H_2 —This gives the leading term in the fourth order nuclear potential which has been obtained by many authors (Graph b).

By the third order perturbation calculations (the second term in (2.10)), (c) the third order perturbation contributions from H_3 and H_2 . This gives one of the most important parts in the fourth order nuclear potential due to the existence of excited states (Graph c).

(b') the third order perturbation contributions from H_1 and H_2 —This gives the next higher order correction to the fourth order potential (b) and it will be proven later that this term is also correctly given by our calculations (Graph b').

By the fourth order perturbation calculation (the third term in (2.10)), (d) the fourth order perturbation contributions from H_3 . This term, being the most complicated one in the evaluation, gives the other part of the most important contributions to the fourth order nuclear potential due to the excited states (Graph d).

(c') the fourth order perturbation contributions from H_1 and H_2 —This gives, as is shown later, the next higher order correction to the fourth order potential (c) and it will be proven later that this term is also correctly given by our calculations (Graph c').

(b'') the fourth order perturbation contributions from H_1 only. This term is, as is seen later, the still higher correction to the fourth order potential (b) (Graph b''). These terms are derived successively in the following subsections.

(a) The second order potential due to H_1

The possible intermediate states are represented diagrammatically in Fig. 1. Note that the diagrams in this paper are not Dyson-Feynman graphs. They only visualize the sequences

of possible virtual processes in perturbation calculations. For instance, Fig. a_1 shows that the nucleon 1 emits a pion of wave number vector \mathbf{k} which is subsequently absorbed by the nucleon 2.

Noting that the energy E of the system in initial, intermediate, and final states is the sum of nucleon rest energies and the total energy of pions, we readily obtain

$$V_a(\mathbf{r}) = -\frac{f^2}{8\pi^2} (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \frac{1}{(2\pi)^3} \times \int d\mathbf{k} \frac{(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k})}{\mu^2 + \mathbf{k}^2} e^{-i\mathbf{k} \cdot \mathbf{r}}, \quad (3.1)$$

where $\mathbf{r} = \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$ is the relative coordinate of two nucleons. Putting $\mu r = x$,

$$V_a(x) = (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{m_\pi}{2M_0} \right)^2 (\boldsymbol{\tau}^{(1)} \cdot \boldsymbol{\tau}^{(2)}) \times \left[\frac{1}{3} (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \frac{1}{x} e^{-x} + \frac{1}{3} S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{1}{x} e^{-x} \right], \quad (3.2)$$

$$S_{12} = (3(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{x})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{x})/x^2) - (\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}).$$

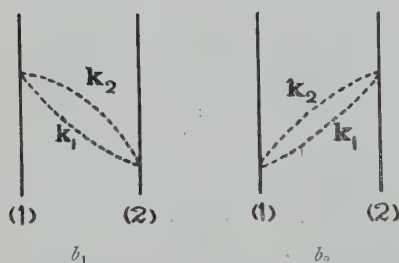


Fig. 2. The virtual processes which give the second order nuclear potential due to H_2 .

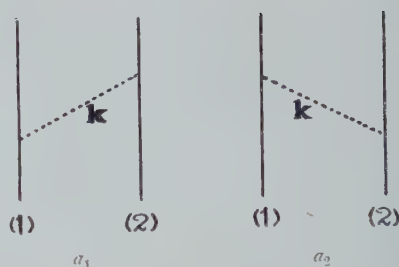


Fig. 1. The virtual processes which give the second order nuclear potential due to H_1 .

This is just the well known second order nuclear potential from the symmetrical pseudoscalar coupling theory. The main purpose of this paper is to estimate the modification of (3.2) up to the fourth order calculations when the effects of excited states are taken into account.

(b) *The second order potential due to H_2*

Next we calculate the second order potential due to H_2 , in which case the possible intermediate states are represented by the diagrams in Fig. 2. If one sums up the contributions from these virtual processes, the result is

$$V_b(\mathbf{r}) = -\left(\frac{f^2}{2\pi} \right)^2 \frac{1}{4\hbar c} \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2 e^{-i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}}}{\omega_1 \omega_2 (\omega_1 + \omega_2)} \times 3 \times 2$$

$$+ \text{exchange term in (1) and (2),} \quad (3.3)$$

$$\omega_i = \sqrt{\mathbf{k}_i^2 + \mu^2} \quad (i=1, 2),$$

where the numerical factor 3 comes from the summation with respect to the isotopic spin index α , and the factor 2 comes from the two possible ways in which two pions of wave vector \mathbf{k}_1 and \mathbf{k}_2 are absorbed.

Though this term is derived here from the second order perturbation calculation,

$V_b(r)$ is of the fourth order in coupling constant since the pair term is already of the second order. Thus this term should be called "the fourth order nuclear potential" according to the conventional language. The calculations of the fourth order nuclear potential in the pseudoscalar theory with pseudoscalar coupling were performed previously by Lévy,⁷⁾ Nakabayashi and Sato,⁸⁾ and several others. The results of their analysis agree, as for the leading term, with ours exactly under the same approximation made here. The next higher order term will be obtained in subsection (b'). By using the integral representation of the Hankel functions of an imaginary argument,⁹⁾

$$K_n(\mu r) = \frac{\Gamma(n+1/2)}{\Gamma(1/2)} \left(\frac{2\mu}{r} \right)^n \int_0^\infty \frac{\cos kr dk}{(k^2 + \mu^2)^{n+1/2}}, \quad (3.4)$$

eq. (3.3) can easily be put into the form

$$V_b(x) = -(m_\pi c^2) (f^2/4\pi\hbar c)^2 (m_\pi/2M_0)^2 (6/\pi) (1/x^2) K_1(2x). \quad (3.5)$$

The contribution of this term must not be neglected near the range of nuclear forces since it is of larger order of magnitude by $f^2/4\pi\hbar c \approx 10$ than the second order one, eq. (3.2). This dominant attractive force of shorter range seems to improve the agreement of the theory with the experiment since it enlarges the attractive central force which is somewhat weak compared with the tensor one in eq. (3.2). Indeed, Lévy,⁷⁾ Martin and Verlet^{10)*} obtained satisfactory results on the low energy two nucleon system by assuming the hard core inside.

Now, in order to examine the physical meaning of pair term H_2 according to the program mentioned in § 2, we compare our second order perturbation calculation with the more detailed fourth order one due to Lévy.⁷⁾ In the Lévy's calculations, eq. (3.3) or (3.5) is just the contributions from such virtual processes as is shown in Fig. 3.

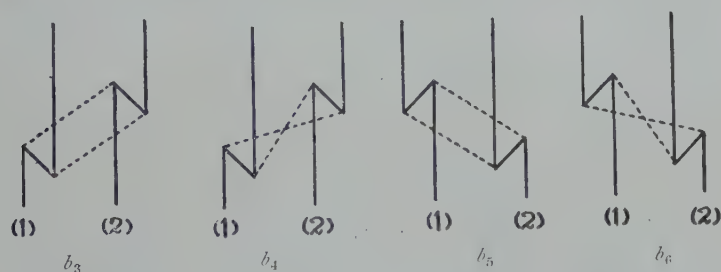


Fig. 3. The virtual processes which give the main contribution to the fourth order nuclear potential in p.s.-p.s. theory.

Comparing with that of ours, and noting that the graph b_1 includes two possible ways in which two pions are absorbed, one could suppose reasonably that the contribution from

* In view of the recent clarifications of certain mistakes involved in Lévy's calculation, their conclusions have become most doubtful, see reference 11. It seems, however, that nothing definite can be said in this connection since those important effects of the "finite self-energy terms" might better the situation. An investigation in this line has recently been made by Jastrow (private communication).

graph b_1 obtained by us is exactly equal to the one by Lévy from graphs $b_2 + b_4$. Actually, if one assumes the p. s. coupling of the form $if\bar{\psi}\gamma_5\tau_a\psi U_a$, the matrix element for the process in which a nucleon pair is created virtually and then one of the pair nucleons annihilates with the initial nucleon together can be given by

$$-\sum_I f^2 (\bar{\psi}_f i\gamma_5 \psi_i) (\bar{\psi}_1 i\gamma_5 \psi_i) \tau_a U_a \tau_b U_b / (E_i - E_1), \quad (3.6)$$

where ψ_i is some negative energy Dirac spinor and ψ_i and ψ_f are positive energy ones of initial and final nucleons. Here the minus sign, which comes from the exchange character of nucleon creation and annihilation operators, is important. If we neglect meson energy compared to the nucleon one in E_i and E_1 and assume nonrelativistic approximation, it can be shown easily that $E_i - E_1 = -2M_0 c^2 = -2\hbar c x$, and $\sum_I (\bar{\psi}_f i\gamma_5 \psi_i) (\bar{\psi}_1 i\gamma_5 \psi_i) \approx (\bar{\psi}_f \psi_i)$. Therefore, in this approximation (3.6) can be replaced effectively by

$$(f^2/2x\hbar c) (\bar{\psi}\psi) (\tau_a U_a)^2 = (f^2/2x\hbar c) (\bar{\psi}\psi) U_a^2, \quad (3.7)$$

which is just the pair term H_2 itself. Thus one can conclude as follows; in the non-relativistic approximation, the simultaneous emission or absorption processes of two pions induced by the pair term is nothing but the virtual processes in which a pair of nucleons is first created and then one of the pair nucleons annihilates together with the one initially present and totally two (one in each transition) pions are emitted or absorbed.

It is to be remarked here that the virtual processes which include two nucleon pairs at the same time are not included in the pair term approximation. Those graphs which contribute to the next higher order terms to $I_b(x)$ (of order $(f^2/4\pi\hbar c)^2 (m_\pi/2M_0)^3$)

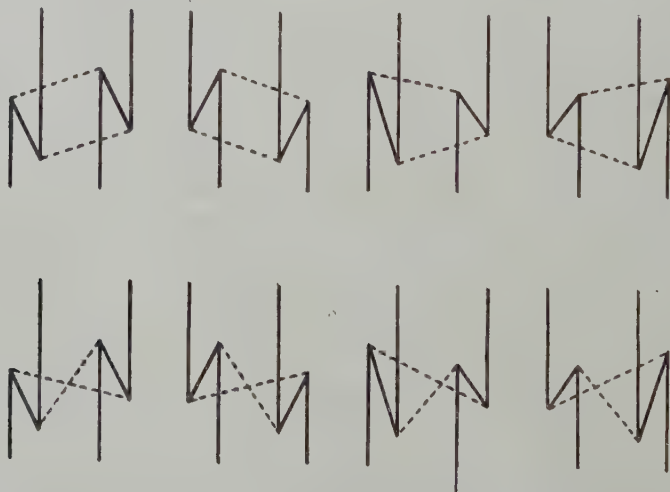


Fig. 4. The typical virtual processes which are not included in the present pair term approximation.

are shown in Fig. 4. However, it is also to be remarked that the pair term neglects the energy of the virtual meson compared to the one of nucleon pair as was mentioned before the equation (3.7), and that this neglect also produces some correction terms of the order

$(f^2/4\pi\hbar c)^2(m_\pi/2M_0)^3$. It is quite easy to show that the correction terms of the order $(f^2/4\pi\hbar c)^2(m_\pi/2M_0)^3$ to $I_b(x)$ arising from the above pair term approximation exactly cancel out the contributions from the graphs in Fig. 4. This exact cancellation is natural since the transformed pair term does not give the above terms entirely. Thus the next higher term of the order $(f^2/4\pi\hbar c)^2(m_\pi/2M_0)^3$ to $I_b(x)$ is correctly given by the contributions in subsection (b'). The similar cancellation will be mentioned in subsection (c').

(c) *The third order potential due to H_2 and H_3*

Now we examine the nuclear potential calculated by the third order perturbation analysis due to H_2 and H_3 . This gives the main contribution to nuclear potential due to the processes in which only one of the two nucleons give rise to the transition to excited state virtually, and thus contributes an important part in nuclear potential due to excited states which is the main purpose of this paper. The possible virtual processes are represented by three diagrams in Fig. 5 and the ones obtained by the interchange of particles (1) and (2) from the former. In the diagrams hereafter the vertical double lines represent the nucleons in excited states. By using eqs. (2.9), (2.10), (2.11), and (2.12) one obtains the following terms corresponding to diagrams c_1 , c_2 , and c_3 , respectively,

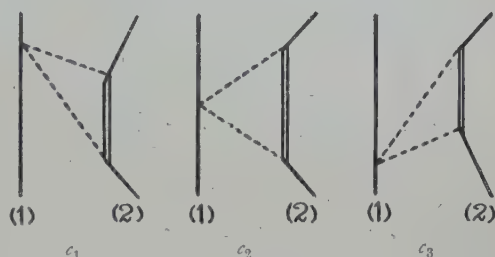


Fig. 5. The virtual processes which give the third order nuclear potential due to H_2 and H_3 .

$$V_{c_1} = \left(\frac{f^2}{2\pi\hbar c} \right) \frac{G^2}{\mu^2} \frac{i^2}{4} \frac{2}{(2\pi)^6} \iint d\mathbf{k}_1 d\mathbf{k}_2 \frac{(T_a^{(2)*} T_a^{(2)}) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1)}{\omega_1 \omega_2 (\omega_1 + \omega_2) (q + \omega_1)} e^{i(k_1 + k_2)r} + \text{exchange term in (1) and (2)}, \quad (3.8)$$

$$V_{c_2} = \left(\frac{f^2}{2\pi\hbar c} \right) \frac{G^2}{\mu^2} \frac{i^2}{4} \frac{2}{(2\pi)^6} \iint d\mathbf{k}_1 d\mathbf{k}_2 \frac{(T_a^{(2)*} T_a^{(2)}) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1)}{\omega_1' \omega_2 (q + \omega_1) (q + \omega_2)} e^{i(k_1 + k_2)r} + \text{exchange term in (1) and (2)}, \quad (3.8)'$$

$$V_{c_3} = \left(\frac{f^2}{2\pi\hbar c} \right) \frac{G^2}{\mu^2} \frac{i^2}{4} \frac{2}{(2\pi)^6} \iint d\mathbf{k}_1 d\mathbf{k}_2 \frac{(T_a^{(2)*} T_a^{(2)}) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1)}{\omega_1 \omega_1 (\omega_1 + \omega_2) (q + \omega_2)} e^{i(k_1 + k_2)r} + \text{exchange term in (1) and (2)}, \quad (3.8)''$$

where q is defined by $\hbar c q = M c^2 - M_0 c^2$, i.e. $\hbar c q$ is the mass difference between excited state nucleon and ground state one. Summing up these terms, one obtains the third order potential corresponding to diagram c ;

$$V_c(\mathbf{r}) = V_{c_1} + V_{c_2} + V_{c_3} = - \left(\frac{f^2}{2\pi\hbar c} \right) \frac{G^2}{4\mu^2} \frac{2}{(2\pi)^6} (T_a^{(2)*} T_a^{(2)}) \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} \iint (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1) e^{i(k_1 + k_2)r}$$

$$\times \left[\frac{1}{(q+\omega_1)(\omega_1+\omega_2)} + \frac{1}{(q+\omega_1)(q+\omega_2)} + \frac{1}{(q+\omega_2)(\omega_1+\omega_2)} \right] \\ + \text{exchange term in (1) and (2)}.$$

In this expression it is required to evaluate the factor $(\Pi^{(s)*} \cdot \mathbf{k}_2)(\Pi^{(s)} \cdot \mathbf{k}_1)$. As is shown in Appendix II, by comparing the matrix elements between various initial and final spin states of nucleon, the factor $(\Pi^{(s)*} \cdot \mathbf{k}_2)(\Pi^{(s)} \cdot \mathbf{k}_1)$ can be replaced exactly by the operator $(\mathbf{k}_1 \cdot \mathbf{k}_2) - (1/3)(\mathbf{k}_2 \cdot \boldsymbol{\sigma}^{(s)})(\mathbf{k}_1 \cdot \boldsymbol{\sigma}^{(s)})$. Thus

$$(\Pi^{(s)*} \cdot \mathbf{k}_2)(\Pi^{(s)} \cdot \mathbf{k}_1) \rightarrow (\mathbf{k}_1 \cdot \mathbf{k}_2) - (1/3)(\mathbf{k}_2 \cdot \boldsymbol{\sigma}^{(s)})(\mathbf{k}_1 \cdot \boldsymbol{\sigma}^{(s)}) \quad (3.9)$$

$$= (2/3)(\mathbf{k}_1 \cdot \mathbf{k}_2) - (i/3)(\boldsymbol{\sigma}^{(s)} \cdot [\mathbf{k}_2 \times \mathbf{k}_1]). \quad (3.10)$$

Since the term $(\boldsymbol{\sigma}^{(s)} \cdot [\mathbf{k}_2 \times \mathbf{k}_1])$, however, has no contribution to the integration in $V_c(\mathbf{r})$, one can replace effectively as follows;

$$(\Pi^{(s)*} \cdot \mathbf{k}_2)(\Pi^{(s)} \cdot \mathbf{k}_1) \rightarrow 2/3(\mathbf{k}_1 \cdot \mathbf{k}_2). \quad (3.11)$$

If further replacements of wave vector \mathbf{k}_1 and \mathbf{k}_2 by the gradient operators are performed, $V_c(\mathbf{r})$ becomes

$$V_c(\mathbf{r}) = (f^2/2\pi\hbar c)(G^2/\mu^2)(1/3)Q_0[(\nabla \cdot \nabla')G_0(\mathbf{r}, \mathbf{r}')]_{\mathbf{r} \rightarrow \mathbf{r}'},$$

where

$$Q_0 = T_\alpha^{(1)*} T_\alpha^{(1)} + T_\alpha^{(2)*} T_\alpha^{(2)}, \quad (3.12)$$

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \\ \times \left[\frac{1}{(q+\omega_1)(q+\omega_2)} + \frac{1}{(q+\omega_1)(\omega_1+\omega_2)} + \frac{1}{(q+\omega_2)(\omega_1+\omega_2)} \right].$$

The isotopic spin factor Q_0 is evaluated as shown in Appendix I,

$$Q_0 = T_\alpha^{(1)*} T_\alpha + T_\alpha^{(2)*} T_\alpha^{(2)} = \sum_\alpha (4/3) \times 2 = 8. \quad (3.13)$$

As for $G^0(\mathbf{r}, \mathbf{r}')$, the angle part integration can be performed easily as follows;

$$\frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} f(\omega_1, \omega_2) \\ = \frac{4}{(2\pi)^4 r r'} \int_0^\infty \int_0^\infty \frac{k_1 dk_1 k_2 dk_2}{\omega_1 \omega_2} \sin(k_1 r) \sin(k_2 r') f(\omega_1, \omega_2). \quad (3.14)$$

Now, concerning the radial part integration, however, the exact evaluation is not possible generally because of the appearance of an extra parameter q in the energy denominators such as $q+\omega$. In order to avoid this difficulty, we adopt the following approximation,

$$1/(q+\omega) = (1/\omega)\omega/(q+\omega) \rightarrow (1/\omega)a(r). \quad (3.15)$$

That is, since the most effective contribution to the integral of (3.14) comes from the neighbourhood of the first maximum point of $\sin kr$, $k=\pi/2r$, the factor $\omega/(q+\omega)$, which varies very slowly with k , can be approximated by a function $a(r)$ that is obtained

from the factor $\omega/(q+\omega)$ by the substitution $k=(\pi/2)(1/r)$. By means of this approximation, the radial part integration of (3.14) can be performed straightforwardly. The function $a(r)$, though it depends really on the relative distance r under consideration, changes very slowly with r in the domain $\infty > r > 1/2$ (Cf. Table I); and, therefore, can be treated as a constant when one perform the differentiation such as $(\nabla \cdot \nabla')$. As the result of this approximation $G_0(r, r')$ can be evaluated easily,

$$(\nabla \cdot \nabla') G_0(r, r') = \frac{1}{(4\pi)^2} [\alpha^2(r) + a(r)] (\nabla \cdot \nabla') \frac{1}{rr'} e^{-\mu r} e^{-\mu r'}.$$

The substitution of these expression into (3.12) gives $(x = \mu r)$

$$V_c(x) = (m_\pi c^2) \left(\frac{m_\pi}{2M_0} \right) \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{G^2}{4\pi\hbar c} \right) \frac{8}{3} (a^2 + a) \frac{1}{x^2} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) e^{-2x}. \quad (3.16)$$

Table I. The values of the function $a(r) = \omega/(q+\omega)$

$x = \mu r$	0.5	1.0	1.5	2.0	3.0
$a(r)$	0.62	0.48	0.41	0.38	0.36

This term represents rather strong central repulsive force of short range, which will be discussed later.

(b') *The third order potential due to H_1 and H_2*

This term is derived in the same way as $V_c(x)$ and is a next higher term to the fourth order one $V_b(x)$. The possible virtual processes are given by the six diagrams in Fig. 6. Summing up all the contributions from these diagrams, one obtains a central repulsive force in the analogous way as (3.16);

$$V_{b'}(x) = (m_\pi c^2) \left(\frac{m_\pi}{2M_0} \right)^3 \left(\frac{f^2}{4\pi\hbar c} \right)^2 6 \frac{1}{x^2} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) e^{-2x}. \quad (3.17)$$

As was discussed by two of the authors in the other paper, the term $V_{b'}(x)$ thus derived is exactly in agreement with the one from other fourth order calculation.¹¹⁾

(d) *The fourth order potential due to H_3*

Next we evaluate the fourth order nuclear potential due to the interaction Hamiltonian H_3 , which is obtained through the virtual processes in which both nucleons undergo

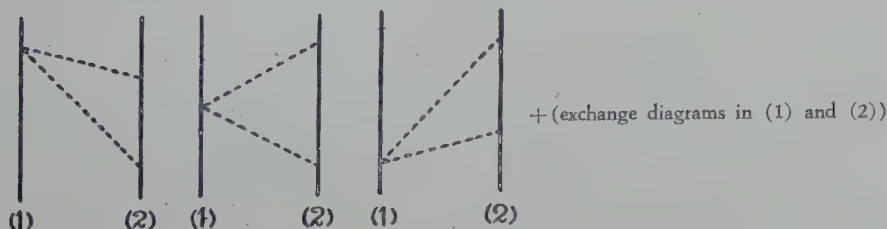


Fig. 6. The virtual processes which give the third order nuclear potential due to interaction Hamiltonian H_1 and H_2 .

transitions to excited states. The possible processes to be considered are represented by six diagrams in Fig. 7 and the ones obtained by the interchange of particles (1) and (2).

The contributions from diagrams d_1 and d_4 are written down straightforwardly as before ;

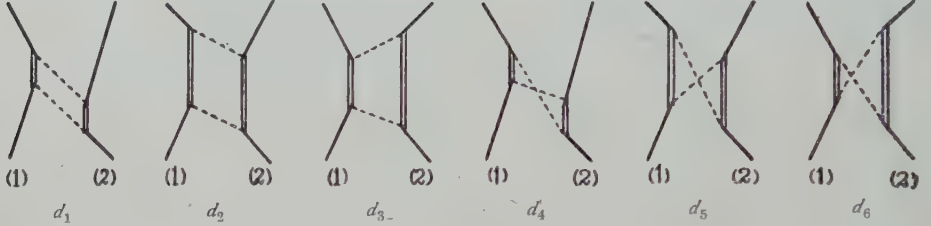


Fig. 7. The virtual processes which give the fourth order nuclear potential due to interaction Hamiltonian H_3 .

$$\begin{aligned}
 I_{d_1}(\mathbf{r}) = & -\left(\frac{G}{\mu}\right)^4 \frac{1}{4\hbar c} \frac{1}{(2\pi)^6} \\
 & \times \iint d\mathbf{k}_1 d\mathbf{k}_2 \frac{(T_{\alpha}^{(1)*} T_{\beta}^{(1)} T_{\alpha}^{(2)*} T_{\beta}^{(2)}) (\Pi^{(1)*} \cdot \mathbf{k}_1) (\Pi^{(1)} \cdot \mathbf{k}_2) (\Pi^{(2)*} \cdot \mathbf{k}_1) (\Pi^{(2)} \cdot \mathbf{k}_2)}{\omega_1 \omega_2 (\omega_1 + \omega_2) (q + \omega_1) (q + \omega_2)} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} \\
 & + \text{exchange term in (1) and (2)}, \quad (3.18)
 \end{aligned}$$

$$\begin{aligned}
 V_{d_4}(\mathbf{r}) = & -\left(\frac{G}{\mu}\right)^4 \frac{1}{4\hbar c} \frac{1}{(2\pi)^6} \\
 & \times \iint d\mathbf{k}_1 d\mathbf{k}_2 \frac{(T_{\alpha}^{(1)*} T_{\beta}^{(1)} T_{\beta}^{(2)*} T_{\alpha}^{(2)}) (\Pi^{(1)*} \cdot \mathbf{k}_1) (\Pi^{(1)} \cdot \mathbf{k}_2) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1)}{\omega_1 \omega_2 (\omega_1 + \omega_2) (q + \omega_1)^2} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} \\
 & + \text{exchange term in (1) and (2)}. \quad (3.19)
 \end{aligned}$$

The contributions from other diagrams can be written down in an analogous way and it is readily seen that the terms corresponding to d_2 and d_3 are of the same form except energy denominators as (3.18) and the ones corresponding to d_5 and d_6 are as (3.19). In the integrand of (3.18) and (3.19) the numerators are treated by the replacement of (3.9) ;

$$\begin{aligned}
 & (\Pi^{(1)*} \cdot \mathbf{k}_1) (\Pi^{(1)} \cdot \mathbf{k}_2) (\Pi^{(2)*} \cdot \mathbf{k}_1) (\Pi^{(2)} \cdot \mathbf{k}_2) \\
 & \rightarrow 1/9 [3(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 + (\sigma^{(1)} \cdot \mathbf{k}_1) (\sigma^{(1)} \cdot \mathbf{k}_2) (\sigma^{(2)} \cdot \mathbf{k}_1) (\sigma^{(2)} \cdot \mathbf{k}_2)], \quad (3.20)
 \end{aligned}$$

$$\begin{aligned}
 & (\Pi^{(1)*} \cdot \mathbf{k}_1) (\Pi^{(1)} \cdot \mathbf{k}_2) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1) \\
 & \rightarrow 1/9 [3(\mathbf{k}_1 \cdot \mathbf{k}_2)^2 + (\sigma^{(1)} \cdot \mathbf{k}_1) (\sigma^{(1)} \cdot \mathbf{k}_2) (\sigma^{(2)} \cdot \mathbf{k}_2) (\sigma^{(2)} \cdot \mathbf{k}_1)], \quad (3.21)
 \end{aligned}$$

in which the terms of no contribution to the integral are omitted. As before, performing the replacement of \mathbf{k}_1 , \mathbf{k}_2 by the gradient operators, it follows that

$$V_d(\mathbf{r}) = - (1/2\hbar c) (G/\mu)^4 [QI(\nabla, \nabla') G(\mathbf{r}, \mathbf{r}') + Q'I'(\nabla, \nabla') G'(\mathbf{r}, \mathbf{r}')]_{\mathbf{r}' \rightarrow \mathbf{r}}, \quad (3.22)$$

$$Q = T_{\alpha}^{(1)*} T_{\beta}^{(1)} T_{\alpha}^{(2)*} T_{\beta}^{(2)}, \quad Q' = T_{\alpha}^{(1)*} T_{\beta}^{(1)} T_{\beta}^{(2)*} T_{\alpha}^{(2)}, \quad (3.23)$$

$$I(\mathbf{r}, \mathbf{r}') = 1/9 [3(\mathbf{r} \cdot \mathbf{r}')^2 + (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r}')(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}')], \quad (3.24)$$

$$I'(\mathbf{r}, \mathbf{r}') = 1/9 [3(\mathbf{r} \cdot \mathbf{r}')^2 + (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r})(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{r}')(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r}')(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{r})], \quad (3.25)$$

$$G(\mathbf{r}, \mathbf{r}') = \frac{-1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \times \left[\frac{1}{(q + \omega_1)(q + \omega_2)(\omega_1 + \omega_2)} + \frac{1}{q(q + \omega_1)(q + \omega_2)} \right], \quad (3.26)$$

$$G'(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \times \left(\frac{1}{2} \right) \left[\frac{1}{(q + \omega_1)^2(\omega_1 + \omega_2)} + \frac{1}{(q + \omega_1)^2(2q + \omega_1 + \omega_2)} + \frac{2}{(q + \omega_1)(q + \omega_2)(2q + \omega_1 + \omega_2)} + \frac{1}{(q + \omega_2)^2(2q + \omega_1 + \omega_2)} + \frac{1}{(q + \omega_2)^2(\omega_1 + \omega_2)} \right]. \quad (3.27)$$

The integrations of $G(\mathbf{r}, \mathbf{r}')$ and $G'(\mathbf{r}, \mathbf{r}')$ can be performed by means of the analogous approximations to (3.15);

$$\begin{aligned} \frac{1}{q + \omega_1} &= \frac{1}{\omega_1} \frac{\omega_1}{q + \omega_1} \rightarrow \frac{1}{\omega_1} a(r), & \frac{1}{q + \omega_2} &\rightarrow \frac{1}{\omega_2} a(r), \\ \frac{1}{2q + \omega_1 + \omega_2} &= \frac{1}{\omega_1 + \omega_2} \frac{\omega_1 + \omega_2}{2q + \omega_1 + \omega_2} \rightarrow \frac{1}{\omega_1 + \omega_2} a(r). \end{aligned} \quad (3.28)$$

Using these approximations one obtains immediately,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \frac{1}{\omega_1 \omega_2} \left(\frac{1}{\omega_1 + \omega_2} + \frac{1}{q} \right) a^2(r), \quad (3.29)$$

where the first term can be integrated by the method in Appendix III and the second term gives directly the usual Yukawa potential. Therefore,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\mu(2\pi)^2} \frac{1}{rr'} \left[\frac{\mu}{q} e^{-\mu(r+r')} - F(r+r') + e^{-\mu r} F(r') + e^{-\mu r'} F(r) \right], \quad (3.30)$$

in which

$$F(r) \equiv \frac{4\mu}{2\pi} \int_0^\infty \frac{k \sin kr}{(k^2 + \mu^2)^{3/2}} dk = \frac{2}{\pi} \mu r K_0(\mu r), \quad (3.31)$$

where $K_n(x)$ is defined by eq. (3.4). As to $G'(\mathbf{r}, \mathbf{r}')$, the same approximations yield

$$G'(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \frac{1}{\omega_1 + \omega_2} \times \frac{1}{2} \left[\left(\frac{1}{\omega_1^2} + \frac{1}{\omega_1 \omega_2} + \frac{1}{\omega_2^2} \right) a^2(1+a) - \frac{1}{\omega_1 \omega_2} a^2(1-a) \right]. \quad (3.32)$$

Further integrations are carried out in Appendix III, and the result is

$$G'(r, r') = \frac{1}{4\mu(2\pi)^2} \frac{1}{rr'} \left[F(r+r')a^2 - \{e^{-\mu r}F(r') + e^{-\mu r'}F(r)\} \frac{a^2(1-a)}{2} \right]. \quad (3.33)$$

Noting that $G(r, r')$ and $G'(r, r')$ become the functions of only r and r' , the differential operators $I(\mathcal{P}, \mathcal{P}')$ and $I'(\mathcal{P}, \mathcal{P}')$ are written down as follows;

$$\begin{aligned} (\mathcal{P} \cdot \mathcal{P}')^2 &= \frac{2}{r^2} \frac{\partial^2}{\partial r \partial r'} + \frac{\partial^4}{\partial r^2 \partial r'^2}, \\ (\sigma^{(1)} \cdot \mathcal{P})(\sigma^{(1)} \cdot \mathcal{P}')(\sigma^{(2)} \cdot \mathcal{P})(\sigma^{(2)} \cdot \mathcal{P}') \\ &= \frac{\partial^4}{\partial r^2 \partial r'^2} + \frac{2}{r^2} \frac{\partial^2}{\partial r \partial r'} - \frac{2}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) \left(\frac{1}{r^2} \frac{\partial^2}{\partial r \partial r'} + \frac{1}{r} \frac{\partial^3}{\partial r^2 \partial r'} + \frac{1}{r} \frac{\partial^3}{\partial r \partial r'^2} \right) \\ &\quad - \frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) S_{12} \left(\frac{2}{r^2} \frac{\partial^2}{\partial r \partial r'} - \frac{1}{r} \frac{\partial^3}{\partial r^2 \partial r'} - \frac{1}{r} \frac{\partial^3}{\partial r \partial r'^2} \right), \end{aligned} \quad (3.34)$$

$$\begin{aligned} (\sigma^{(1)} \cdot \mathcal{P})(\sigma^{(1)} \cdot \mathcal{P}')(\sigma^{(2)} \cdot \mathcal{P}')(\sigma^{(2)} \cdot \mathcal{P}) \\ &= \frac{\partial^4}{\partial r^2 \partial r'^2} + \frac{2}{r^2} \frac{\partial^2}{\partial r \partial r'} + \frac{2}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) \left(\frac{1}{r^2} \frac{\partial^2}{\partial r \partial r'} + \frac{1}{r} \frac{\partial^3}{\partial r^2 \partial r'} + \frac{1}{r} \frac{\partial^3}{\partial r \partial r'^2} \right) \\ &\quad + \frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) S_{12} \left(\frac{2}{r^2} \frac{\partial^2}{\partial r \partial r'} - \frac{1}{r} \frac{\partial^3}{\partial r^2 \partial r'} - \frac{1}{r} \frac{\partial^3}{\partial r \partial r'^2} \right), \end{aligned}$$

where, in the right hand sides, r' has been equated with r after the differentiation have been performed. As can be seen from these expressions the calculations of $I(\mathcal{P}, \mathcal{P}')G(r, r')$ and $I'(\mathcal{P}, \mathcal{P}')G(r, r')$ yield many higher order derivatives of $F(r)$ or $K_0(\mu r)$, which, however, can be reduced to the expressions including only the functions $K_1(x)$ and $K_0(x)$ but not their derivatives at all, by means of the recurrence formula⁽¹²⁾ with respect to differentiation of $K_n(x)$;

$$xK_n'(x) + nK_n(x) = -xK_{n-1}(x), \quad K_0'(x) = -K_1(x). \quad (3.35)$$

Carrying out all of these tedious and elementary calculations and equating r' to r finally, one obtains as the final result ($x = \mu r$);

$$V_a(x) = -(m_\pi c^2) \left(\frac{G^2}{4\pi\hbar c} \right)^2 [R_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)})R_2(x) + S_{12}R_3(x)], \quad (3.36)$$

$$\begin{aligned} R_1(x) &= \frac{2}{9} \left[\frac{2}{\pi} a^2(Q-Q') \left\{ - \left(\frac{6}{x^2} + \frac{23}{2x^4} \right) K_1(2x) - \left(\frac{2}{x} + \frac{23}{2x^3} \right) K_0(2x) \right\} \right. \\ &\quad + \frac{4}{\pi} a^2 Q \left\{ \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4} \right) K_1(x) + \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3} \right) K_0(x) \right\} e^{-x} \\ &\quad + \frac{2}{\pi} (a^3 - a^2) Q' \left\{ \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4} \right) K_1(x) + \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3} \right) K_0(x) \right\} e^{-x} \\ &\quad \left. + \frac{\mu}{Q} a^2 Q \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{10}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x} \right], \end{aligned} \quad (3.37)$$

$$\begin{aligned}
 R_2(x) = & \frac{1}{27} \left[\frac{4}{\pi} a^2 (Q + Q') \left\{ - \left(\frac{2}{x^2} + \frac{3}{x^4} \right) K_1(2x) - \frac{3}{x^3} K_0'(2x) \right\} \right. \\
 & + \frac{4}{\pi} a^2 Q \left\{ \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0'(x) \right\} e^{-x} \\
 & - \frac{2}{\pi} (a^3 - a^2) Q' \left\{ \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0'(x) \right\} e^{-x} \\
 & \left. + \frac{\mu}{q} a^2 Q \left(\frac{2}{x^3} + \frac{5}{x^4} + \frac{6}{x^5} + \frac{3}{x^6} \right) e^{-2x} \right], \quad (3.38)
 \end{aligned}$$

$$\begin{aligned}
 R_3(x) = & \frac{1}{27} \left[\frac{2}{\pi} a^2 (Q + Q') \left\{ \left(\frac{2}{x^2} + \frac{15}{2x^4} \right) K_1(2x) + \frac{6}{x^3} K_0'(2x) \right\} \right. \\
 & - \frac{2}{\pi} a^2 Q \left\{ \left(\frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0'(x) \right\} e^{-x} \\
 & + \frac{1}{\pi} (a^3 - a^2) Q' \left\{ \left(\frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) K_1(x) + \left(\frac{1}{x^2} + \frac{1}{x^3} \right) K_0'(x) \right\} e^{-x} \\
 & \left. - \frac{\mu}{q} a^2 Q \left(\frac{1}{x^3} + \frac{4}{x^4} + \frac{6}{x^5} + \frac{3}{x^6} \right) e^{-2x} \right], \quad (3.39)
 \end{aligned}$$

where the factor $(\sigma^{(1)} \cdot \sigma^{(2)})$ multiplied by S_{12} is already put equal to 1. In these expressions the isotopic spin matrices Q and Q' are, as are calculated in Appendix I;

$$\begin{aligned}
 Q = \frac{40}{9}, \quad Q' = \frac{56}{9} & \quad \text{for isotopic spin triplet,} \\
 Q = \frac{72}{9}, \quad Q' = \frac{24}{9} & \quad \text{for isotopic spin singlet.} \quad (3.40)
 \end{aligned}$$

And the spin factor $(\sigma^{(1)} \cdot \sigma^{(2)})$ takes the well known eigenvalue;

$$\begin{aligned}
 (\sigma^{(1)} \cdot \sigma^{(2)}) &= 1 \quad \text{for spin triplet,} \\
 &= -3 \quad \text{for spin singlet.} \quad (3.31)
 \end{aligned}$$

The term $V_a(x)$ thus obtained is another part of the contributions to the nuclear potential due to the excited states, that is due to the interaction in which both nucleons pass through the excited states.

(c') *The fourth order potential due to H_1 and H_3*

The possible virtual processes are represented by the twelve diagrams in Fig. 8. and the same number of diagrams obtained from them by the interchange of particles (1) and (2). The contributions from these diagrams are readily seen to be of the order $(m_\pi/2M_0)^2 (f^2/4\pi\hbar c) (G^2/4\pi\hbar c)$ and therefore the next higher order terms to $V_c(x)$ which is of the order $(m_\pi/2M_0) (f^2/4\pi\hbar c) (G^2/4\pi\hbar c)$. Again these corrections prove to be very large compared to $V_c(x)$ quite the same way as $V_{b'}(x)$ compared with $V_b(x)$. We will consistently consider up to the contributions $V_{c'}(x)$ (which will be calculated below), because we took into account up to $V_{b'}(x)$ before. Quite the same way as before, one can immediately obtain

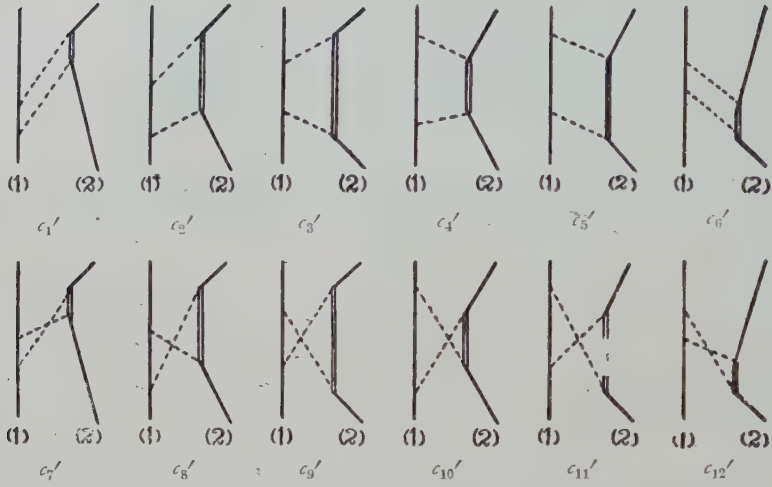


Fig. 8. The virtual process which give the fourth order nuclear potential due to interaction Hamiltonian H_1 and H_3 .

$$\begin{aligned}
 V_{c'}(\mathbf{r}) = & -\left(\frac{f}{2\pi}\right)^2 \left(\frac{G}{\mu}\right)^2 \frac{1}{4\hbar c} \frac{1}{(2\pi)^6} \iint d\mathbf{k}_1 d\mathbf{k}_2 (\tau_{\alpha}^{(1)} \tau_{\beta}^{(1)} T_{\alpha}^{(2)*} T_{\beta}^{(2)}) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) \\
 & \times (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1) \frac{1}{D} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} + \text{exchange term in (2) and (2)} \\
 & -\left(\frac{f}{2\pi}\right)^2 \left(\frac{G}{\mu}\right)^2 \frac{1}{4\hbar c} \frac{1}{(2\pi)^6} \iint d\mathbf{k}_1 d\mathbf{k}_2 (\tau_{\alpha}^{(1)} \tau_{\beta}^{(1)} T_{\beta}^{(2)*} T_{\alpha}^{(2)}) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) \\
 & \times (\Pi^{(2)*} \cdot \mathbf{k}_1) (\Pi^{(2)} \cdot \mathbf{k}_2) \frac{1}{D'} e^{i(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{r}} + \text{exchange term in (1) and (2)}, \quad (3.42)
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{D} = & \frac{1}{\omega_1(\omega_1 + \omega_2)(q + \omega_2)} + \frac{1}{\omega_1 q(q + \omega_2)} + \frac{1}{(q + \omega_1)q(q + \omega_2)} \\
 & + \frac{1}{\omega_1 q \omega_2} + \frac{1}{(q + \omega_1)q \omega_2} + \frac{1}{(q + \omega_1)(\omega_1 + \omega_2)\omega_2}, \\
 \frac{1}{D'} = & \frac{1}{\omega_1(\omega_1 + \omega_2)(q + \omega_1)} + \frac{1}{\omega_1(q + \omega_1 + \omega_2)(q + \omega_1)} + \frac{1}{(q + \omega_2)(q + \omega_1 + \omega_2)(q + \omega_1)} \\
 & + \frac{1}{\omega_1(q + \omega_1 + \omega_2)\omega_2} + \frac{1}{(q + \omega_2)(q + \omega_1 + \omega_2)\omega_2} + \frac{1}{(q + \omega_2)(\omega_1 + \omega_2)\omega_2}.
 \end{aligned}$$

By the replacement (3.9), one obtains, dropping the terms which contribute nothing to the above integral,

$$\begin{aligned}
 & (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\Pi^{(2)*} \cdot \mathbf{k}_2) (\Pi^{(2)} \cdot \mathbf{k}_1) \rightarrow (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \\
 & \quad - (1/3) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_1), \\
 & (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\Pi^{(2)*} \cdot \mathbf{k}_1) (\Pi^{(2)} \cdot \mathbf{k}_2) \rightarrow (\mathbf{k}_1 \cdot \mathbf{k}_2)^2 \\
 & \quad - (1/3) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_2) (\boldsymbol{\sigma}^{(1)} \cdot \mathbf{k}_1) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_1) (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{k}_2). \quad (3.43)
 \end{aligned}$$

Now we must make the similar approximations as (3.15) in the energy denominators $1/D$ and $1/D'$;

$$\begin{aligned} 1/(q+\omega_1) &\rightarrow 1/\omega_1 \cdot a(r), \quad 1/(q+\omega_2) \rightarrow 1/\omega_2 \cdot a(r), \\ 1/(q+\omega_1+\omega_2) &= 1/(\omega_1+\omega_2) \cdot (\omega_1+\omega_2)/(q+\omega_1+\omega_2) \rightarrow 1/(\omega_1+\omega_2) \cdot b(r), \end{aligned} \quad (3.44)$$

where $a(r)$ is defined before and $b(r)$ is a function of r which is obtained from $(\omega_1+\omega_2)/(q+\omega_1+\omega_2)$ by the substitution $k_1=k_2=(\pi/2)(1/r)$, which is again a slowly varying function of r and is assumed to be treated as a constant when performing the differentiation with respect to r . The other calculations are quite similar as those in subsection (d) and the result is as follows ($x=pr$);

$$U_{e1}(x) = -(m_\pi c^2) \left(\frac{m_\pi}{2M_0} \right)^2 \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{G^2}{4\pi\hbar c} \right) [U_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_2(x) + S_{12} U_3(x)], \quad (3.45)$$

$$\begin{aligned} U_1(x) = & \frac{1}{3} \left[-\frac{2}{\pi} \{ 2aP - (2a(1+b) - b(1+a^2))P' \} \left\{ \left(\frac{6}{x^2} + \frac{23}{2x^4} \right) K_1(2x) \right. \right. \\ & + \left(\frac{2}{x} + \frac{23}{2x^3} \right) K_0(2x) - 2 \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4} \right) e^{-x} K_1(x) - 2 \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3} \right) \\ & \times e^{-x} K_0(x) \left. \right\} + \frac{2}{\pi} a(1+b)P' \left\{ 2 \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{4}{x^4} \right) e^{-x} K_1(x) \right. \\ & + 2 \left(\frac{1}{x} + \frac{2}{x^2} + \frac{2}{x^3} \right) e^{-x} K_0(x) \left. \right\} + \frac{\mu}{q} (1+a)^2 P \left\{ \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{10}{x^4} \right. \right. \\ & \left. \left. + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x} \right\} \left. \right], \quad (3.46) \end{aligned}$$

$$\begin{aligned} U_2(x) = & -\frac{1}{9} \left[\frac{2}{\pi} \{ 2aP + (2a(1+b) - b(1+a^2))P' \} \left\{ - \left(\frac{4}{x^2} + \frac{6}{x^4} \right) K_1(2x) \right. \right. \\ & - \frac{6}{x^3} K_0(2x) + 2 \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) e^{-x} K_1(x) + 2 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{-x} K_0(x) \left. \right\} \\ & - \frac{2}{\pi} a(1+b)P' \left\{ 2 \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{2}{x^4} \right) e^{-x} K_1(x) + 2 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{-x} K_0(x) \right\} \\ & + \frac{\mu}{q} (1+a)^2 P \left\{ \left(\frac{2}{x^3} + \frac{5}{x^4} + \frac{6}{x^5} + \frac{3}{x^6} \right) e^{-2x} \right\} \left. \right], \quad (3.47) \end{aligned}$$

$$\begin{aligned} U_3(x) = & -\frac{1}{18} \left[\frac{2}{\pi} \{ 2aP + (2a(1+b) - b(1+a^2))P' \} \left\{ \left(\frac{4}{x^2} + \frac{15}{x^4} \right) K_1(2x) \right. \right. \\ & + \frac{12}{x^3} K_0(2x) - 2 \left(\frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) e^{-x} K_1(x) - 2 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{-x} K_0(x) \left. \right\} \\ & + \frac{2}{\pi} a(1+b)P' \left\{ 2 \left(\frac{1}{x^2} + \frac{5}{x^3} + \frac{5}{x^4} \right) e^{-x} K_1(x) + 2 \left(\frac{1}{x^2} + \frac{1}{x^3} \right) e^{-x} K_0(x) \right\} \\ & - \frac{\mu}{q} (1+a)^2 P \left\{ \left(\frac{2}{x^3} + \frac{8}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x} \right\} \left. \right], \quad (3.48) \end{aligned}$$

where the isotopic spin factor P and P' have the following values, as is calculated in Appendix I;

$$P \equiv \tau_{\alpha}^{(1)} \tau_{\beta}^{(1)} T_{\alpha}^{(2)*} T_{\beta}^{(2)} = 13/6, \quad P' \equiv \tau_{\alpha}^{(1)} \tau_{\beta}^{(1)} T_{\alpha}^{(2)*} T_{\beta}^{(2)} = 8/3, \quad \text{for isotopic spin triplet,} \\ P = 0, \quad P' = 8, \quad \text{for isotopic spin singlet.} \quad (3.49)$$

Thus the required quantities are all derived except the next one. The effect on nuclear potential due to the excited states of nucleons is represented by the sum of $I_e(x)$, $I_{e'}(x)$, and $I_a(x)$. The purpose of this paper is to investigate quantitatively how these terms modify the usual nuclear potential $I_a(x) + I_b(x) + I_{b'}(x) + I_{b''}(x)$, which, however, will be discussed in detail in the following paper (Part II).

(b'') *The fourth order potential due to H_1 only*

It is readily seen that these terms are of the order $(f^2/4\pi\hbar c)^2(m_{\pi}/2M_0)^4$ and thus the highest contributions to the usual fourth order nuclear potential in p.s.-p.s. theory and have the same form as the usual fourth order potentials due to p.v. coupling, which will be given in the next section.

§ 4. Calculation of nuclear potential for p.v. coupling

For this type of coupling, the static nuclear potential can be obtained more easily and unambiguously than for p.s. coupling. The second and fourth order potentials due to p.v. coupling in (2.2) have been derived by many authors and the results are given in the following;

The second order potential:

$$V_a(x) = (m_{\pi}c^2) \left(\frac{g^2}{4\pi\hbar c} \right) (\tau^{(1)} \cdot \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) \frac{1}{x} e^{-x} \right. \\ \left. + \frac{1}{3} S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{1}{x} e^{-x} \right], \quad (4.1)$$

The fourth order potential¹³⁾:

$$I_b(x) = (m_{\pi}c^2) \left(\frac{g^2}{4\pi\hbar c} \right)^2 [(\tau^{(1)} \cdot \tau^{(2)}) U_{\tau}(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_{\sigma}(x) + S_{12} U_T(x)], \quad (4.2)$$

where*

$$U_{\tau}(x) = -\frac{8}{\pi} \left\{ \left(\frac{1}{x} + \frac{23}{4x^3} \right) K_0(2x) + \left(\frac{3}{x^2} + \frac{23}{4x^4} \right) K_1(2x) \right\}, \quad (4.3)$$

$$U_{\sigma}(x) = \frac{8}{\pi} \left\{ \frac{3}{x^3} K_0(2x) + \left(\frac{2}{x^2} + \frac{3}{x^4} \right) K_1(2x) \right\}, \quad (4.4)$$

$$U_T(x) = -\frac{8}{\pi} \left\{ \frac{3}{x^3} K_0(2x) + \left(\frac{1}{x^2} + \frac{15}{4x^4} \right) K_1(2x) \right\}. \quad (4.5)$$

* The two mistakes in the eqs. (28) and (29) in Taketani, Machida and Ônuma's paper (Prog. Theor. Phys. 7 (1952), 45) are corrected in our eqs. (4.3) and (4.4).

The contributions of excited states of nucleons to nuclear potential consist of two parts, one of which will be denoted by $V_e(x)$ and is due to the virtual processes in which only one of the nucleons passes through the excited states and the other of which will be denoted by $V_d(x)$ and is due to the virtual processes in which both of the nucleons pass through the excited states. It is readily seen that $V_e(x)$ can be obtained from $V_o(x)$ in (3.45) only by the replacement of the factor $(m_\pi/2M_0)^2(f^2/4\pi\hbar c)$ by $(g^2/4\pi\hbar c)^2$ and $V_d(x)$ is quite the same as was calculated in (3.36). Thus we can readily write down them:

$$V_e(x) = -(m_\pi c^2)(g^2/4\pi\hbar c)(G^2/4\pi\hbar c)[U_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)})U_2(x) + S_{12}U_3(x)], \quad (4.6)$$

where $U_1(x)$, $U_2(x)$, and $U_3(x)$ are given by (3.46), (3.47), and (3.48).

$$V_d(x) = -(m_\pi c^2)(G^2/4\pi\hbar c)^2[R_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)})R_2(x) + S_{12}R_3(x)] \quad (4.7)$$

where $R_1(x)$, $R_2(x)$, and $R_3(x)$ are given by (3.37), (3.38), and (3.39).

§ 5. Qualitative conclusions

In the preceding two sections the nuclear potentials are derived for p.s. and p.v. couplings respectively. We will summarize them below. For the p.s. coupling:

$$\begin{aligned} V_s(x) &= V_a(x) + V_b(x) + V_{b'}(x) + V_{b''}(x) + V_o(x) + V_o'(x) + V_d(x) \\ &= (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{m_\pi}{2M_0} \right)^2 (\tau^{(1)} \cdot \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) \frac{1}{x} e^{-x} \right. \\ &\quad \left. + \frac{1}{3} S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{1}{x} e^{-x} \right] - (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right)^2 \left(\frac{m_\pi}{2M_0} \right)^2 \frac{6}{\pi} \frac{1}{x^2} K_1(2x) \\ &\quad + (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right)^2 \left(\frac{m_\pi}{2M_0} \right)^3 6 \frac{1}{x^2} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) e^{-2x} \\ &\quad + (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right)^2 \left(\frac{m_\pi}{2M_0} \right)^4 [(\tau^{(1)} \cdot \tau^{(2)}) U_\tau(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_\sigma(x) + S_{12} U_T(x)] \\ &\quad + (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{G^2}{4\pi\hbar c} \right) \left(\frac{m_\pi}{2M_0} \right) \frac{8}{3} (a^2 + a) \frac{1}{x^2} \left(1 + \frac{2}{x} + \frac{1}{x^2} \right) e^{-2x} \\ &\quad - (m_\pi c^2) \left(\frac{f^2}{4\pi\hbar c} \right) \left(\frac{G^2}{4\pi\hbar c} \right) \left(\frac{m_\pi}{2M_0} \right)^2 [U_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_2(x) + S_{12} U_3(x)] \\ &\quad - (m_\pi c^2) \left(\frac{G^2}{4\pi\hbar c} \right)^2 [R_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) R_2(x) + S_{12} R_3(x)]. \end{aligned} \quad (5.1)$$

For the p.v. coupling:

$$\begin{aligned} V_v(x) &= V_a(x) + V_b(x) + V_o(x) + V_d(x) \\ &= (m_\pi c^2) \left(\frac{g^2}{4\pi\hbar c} \right) (\tau^{(1)} \cdot \tau^{(2)}) \left[\frac{1}{3} (\sigma^{(1)} \cdot \sigma^{(2)}) \frac{1}{x} e^{-x} + \frac{1}{3} S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \frac{1}{x} e^{-x} \right] \end{aligned}$$

$$\begin{aligned}
& + (m_\pi c^2) \left(\frac{g^2}{4\pi\hbar c} \right)^2 [(\tau^{(1)} \cdot \tau^{(2)}) U_\tau(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_\sigma(x) + S_{12} U_T(x)] \\
& - (m_\pi c^2) \left(\frac{g^2}{4\pi\hbar c} \right) \left(\frac{G^2}{4\pi\hbar c} \right) [U_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) U_2(x) + S_{12} U_3(x)] \\
& - (m_\pi c^2) \left(\frac{G^2}{4\pi\hbar c} \right)^2 [R_1(x) + (\sigma^{(1)} \cdot \sigma^{(2)}) R_2(x) + S_{12} R_3(x)]. \quad (5.2)
\end{aligned}$$

The rather complicated expressions of R 's and U 's are not given here again. It is only noted that they all behave like $\exp(-2x)$ for large x and like x^{-5} or x^{-6} for very small x . Thus we can conclude that the nuclear potential due to the excited states have approximately the same force range as the usual fourth order ones, and the more strong singularities than the usual fourth order potentials. By investigating the functions U 's and R 's, which are all characteristic of terms due to the virtual processes through the excited states, it can be shown that the terms containing the factor μ/q are at least several times larger than all the remaining terms in each of the U 's and R 's. These main terms are the contributions from the processes involving the virtual states in which either or both nucleons are raised up to the excited states with no meson present, the typical diagrams of which are represented in Fig. 9. Thus we obtain another important conclusion that the virtual processes in which one or two of the nucleons pass through the excited states without meson accompanied make the main contributions to nuclear potential due to excited states, and that, therefore, a resonance scattering phenomenon may be expected to occur for sufficiently high energy nucleon scatterings. We can give, therefore, approximately

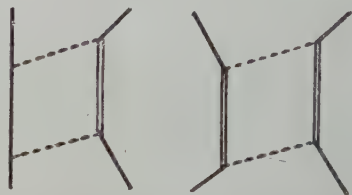


Fig. 9. The typical two diagrams which make the main contributions to the nuclear potential due to the excited states.

$$\begin{aligned}
U_1(x) & \approx \frac{1}{3} \frac{\mu}{q} (1+a)^2 P \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{10}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}, \\
U_2(x) & \approx -\frac{1}{18} \frac{\mu}{q} (1+a)^2 P \left(\frac{4}{x^3} + \frac{10}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}, \\
U_3(x) & \approx \frac{1}{18} \frac{\mu}{q} (1+a)^2 P \left(\frac{2}{x^3} + \frac{8}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}, \quad (5.3)
\end{aligned}$$

$$\begin{aligned}
R_1(x) & \approx \frac{2}{9} \frac{\mu}{q} a^2 Q \left(\frac{1}{x^2} + \frac{4}{x^3} + \frac{10}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}, \\
R_2(x) & \approx \frac{1}{54} \frac{\mu}{q} a^2 Q \left(\frac{4}{x^3} + \frac{10}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}, \\
R_3(x) & \approx -\frac{1}{54} \frac{\mu}{q} a^2 Q \left(\frac{2}{x^3} + \frac{8}{x^4} + \frac{12}{x^5} + \frac{6}{x^6} \right) e^{-2x}. \quad (5.4)
\end{aligned}$$

Furthermore we must take from pion-nucleon scattering data

$$\mu/q \cong 1/2, \quad (5.5)$$

and from eq. (3.15) for $x=1$

$$a \cong 1/2. \quad (5.6)$$

Thus it is seen that U_i is 14 or 27-times larger than R_i (for each $i=1, 2, 3$) except for the factor P and Q and qualitatively the R 's can be neglected entirely compared with U 's. Remembering that the terms containing R 's, or the ones containing U 's, are due to the virtual processes in which both, or either, nucleons pass through the excited states, respectively, we can conclude that the effects of the virtual processes in which both nucleons pass through the excited states are much smaller than the effects of the ones in which only one nucleon passes through the excited state if the coupling constants are chosen so that

$$(f^2/4\pi\hbar c)(m_\pi/2M_0)^2 \gg (G^2/4\pi\hbar c), \text{ and } g^2/4\pi\hbar c \gg G^2/4\pi\hbar c, \quad (5.8)$$

which are very natural from theoretical considerations.

From eqs. (3.40) and (3.49),

$$\begin{aligned} P=0, \quad Q=8, \quad & \text{for isotopic spin singlet,} \\ P \approx Q \approx 5, \quad & \text{for isotopic spin triplet.} \end{aligned} \quad (5.8)$$

It is concluded, therefore, that the effect of nucleon isobars is very large only for charge triplet states; for charge singlet states it is much smaller. Thus the effect upon deuteron problems will rather be small; but the effect upon proton-proton scatterings would be much larger. The details will be discussed in the subsequent paper (Part II).

Finally we will summarize in the following several qualitative conclusions thus far obtained:

(i) The nuclear potential due to excited states have approximately the same force range as the usual fourth order ones, which is due to the circumstances that the effects of excited states are calculated in our paper as the fourth order perturbation, and have more strong singularities ($\approx r^{-6}$) than the usual ones of the same order ($\approx r^{-4}$ or r^{-5}).

(ii) The main contributions due to excited states come from the virtual processes in which one or the two of the nucleons can be raised up to their excited states with no mesons present and therefore a resonance scattering may be expected to occur for sufficiently high energy nucleon-nucleon scatterings.

(iii) If the coupling constants are chosen reasonably, the contributions of virtual processes in which both nucleons pass through excited states are much smaller than that of virtual processes in which only one of nucleons passes through excited states.

(iv) The above contributions vanish identically for charge singlet states owing to the identically vanishment of the isotopic spin factor. Thus the effect of nucleon isobars is very large only for charge triplet states; it is much smaller for charge singlet states.

(v) The nuclear potential due to nucleon isobars consist of a very large central attractive force for all states and a very small tensor force of various sign.

In conclusion, the authors wish to express their sincere thanks to Professor M. Taketani and several other members who gathered at the Yukawa Hall in May for their valuable

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Appendix I

The method of treating the isotopic spin matrix T_α

We assumed in (2.4) the new interaction Hamiltonian

$$(G/\mu)\bar{\Psi}_\mu T_\alpha \psi (\partial U_\alpha / \partial x_\mu) + \text{complex conjugate}, \quad (\text{A} \cdot 1)$$

where

$$\Psi_\mu = \begin{bmatrix} \Psi_\mu^{++} \\ \Psi_\mu^+ \\ \Psi_\mu^0 \\ \Psi_\mu^- \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} \psi_P \\ \psi_N \end{bmatrix}. \quad (\text{A} \cdot 2)$$

and

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1/\sqrt{3} \\ -1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -i & 0 \\ 0 & i/\sqrt{3} \\ -i/\sqrt{3} & 0 \\ 0 & i \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 \\ 2/\sqrt{3} & 0 \\ 0 & -2/\sqrt{3} \\ 0 & 0 \end{bmatrix}. \quad (\text{A} \cdot 3)$$

We will investigate here various properties of these three matrices and their hermite conjugates. From eq. (A.3) and their conjugates

$$T_1^* = \begin{bmatrix} 1 & 0 & -1/\sqrt{3} & 0 \\ 0 & -1/\sqrt{3} & 0 & 1 \end{bmatrix}, \quad T_2^* = \begin{bmatrix} i & 0 & i/\sqrt{3} & 0 \\ 0 & -i/\sqrt{3} & 0 & -i \end{bmatrix}, \\ T_3^* = \begin{bmatrix} 0 & 2/\sqrt{3} & 0 & 0 \\ 0 & 0 & -2/\sqrt{3} & 0 \end{bmatrix}, \quad (\text{A} \cdot 4)$$

it can be derived by direct multiplications that

$$T_1^* T_1 = T_2^* T_2 = T_3^* T_3 = \frac{4}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{4}{3}, \quad (\text{A} \cdot 5)$$

and

$$T_1^* T_2 = -T_2^* T_1 = -\frac{2}{3} i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -\frac{2}{3} i \tau_3, \\ T_2^* T_3 = -T_3^* T_2 = -\frac{2}{3} i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\frac{2}{3} i \tau_1, \\ T_3^* T_1 = -T_1^* T_3 = -\frac{2}{3} i \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\frac{2}{3} i \tau_2, \quad (\text{A} \cdot 6)$$

which are to be compared with the usual relations among τ_α 's:

$$\tau_1^2 = \tau_2^2 = \tau_3^2 = 1 \quad (\text{A} \cdot 5)'$$

$$\tau_1 \tau_2 = -\tau_2 \tau_1 = i \tau_3, \text{ etc.} \quad (\text{A} \cdot 6)'$$

The value of Q_0 in (3.12) can readily be obtained by eq. (A.5) and is given by

(3·13). The other isotopic spin matrices appearing in the calculations, $T_\alpha^{(1)*} T_\beta^{(1)} T_\alpha^{(2)*} T_\beta^{(2)}$, $T_\alpha^{(1)*} T_\beta^{(1)} T_\beta^{(2)*} T_\alpha^{(2)}$, $T_\alpha^{(1)*} T_\beta^{(1)} \tau_\alpha^{(2)} \tau_\beta^{(2)}$, and $T_\alpha^{(1)*} T_\beta^{(1)} \tau_\beta^{(2)} \tau_\alpha^{(2)}$, can be modified by eqs. (A·5), (A·6) and are given by :

$$\begin{aligned} Q &\equiv T_\alpha^{(1)*} T_\beta^{(1)} T_\alpha^{(2)*} T_\beta^{(2)} = \frac{16}{3} - \frac{8}{9} (\tau^{(1)} \cdot \tau^{(2)}), \\ Q' &\equiv T_\alpha^{(1)*} T_\beta^{(1)} T_\beta^{(2)*} T_\alpha^{(2)} = \frac{16}{3} + \frac{8}{9} (\tau^{(1)} \cdot \tau^{(2)}), \\ P &\equiv \tau_\alpha^{(1)} \tau_\beta^{(1)} T_\alpha^{(1)*} T_\beta^{(2)*} = 4 + \frac{4}{3} (\tau^{(1)} \cdot \tau^{(2)}), \\ P' &\equiv \tau_\alpha^{(1)} \tau_\beta^{(1)} T_\beta^{(2)*} T_\alpha^{(2)*} = 4 - \frac{4}{3} (\tau^{(1)} \cdot \tau^{(2)}), \end{aligned} \quad (\text{A} \cdot 7)$$

whose values are readily obtained as :

For isotopic triplet $((\tau^{(1)} \cdot \tau^{(2)}) = 1)$:

$$Q = \frac{40}{9}, \quad Q' = \frac{56}{9}, \quad P = \frac{16}{3}, \quad P' = \frac{8}{3}, \quad (\text{A} \cdot 8)$$

For isotopic singlet $((\tau^{(1)} \cdot \tau^{(2)}) = -3)$:

$$Q = \frac{72}{9}, \quad Q' = \frac{24}{9}, \quad P = 8, \quad P' = 0, \quad (\text{A} \cdot 9)$$

thus guarantying that nuclear interaction induced by (A·1) does really satisfy the charge independence requirement.

If we introduce the usual complex meson fields, using the relations

$$U = (U_1 - iU_2)/\sqrt{2} \quad \text{and} \quad U^* = (U_1 + iU_2)/\sqrt{2}, \quad (\text{A} \cdot 10)$$

we obtain from (A·1)

$$\begin{aligned} &\sqrt{2} \frac{G}{\mu} \bar{\Psi}_\mu \begin{bmatrix} 1 & 0 \\ 0 & -1/\sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \psi \frac{\partial U}{\partial x_\mu} + \sqrt{2} \frac{G}{\mu} \bar{\Psi}_\mu \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1/\sqrt{3} & 0 \\ 0 & 1 \end{bmatrix} \\ &\times \psi \frac{\partial U^*}{\partial x_\mu} + \sqrt{2} \frac{G}{\mu} \bar{\Psi}_\mu \begin{bmatrix} 0 & 0 \\ \sqrt{2}/\sqrt{3} & 0 \\ 0 & -\sqrt{2}/\sqrt{3} \\ 0 & 0 \end{bmatrix} \psi \frac{\partial U_3}{\partial x_\mu} + \text{Comp. Conj.} \end{aligned} \quad (\text{A} \cdot 11)$$

or*

$$\begin{aligned} &\sqrt{2} \frac{G}{\mu} \left[\left(\bar{\Psi}_\mu^+ \psi_P - \frac{1}{\sqrt{3}} \bar{\Psi}_\mu^+ \psi_N \right) \frac{\partial U}{\partial x_\mu} + \left(-\frac{1}{\sqrt{3}} \bar{\Psi}_\mu^0 \psi_P + \bar{\Psi}_\mu^- \psi_N \right) \frac{\partial U^*}{\partial x_\mu} \right. \\ &\quad \left. + \left(\frac{\sqrt{2}}{\sqrt{3}} \bar{\Psi}_\mu^+ \psi_P - \frac{\sqrt{2}}{\sqrt{3}} \bar{\Psi}_\mu^0 \psi_N \right) \frac{\partial U_3}{\partial x_\mu} \right] + \text{Comp. Conj.} \end{aligned} \quad (\text{A} \cdot 12)$$

* The choice of Minami *et. al.* of reference 3) (see their eq. (4·5) or (4·6)) is unfortunately mistaken. Only our expression (A·1) or (A·2) does satisfy the charge independence requirement.

Appendix II

The evaluation of the matrix element

We will verify the replacement of (3.9), namely

$$(\Pi^* \cdot \mathbf{k}_2) (\Pi \cdot \mathbf{k}_1) \rightarrow (\mathbf{k}_1 \cdot \mathbf{k}_2) - (1/3) (\boldsymbol{\sigma} \cdot \mathbf{k}_2) (\boldsymbol{\sigma} \cdot \mathbf{k}_1). \quad (\text{A} \cdot 13)$$

By definition in (2.12),

$$\Pi_j = (\bar{\Psi}_j \psi_j), \quad \Pi_j^* = (\bar{\psi}_j \Psi_j), \quad (\text{A} \cdot 14)$$

and the four independent solutions for Ψ_j for a rest nucleon are from eq. (2.8)

$$\begin{aligned} \vec{\Psi}^{(1)} &= e_1 \psi_\uparrow, \quad \vec{\Psi}^{(2)} = e_2 \psi_\downarrow, \\ \vec{\Psi}^{(3)} &= \frac{1}{\sqrt{3}} e_2 \psi_\uparrow + \sqrt{\frac{2}{3}} e_3 \psi_\downarrow, \quad \vec{\Psi}^{(4)} = \frac{1}{\sqrt{3}} e_1 \psi_\downarrow - \sqrt{\frac{2}{3}} e_3 \psi_\uparrow, \end{aligned} \quad (\text{A} \cdot 15)$$

where

$$\begin{aligned} e_1 &= \left\{ \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0 \right\}, \quad e_2 = \left\{ \frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}}, 0 \right\}, \quad e_3 = \{0, 0, 1\}, \\ \psi_\uparrow &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \psi_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Rewriting more rigorously, we obtain

$$(\Pi^* \cdot \mathbf{k}_2) (\Pi \cdot \mathbf{k}_1) = \sum_{s=1}^4 [\bar{\psi}(\mathbf{k}_2 \vec{\Psi}^{(s)})] [(\mathbf{k}_1 \vec{\Psi}^{(s)}) \psi] = \sum_{s=1}^4 [\psi^* (\mathbf{k}_2 \vec{\Psi}^{(s)})] [(\mathbf{k}_1 \cdot \vec{\Psi}^{(s)*}) \psi],$$

whose value can be evaluated using (A.15) if we fix the two possible spin orientations in ψ^* and ψ . For these four possible combinations, the following values are obtained:

$$\begin{aligned} \sum_{s=1}^4 (\bar{\psi}_\uparrow \cdots \psi_\uparrow) &= (2/3) (\mathbf{k}_2 \cdot \mathbf{k}_1) - (i/3) [\mathbf{k}_2 \times \mathbf{k}_1]_z, \\ \sum_{s=1}^4 (\bar{\psi}_\downarrow \cdots \psi_\downarrow) &= (2/3) (\mathbf{k}_2 \cdot \mathbf{k}_1) + (i/3) [\mathbf{k}_2 \times \mathbf{k}_1]_z, \\ \sum_{s=1}^4 (\bar{\psi}_\uparrow \cdots \psi_\downarrow) &= (1/3) \{ -[\mathbf{k}_2 \times \mathbf{k}_1]_y - i[\mathbf{k}_2 \times \mathbf{k}_1]_x \}, \\ \sum_{s=1}^4 (\bar{\psi}_\downarrow \cdots \psi_\uparrow) &= (1/3) \{ [\mathbf{k}_2 \times \mathbf{k}_1]_y - i[\mathbf{k}_2 \times \mathbf{k}_1]_x \}. \end{aligned} \quad (\text{A} \cdot 16)$$

If on the other hand we evaluate the expectation values of $(\boldsymbol{\sigma} \cdot \mathbf{k}_2) (\boldsymbol{\sigma} \cdot \mathbf{k}_1)$ for the above four combinations of ψ_\uparrow and ψ_\downarrow , we obtain

$$\begin{aligned} (\psi_\uparrow \cdots \psi_\uparrow) &= (\mathbf{k}_2 \cdot \mathbf{k}_1) + i[\mathbf{k}_2 \times \mathbf{k}_1]_z, \\ (\psi_\downarrow \cdots \psi_\downarrow) &= (\mathbf{k}_2 \cdot \mathbf{k}_1) - i[\mathbf{k}_2 \times \mathbf{k}_1]_z, \\ (\psi_\uparrow \cdots \psi_\downarrow) &= [\mathbf{k}_2 \times \mathbf{k}_1]_y + i[\mathbf{k}_2 \times \mathbf{k}_1]_x, \\ (\psi_\downarrow \cdots \psi_\uparrow) &= -[\mathbf{k}_2 \times \mathbf{k}_1]_y + i[\mathbf{k}_2 \times \mathbf{k}_1]_x. \end{aligned} \quad (\text{A} \cdot 17)$$

By comparing (A.16) with (A.17), it is readily seen that the replacement in (A.13) is permissible.

Appendix III

The evaluations of the integrals in $G(\mathbf{r}, \mathbf{r}')$ and $G'(\mathbf{r}, \mathbf{r}')$

In this section we will evaluate the integrals in $G(\mathbf{r}, \mathbf{r}')$ in eq. (3.29) and in $G'(\mathbf{r}, \mathbf{r}')$ in eq. (3.32). We have only to show how to carry out the following two integrations :

$$I_1 = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \frac{1}{\omega_1 \omega_2 (\omega_1 + \omega_2)}, \quad (\text{A} \cdot 18)$$

$$I_2 = \frac{1}{(2\pi)^6} \iint \frac{d\mathbf{k}_1 d\mathbf{k}_2}{\omega_1 \omega_2} e^{i\mathbf{k}_1 \mathbf{r}} e^{i\mathbf{k}_2 \mathbf{r}'} \frac{1}{\omega_1^2 (\omega_1 + \omega_2)}. \quad (\text{A} \cdot 19)$$

Integrating over directions of \mathbf{k}_1 and \mathbf{k}_2 , one obtains

$$I_1 = \frac{4}{(2\pi)^2} \frac{1}{rr'} \int_0^\infty \int_0^\infty \frac{k_1 dk_1 k_2 dk_2}{\omega_1 \omega_2} \sin k_1 r \sin k_2 r' \frac{1}{\omega_1 \omega_2 (\omega_1 + \omega_2)}, \quad (\text{A} \cdot 20)$$

$$I_2 = \frac{4}{(2\pi)^2} \frac{1}{rr'} \int_0^\infty \int_0^\infty \frac{k_1 dk_1 k_2 dk_2}{\omega_1 \omega_2} \sin k_1 r \sin k_2 r' \frac{1}{\omega_1^2 (\omega_1 + \omega_2)}. \quad (\text{A} \cdot 21)$$

Furthermore, one has

$$\begin{aligned} I_1 &= \frac{4}{(2\pi)^2} \frac{1}{rr'} \int_0^\infty \int_0^\infty \frac{k_1 dk_1 k_2 dk_2}{\omega_1 \omega_2} \sin k_1 r \sin k_2 r' \frac{\omega_1 - \omega_2}{\omega_1 \omega_2} \frac{1}{k_1^2 - k_2^2} \\ &= \frac{1}{(2\pi)^2} \frac{1}{rr'} \int_{-\infty}^\infty \int_{-\infty}^\infty k_1 dk_1 k_2 dk_2 \sin k_1 r \sin k_2 r' \left(\frac{1}{k_1^2 - k_2^2} \right) \left(\frac{1}{\omega_1 \omega_2^2} \right) \\ &\quad + \text{exchange term in } r \text{ and } r' \\ &= \frac{1}{(2\pi)^2} \frac{1}{rr'} \int_{-\infty}^\infty k_1 dk_1 \sin k_1 r \frac{1}{\omega_1^3} (\pi e^{-\mu r'} - \pi \cos k_1 r') \\ &\quad + \text{exchange term in } r \text{ and } r', \end{aligned}$$

where the use is made of the formula,

$$\begin{aligned} &\int_{-\infty}^\infty k_2 dk_2 \sin k_2 r' \frac{1}{\omega_2^2 (k_2^2 - k_1^2)} \\ &= \frac{1}{2i} \int_{-\infty}^\infty k_2 dk_2 (e^{ik_2 r'} - e^{-ik_2 r'}) \frac{1}{(k_2 + i\mu)(k_2 - i\mu)(k_2 + k_1)(k_2 - k_1)} \\ &= \frac{\pi}{\omega_1^2} (\cos k_1 r' - e^{-\mu r'}), \end{aligned} \quad (\text{A} \cdot 22)$$

which is obtained by summing the contributions from various poles. The remaining integral can be carried out quite straightforwardly and finally one gets, using the definition of $F(r)$ in (3.31),

$$I_1 = \frac{1}{4\mu(2\pi)^2} \frac{1}{rr'} \{e^{-\mu r} F(r') + e^{-\mu r'} F(r) - F(r+r')\}, \quad (\text{A} \cdot 23)$$

from which one readily obtains (3.30). As for I_2 , one has

$$I_2 = \frac{1}{(2\pi)^2} \frac{1}{rr'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_1 dk_1 k_2 dk_2 \sin k_1 r \sin k_2 r' \left(\frac{1}{k_1^2 - k_2^2} \right) \left(\frac{1}{\omega_1^2 \omega_2} - \frac{1}{\omega_1^3} \right), \quad (\text{A} \cdot 24)$$

where the first term can be integrated in quite the same way as I_1 and the second term can be integrated by using the formula:

$$\int_{-\infty}^{\infty} k_2 dk_2 \sin k_2 r' \frac{1}{k_2^2 - k_1^2} = \pi \cos k_1 r', \quad (\text{A} \cdot 25)$$

thus obtaining the final results (3.33).

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A Speculation on V -spin*

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A specific example is exhibited to show that it is necessary to assume complete pair production of V^0 particles in order to understand their copious production versus slow decay in isolation. This argument can be extended to V_2^0 as well as V_1^0 . It also applies to the heavy " V -fragments" recently observed, which do not seem to be compatible with the pair production hypothesis.

The idea of v -spin provides a convenient though not necessarily correct vehicle for the discussion.

§ 1. Introduction and summary

The metastable heavy particle best studied to date is the V_1^0 . Its most obvious feature is its copious production in nuclear encounters relative to its slow decay in isolation. Furthermore, recent observations suggest that the V^0 particles are not always produced in pairs.

Additional quantum numbers can be ascribed to the nucleon (N), π meson, and V^0 particles in such a way that in isolation the V^0 decays slowly while in nuclear reactions it is produced copiously but not necessarily in pairs. A possible quantum number of this type is " v -spin": $v=1/2$ for N ; $v=0$ for π ; $v=3/2$ for V_1^0 ; $v=1$ for V_2^0 . All strong nuclear forces are postulated to conserve v -spin absolutely, and those violating v -spin are supposedly very weak.

The interpretation by means of v -spin can immediately be extended to include not only V_1^0 and V_2^0 , but also the " V -fragments" recently pointed out in cosmic ray stars. These are heavy nuclear fragments that behave as if one nucleon were replaced by a V_1^0 particle; they are not readily explained by the pair production hypothesis for the V^0 .

There is of course no real necessity at this point to choose the new quantum number of fundamental particles to behave exactly like a spin, and later data may force abandonment of this formalism. It is sufficiently flexible to be consistent with present observations, however, and could readily be incorporated into a more complete theory of elementary particles if one should evolve.

§ 2. V_1^0 production and decay

Although we cannot assign too precise a value to the half-life τ_p for production of V_1^0 in nuclear encounters, it is at once obvious from the relatively copious production and slow decay in isolation ($\tau_d \sim 10^{-10}$ sec), that $\log(\tau_d/\tau_p) \sim 10$. To explain this extremely

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large ratio, the only means suggested by previous experience is to look for new quantum numbers of I_1° and the nuclear particles. A scheme of this sort has been proposed⁽¹⁾ that requires the V_1° particles always to be produced in pairs with some other I° particle, distinct in type from a nucleon (N) or π meson. Recent experimental evidence^(2,3,4) does not seem to favor 100% pair production of I_1° . It may therefore be profitable to look for other quantum number schemes that forbid isolated decay while permitting copious single production.

As an example, consider a very familiar type of quantum number; namely, that associated with a spin, designated by v to distinguish it from isotopic spin t and real spin s . If we assign $v=1/2$ to N and $v=3/2$ to V_1° then the decay of an isolated V_1° is

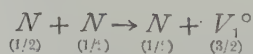
$$V_1^\circ \rightarrow N + X \quad (1)$$

$(3/2) \quad (1/2) \quad (1, 2)$

where X is any number of real product particles, and the total v -spin of each group is indicated in parentheses. Equation (1) refers only to the real initial and ultimate final particles of the decay process. An arbitrarily large number and variety of virtual intermediate states may exist, but the argument is independent of these states, the details of which may therefore be ignored. Energy considerations immediately restrict the composition of eq. (1). The total energy (rest mass plus kinetic) available for X is known to be on the order of 200 Mev. Hence X can contain at most one π , one or two μ , a large number of γ , ν , e .

We now assume that the spin v is an absolute constant of the motion; that is, none of the possible interactions leading to virtual intermediate states can alter the total v of the system. Therefore, unless the v -spin conservation indicated in eq. (1) can be satisfied, the I_1° decay is absolutely forbidden. We can now assume without difficulty that $v=0$ for ν , γ , e , μ and π , resulting in stability for the I_1° . The absolute stability is converted to metastability by adding to the strong v -conserving interaction already assumed a weak interaction that violates v -spin conservation. This interaction could be just strong enough to induce the observed lifetime for isolated decay without having an observable effect in other processes.

There is no difficulty about copious production of I_1° in nuclear collisions: the reaction



conserves v -spin, where eq. (2) again refers to real initial and final particles, omitting all virtual intermediate states. Interactions that conserve v -spin can be arbitrarily strong relative to those violating v -spin, so the experimental ratio τ_d/τ_p can be fitted. Furthermore, there is nothing in eq. (2) to require the production of I° particles in pairs. Of course, pair production is in principle possible by substituting I_1° for N on the right side of eq. (2); but the ratio of pair to single production depends on the explicit details of the interaction, which could presumably be varied to fit a wide range of observed ratios. In any case, the V_1° are not exclusively produced in pairs.

Of course, the new quantum number v need not actually have properties identical with a spin, which has only the recommendation that it is familiar and easy to handle. If further experimental facts show that v does not have all the properties of a spin, it can presumably be amended to have the properties essential to eqs. (1) and (2), while still fitting the additional experimental facts. For instance, if the coupling constant g in eq. (2) turns out to be appreciably smaller than for N - N scattering, we would have to make g depend on the v 's of the individual particles in the interaction, as well as the total v of the reaction. In this case, it might ultimately prove more satisfactory to regard v as something other than a spin quantum number.

We have at this point no particular use for the spin component v_z , which ought also to be a good quantum number. In order to keep ordinary nuclear reactions free from statistical weight factors of v -spin we must assume that the neutron and proton both have the same v_z value. If this is taken by convention to be $v_z = +1/2$, then the $v_z = -1/2$ values would be assumed by the anti-neutron and anti-proton, so that present experiments would show no statistical factors. This choice of v_z implies that a nucleus containing A nucleons must have a total v -spin given by $A/2 \geq v \geq v_z = A/2$; or hence

$$v = A/2. \quad (3)$$

§ 3. V -fragments

When two different hypotheses (pair production and v -spin) are postulated to explain the same observation ($\tau_n/\tau_p \gg 1$), it is natural to seek for some independent phenomenon against which to test them. Fortunately there has recently come to light such an independent fact, which we may call the V -fragments in cosmic ray stars.

Nuclear stars in emulsions frequently eject coherent fragments of nuclear matter heavier than an α -particle, like Li^8 , C^{12} , etc. Occasionally one of these fragments appears to come to rest in the emulsion and then explode with the emission of an energy around 10^2 mev. The slowing-down time of the fragment can be estimated from its mass, charge, and track characteristics. This time is a lower limit for the decay time of the fragment and is generally on the order of 10^{-12} sec. An ordinary nuclear fragment excited to 10^2 Mev would decay in something on the order of 10^{-20} sec. It is therefore suggested⁵⁾ that these long-delayed fragments have a V_1^0 particle replacing a nucleon and that the fragment explosion occurs upon self-absorption of the V_1^0 decay products.

This picture of the V -fragments appears at first sight to be incompatible with the pair production hypothesis.¹⁾ On this scheme, the stability of the V_1^0 against decay into $p + \pi$ is achieved by its very weak interaction with these particles. This means, however, that the V_1^0 is extremely unlikely to remain bound to any fragment of ordinary nuclear matter. There are no Coulomb forces on the V_1^0 and the specific nuclear forces are about the same as those felt by a neutrino. The association of a V_1^0 with a heavy fragment would then be entirely a matter of chance, and the experimental analyses indicate that the observed frequency of V -fragments is several orders of magnitude larger than expected by chance,

On the other hand, the I^- -fragments fit quite naturally into the v -spin formalism. If a large nucleus of mass number A and $v=A/2$ breaks up into two fragments of mass numbers $(A-A')$ and A' , it is possible to conserve v -spin by assigning total spins $v=(A-A')/2$ and $v=(A'/2+1)$ to the respective fragments in an obvious generalization of eq. (2). The fragment with $v=(A'/2+1)$ is not normal nuclear matter as specified by eq. (3); it can become a normal fragment only by a transition with $\Delta v=1$, which may be energetically impossible. It therefore decays very slowly ($\sim 10^{-10}$ sec) in violation of v -spin just as the I_1^0 particles themselves. This description of a I^- -fragment is in many respects equivalent to saying that one nucleon in the fragment is replaced by a I_1^0 particle. Furthermore, the I_1^0 particle is expected to experience a binding force comparable with that of a nucleon, because of the strong interaction of a single I_1^0 with nucleons, as required for copious production by eq. (2).

§ 4. The V_2^0

It also appears possible to incorporate into the present scheme another V^0 particle with $\tau_a/\tau_p \ll 1$, rest mass $\gtrsim 1000 m_e$, and two-body decay into two π mesons. If we call this particle V_2^0 and assign to it a spin of $v=1$, then the decay process

$$V_2^0 \rightarrow \pi^+ + \pi^- \quad (4)$$

violates conservation of v -spin and can be made arbitrarily slow. The I_2^0 is produced copiously in cosmic ray encounters with nuclei by the v -conserving reactions

$$X + N \rightarrow X + V_2^0 + \left(\frac{V_1^0}{N} \right) \quad (5)$$

where $X=\pi$ or N . The ratio of pair production with I_1^0 to single production depends on the details of the interaction, which could presumably be adjusted to fit a wide range of experimental values.

The actual decay operators for isolated V_1^0 and V_2^0 are unknown, including in particular their effect on the isotopic spins involved. *A priori* we should expect that both I^- particles decay into neutral products ($n+\pi^0$, $\pi^0+\pi^0$) about as often as into charged products. If definite experimental evidence to the contrary is found, it will always be possible to account for this by emending the decay operator and making no other changes in the scheme.

The author wishes to thank Prof. H. Yukawa for helpful comments.

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Letters to the Editor

On the Solution of the Equation of Motion

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June 5, 1953

In quantized field theories it is often useful to write the equations of motion in the form of integral equations.¹⁾ There occurs no trouble when such integral equations are used as an initial value problem starting from a finite time, whereas some care must be taken when they are used as an initial value problem starting from an infinite past or future. In the latter case, strictly speaking, there is no initial value problem in its proper sense. In the present note we point out the following two points which are to be carefully treated.

As an example let us consider the field equations in quantum electrodynamics:

$$\left. \begin{aligned} (\gamma_{\mu} \cdot \partial / \partial x_{\mu} + \kappa) \psi(x) \\ &= ie \gamma_{\mu} A_{\mu}(x) \psi(x) \equiv B(x), \\ \partial \bar{\psi}(x) / \partial x_{\mu} \cdot \gamma_{\mu} - \kappa \bar{\psi}(x) \\ &= -ie \bar{\psi}(x) \gamma_{\mu} A_{\mu}(x) \equiv \bar{B}(x), \\ \square A_{\mu}(x) = -(ie/2) [\bar{\psi}(x) \gamma_{\mu} \psi(x) \\ &\quad - \psi(x) \gamma_{\mu} \bar{\psi}(x)] \equiv J_{\mu}(x). \end{aligned} \right\} \quad (1)$$

The field quantities $\psi(x)$, $\bar{\psi}(x)$ and $A_{\mu}(x)$ are operators in the Heisenberg representation, that is, they are quantized in such a way that the equations of motion

$$\left. \begin{aligned} \partial \psi(x) / \partial t &= i [H, \psi(x)], \\ \partial \bar{\psi}(x) / \partial t &= i [H, \bar{\psi}(x)], \\ \partial A_{\mu}(x) / \partial t &= i [H, A_{\mu}(x)], \end{aligned} \right\} \quad (2)$$

reduce to equations (1), where H is the total Hamiltonian of the system. Usually the equations (1) are integrated in the form of integral equations:

$$\left. \begin{aligned} \psi(x) &= \psi^0(x) + \int_{-\infty}^t dt' \int d\vec{x}' S(x-x') B(x'), \\ \bar{\psi}(x) &= \bar{\psi}^0(x) - \int_{-\infty}^t dt' \int d\vec{x}' \bar{B}(x') S(x'-x), \\ A_{\mu}(x) &= A_{\mu}^0(x) + \int_{-\infty}^t dt' \int d\vec{x}' D(x-x') J_{\mu}(x'), \end{aligned} \right\}$$

where $S(x)$ and $D(x)$ are ordinary propagation functions of the free electron and the free photon field respectively, and $\psi^0(x)$, $\bar{\psi}^0(x)$ and $A_{\mu}^0(x)$ are the free field operators in the interaction representation. In the above equations (3), first, we must pay attention to the integrals appearing on the right hand side. These integrals involving $S(x)$ or $D(x)$ over an infinite region $(t, -\infty)$ are improper. As has been pointed out by Glaser and Zimmermann²⁾, it is necessary from the mathematical viewpoint to replace them with Abel's limiting values,

$$\lim_{\epsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\epsilon t'} \dots \quad (4)^3$$

Second, it is not generally permissible to assume the inhomogeneous terms to be the free field operators $\psi^0(x)$, $\bar{\psi}^0(x)$ and $A_{\mu}^0(x)$ in the interaction representation respectively. In order to demonstrate these circumstances we define an operator $X(x)$ by the following equation:

$$\begin{aligned} X(x) &= \psi(x) - \lim_{\epsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\epsilon t'} \int d\vec{x}' \times \\ &\quad \times S(x-x') B(x'). \end{aligned} \quad (5)$$

Obviously this operator $X(x)$ satisfies the free equation

$$(\gamma_{\mu} \cdot \partial / \partial x_{\mu} + \kappa) X(x) = 0. \quad (I)$$

Moreover, by virtue of the equations (4) and the above mentioned meaning of integral, it can be shown that $X(x)$ satisfies the equation

$$\partial X(x) / \partial t - i [H, X(x)] = 0. \quad (II)$$

Accordingly one cannot assume as $X(x)$ the free field operator $\psi^0(x)$ in the interaction representation, because it does not satisfy the condition (II).

There exists, however, an operator satisfying both the conditions (I) and (II). That is an operator $W\psi^0(x)W^+$ which is obtained from $\psi^0(x)$ by using the so-called Heisenberg-Møller's wave matrix⁴⁾ W . The wave matrix has the following properties:

$$\left. \begin{aligned} IIW &= WH_0, \\ WW^+ &= 1, \quad WW^+ \neq 1_f, \\ W\Phi_s &= \Psi_s, \quad W^+\Psi_s = 0, \end{aligned} \right\} \quad (6)$$

where II_0 is the unperturbed free field Hamiltonian and Φ_s is an eigen-state of H_0 with eigen-value E_s ,

and Ψ_s and Ψ_b are eigen-states of H corresponding to the scattering and the bound states respectively. Therefore we may write down the integral equations,

$$\left. \begin{aligned} \psi(x) &= W\psi^0(x)W^\dagger \\ &+ \lim_{\varepsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\varepsilon t'} \int d\vec{x}' S(x-x') B(x'), \\ \bar{\psi}(x) &= W\bar{\psi}^0(x)W^\dagger \\ &- \lim_{\varepsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\varepsilon t'} \int d\vec{x}' \bar{B}(x') S(x'-x), \\ A_\mu(x) &= W A_\mu^0(x) W^\dagger \\ &+ \lim_{\varepsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\varepsilon t'} \int d\vec{x}' D(x-x') J_\mu(x'). \end{aligned} \right\} \quad (7)$$

The role and the meaning of the inhomogeneous terms of (7) are illustrated by the following simple example. Let us consider the system, in which $A_\mu(x)$ is a given time-independent external field. By virtue of (4) the matrix element $(\Psi_0, \psi(x) \Psi_n)$ has the well-known time dependency,⁵⁾ $(\Psi_0, \psi(x) \Psi_n) = \exp(-iE_n t) \varphi_n(\vec{x})$, where Ψ_0 is the true vacuum state and Ψ_n is an eigen-state of H with eigen-value E_n . Then the first equation of (7) gives an integral equation for $\varphi_n(\vec{x})$:

$$\begin{aligned} e^{-iE_n t} \varphi_n(\vec{x}) &= (\Psi_0, W\psi^0(x)W^\dagger \Psi_n) \\ &+ \lim_{\varepsilon \rightarrow +0} \lim_{t_1 \rightarrow -\infty} \int_{t_1}^t dt' e^{\varepsilon t'} \int d\vec{x}' (ie) \gamma_\mu \times \\ &\times A_\mu(\vec{x}') e^{-iE_n t'} \varphi_n(\vec{x}'). \end{aligned} \quad (8)$$

When Ψ_n is a scattering state Ψ_s , then in virtue of the properties of W the inhomogeneous term becomes $(W^\dagger \Psi_0, \psi_0(x) W^\dagger \Psi_s) = (\Phi_0, \psi^0(x) \Phi_s) = e^{-iE_s t} \times \varphi^0(\vec{x})$, which represents an incident wave. On the other hand, when Ψ_n is a bound state, then the inhomogeneous term $(\Psi_0, W\psi^0(x)W^\dagger \Psi_b)$ vanishes by virtue of (6). Therefore the equation (8) gives whole equations of the stationary state problem for this special system in the ordinary configuration space. From the above example we may infer the general character of the inhomogeneous terms of the integral equations (7). Incidentally it can be seen that the equation (7) must not be solved by successive approximation method, because if it were done all the bound states would be swept away by virtue of the last relation of (6).

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On the Theory of Beta-decay

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A formulation of the theory of β -decay can be given on the basis of the following postulates: (1) The primary interactions between nucleons, leptons and bosons, by which the β -decay interactions are derived, are renormalizable.^{1), 2)} (2) The resulting matrix elements give rise to transitions of the Fermi type and the Gamow-Teller type which are of about same order of ratio.

Neither Fermi's theory nor Yukawa's theory of β -decay satisfied these postulates. It was recognized that the Fermi direct interactions between nucleons and leptons, the Yukawa tensor interaction of vector meson and pseudovector interaction of pseudoscalar meson with nucleons or leptons, giving rise to the $G-T$ selection rule of β -decay, are not renormalizable.

One of us proposed an alternative theory of β -decay some years ago³⁾ and suggested that this theory would satisfy the above mentioned postulates, on one occasion. Recently, Umezawa,⁴⁾ Tanaka and Ito⁵⁾ cited and discussed our theory on almost the same point of view of the postulates (1) and (2). We have, therefore, reexamined the theory in order to establish a closer comparison with the recent experimental evidence of β -decay.

The basic idea of our theory consists in the assumption that as sources of a Bose field there exist

not usual pairs of nucleons or leptons, but pairs of a nucleon and a lepton. The primary interactions of scalar or pseudoscalar charged field with nucleons and leptons are given by

$$H_s' = g(\bar{\Psi}\varphi)U^* + f(\bar{\psi}\phi)U + \text{compl. conj.}, \quad (1)$$

or

$$H_{ps}' = g'(\bar{\Psi}\gamma_5\varphi)U^* + f'(\bar{\psi}\gamma_5\phi)U' + \text{compl. conj.}, \quad (2)$$

where Ψ , ϕ , ψ and φ are proton, neutron, electron and neutrino wave functions, U and U' denotes scalar and pseudoscalar field respectively. These interactions are renormalizable. In order to secure the stability of nucleons, we assume that the U field particles have masses larger than that of nucleons, so they will provisionally be called heavy mesons.⁹⁾ From (1) or (2), we obtain the β -decay interaction Hamiltonian

$$H_{\beta} = G \int \int (\bar{\Psi}(\mathbf{r}_1) O \varphi(\mathbf{r}_1)) F(r) \times (\bar{\psi}(\mathbf{r}_2) O \phi(\mathbf{r}_2)) d\mathbf{r}_1 d\mathbf{r}_2, \quad (3)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the difference of the two pair coordinates vectors. (G represents an interaction constant and O a γ -matrix.) We have

$$O=1, \quad G=gf \quad \text{for a scalar interaction,}$$

or

$$O=\gamma_5, \quad G=g'f' \quad \text{for a pseudoscalar interaction.}$$

The potential function $F(r)$ is given by

$$(e^{-\lambda r} + e^{-\lambda' r})/r \quad \text{for } \mu \simeq M$$

or

$$1/2 \cdot (e^{-\lambda r} + e^{-\lambda' r})/r \quad \text{for } \mu \gg M,$$

with $\lambda = \sqrt{\mu^2 - \omega^2}$ and $\lambda' = \sqrt{\mu'^2 - \omega'^2}$, μ and M being heavy meson and nucleon mass, ω , ω' denoting the differences of a nucleon and a lepton energy (in the unit of $\hbar = c = 1$).

Fierz-Pauli's identity of γ -matrix⁷⁾ gives

$$(\bar{\Psi} O \varphi)(\bar{\psi} O \phi) = 1/4 \cdot (S \pm V + T \pm A + P) \quad (5)$$

with the following notations

$$S \equiv (\bar{\Psi} \phi)(\bar{\psi} \varphi), \quad V \equiv (\bar{\Psi} \gamma_\mu \phi)(\bar{\psi} \gamma_\mu \varphi),$$

$$T \equiv (\bar{\Psi} \gamma_{\mu\nu} \phi)(\bar{\psi} \gamma_{\mu\nu} \varphi),$$

$$A \equiv (\bar{\Psi} \gamma_{\mu\nu\lambda} \phi)(\bar{\psi} \gamma_{\mu\nu\lambda} \varphi),$$

$$P \equiv (\bar{\Psi} \gamma_5 \phi)(\bar{\psi} \gamma_5 \varphi),$$

where γ 's are usual γ -matrices and we take, at the right hand side of (5), the upper sign for $O=1$ and the lower sign for $O=\gamma_5$.

Introducing (5) into (3), we obtain an expression of β -decay similar to that of Yukawa's theory.⁸⁾ We must, however, note that space-coordinates dependence of sources is different from that in Yukawa's theory as we see in (3). For the case of $\mu \gg M$, $F(r)$ may be replaced by the delta-function

$$F(r) = 4\pi/(\mu^2 - M^2) \cdot \delta(\mathbf{r}_1 - \mathbf{r}_2),$$

then (3) reduces to an identical interaction Hamiltonian of the Fermi theory with a definite linear combination of five types of direct nucleon and lepton interactions. It is well known that $S+V$ or $T+A$ combinations give rise to the Fierz interference factor of electron energy distribution and the recent experimental evidences seem to exclude the existence of that factor.⁹⁾

In our theory, however, the Fierz factor can be cancelled out, if we assume a mixed theory of scalar and pseudoscalar interactions (1) and (2). With the condition

$$gf = g'f' \quad (6)$$

and for the equal masses of scalar and pseudoscalar mesons, the mixed theory cancels out V and A interactions as we see from (5). Then, only S , T and P interactions remain and they have equal weight. Several authors have suggested that the β -decay evidence seems to be in favor of this combinations.^{10), 11)} Our theory seems to give a reasoning of the selection of the $(S+T+P)$ β -decay interaction on a general standpoint of view.

Even the Fierz interference vanishes in our mixed theory, but there remains another electron energy dependent factor which comes from the meson potential $F(r)$. We can directly see this feature in the neutron β -decay. The energy distribution function of neutron β -decay is given by

$$f(\epsilon) = \frac{\epsilon(\epsilon^2 - m^2)^{1/2}(\epsilon_0 - \epsilon)^2}{(\mu^2 - M^2 + 2M\epsilon)^2}, \quad (7)$$

where ϵ , ϵ_0 and m are the energy, the maximum energy and the mass of electron respectively. The denominator of (7) depends on the electron energy. The neutron β -decay evidence¹²⁾, however, seems to exclude other energy dependence than the statistical Fermi's factor in the numerator of (7), and therefore we must assume

$$\mu^2 \gg M^2 + 2M\epsilon_0. \quad (8)$$

This is sufficiently satisfied by the meson mass

$$\mu \geq 2000 m. \quad (9)$$

With this value of heavy meson mass and neutron life-time 12.8 min., we obtain, assuming $g=f=g'=f'$,

$$g^2 = 3 \times 10^{-7}. \quad (10)$$

We finally obtain the life-time of a heavy meson at rest decaying into a nucleon and a lepton

$$1/\tau\mu = g^2\mu \{1 - (M/\mu)^2\}^2, \quad (11)$$

then we have

$$\tau\mu = 8 \times 10^{-17} \text{ sec.} \quad (12)$$

for the above defined values of constants.

The analysis of nuclear β -decay will be published in near future in a latter issue of this journal.

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On the Spin-lattice Relaxation at Extremely Low Temperatures

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It may be expected that the spin-spin relaxation time is usually shorter than the spin-lattice relaxation time at least at not too high temperatures, hence the

spin waves, absorbing the r.f. field, immediately come into equilibrium with other spin waves.

Therefore, we can treat the spin-lattice relaxation in the thermal equilibrium between the spin systems. Recently, Kittel-Abrahams¹⁾ calculated the spin-lattice relaxation time introducing the macroscopic magneto-elastic energy.

They obtained for nickel, $\tau = 6 \times 10^{-7}$ sec. at 390°K and $\tau \approx 1$ sec. at 1°K . The interaction Hamiltonian²⁾ between the spin waves and the magnetic field induced by the transverse vibration is

$$H' = -g\mu_B \sum_i S_i \cdot H(r_i). \quad (1)$$

Introducing the spin wave and the phonon operators, (1) reduces to

$$\begin{aligned} H' = & \frac{4\pi n e g \mu_B}{c} \sqrt{\frac{\hbar}{2M}} \sum_{K\lambda} \sum_{\lambda=1,2} \left(\frac{S}{2}\right)^{1/2} \left(\frac{u}{K}\right)^{1/2} \\ & \times \{e_{K\lambda}^x (a_K^* b_{K\lambda} + a_K b_{K\lambda}^* + a_{-K}^* b_{K\lambda}^* + a_{-K} b_{K\lambda}) \\ & + e_{K\lambda}^y (a_K^* b_{K\lambda}^* - a_K b_{K\lambda} + a_{-K}^* b_{K\lambda}^* - a_{-K} b_{K\lambda})\} \\ & - \frac{4\pi n e g \mu_B}{c} \sqrt{\frac{\hbar}{2NM}} \sum_{K,K'} \sum_{\lambda=1,2} \left(\frac{\mu}{K}\right)^{1/2} \\ & \times e_{K\lambda}^x (a_{K+K'} a_{K'} b_{K\lambda} + a_{K'}^* a_{K+K'} b_{K\lambda}) \\ & - g\mu_B \sum_i \left(\frac{S}{2}\right)^{1/2} S H_i^2. \end{aligned} \quad (2)$$

In the above, n is the number of atoms per unit volume, u the sound velocity of the transverse wave, $e_{K\lambda}$ ($e_{K\lambda}^x, e_{K\lambda}^y, e_{K\lambda}^z$) the unit vector of the polarization of the transverse wave, and a_K^*, a_K ($b_{K\lambda}^*, b_{K\lambda}$) the creation and the destruction operators of the spin wave quanta (phonon quanta). At extremely low temperatures the operators $a_K^* b_{K\lambda}$ and $a_K b_{K\lambda}^*$ play the dominant role. We define the relaxation time according to $K-A$ as follows;

$$\tau = \Delta T / W(1/C_S + 1/C_L), \quad \Delta T = T_S - T_L, \quad (3)$$

where T_S and C_S (T_L and C_L) are the temperature and the heat capacity of the spin system (lattice system), W the energy transferred per unit time from the spin system to the lattice system. W can be easily obtained as follows;

$$W = \frac{AS\hbar u^3}{3\pi B} \frac{V\Delta T}{kT^2} \sum_{i=1,2} K_i^2 \frac{\exp(\hbar u K_i/kT)}{[\exp(\hbar u K_i/kT) - 1]^2}, \quad (4)$$

where

$$A = \left(\frac{4\pi n e g \mu_B}{c} \sqrt{\frac{\hbar}{2M}} \right)^2, \quad B = 2^2 J S a^2$$

and K_i the roots of energy equation $BK^2 + g\mu_B H = \hbar u K$.

Using (3) and (4), $\tau \approx 10^{-3}$ sec. at 1°K . τ becomes larger with increasing temperature in the region above a certain degree.

This tendency will be controlled by the interactions of the type $a_{K+K'}^* a_{K'} b_{K\lambda}$ and $a_{K'}^* a_{K+K'} b_{K\lambda}$ which are important at the higher temperatures.

Secondarily, we investigate the interaction with the electric field induced by the longitudinal vibration. The interaction Hamiltonian is

$$H' = \sum (\mu_B / 2mc) S_i (\mathbf{E}(\mathbf{r}_i) \times \mathbf{P}), \quad (5)$$

where \mathbf{P} is the average momentum of a d -electron.

As above mentioned, we can easily carry out the calculation and obtain the relaxation time $\tau \approx 10^2$ sec. at 1°K .

Thus temperature dependency of τ is the same as in the case of transverse vibrations. Then we can conclude that in the spin-lattice relaxation phenomena at extremely low temperatures the interaction with the transverse lattice waves is important. On the interaction between the conduction electrons and the spin waves³⁾, the indirect process through the medium of the lattice waves is less important in comparison with the direct process. Recently Kondoh⁴⁾ calculated the latter process in consideration of the dipole energies of the conduction electrons.

We wish to express our sincere thanks to Professor K. Ariyama for his kind guidance and Dr. S. Nakajima for his helpful discussions.

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Nonlinear Pseudoscalar Meson Theory

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It seems to be almost sure that the field of nuclear forces is a pseudoscalar one. Japanese scientists have shown¹⁾ that the second and fourth order

potentials of the symmetric pseudoscalar meson theory agree very well with experimental evidences of low energies. For high energies, however, there exist some difficulties. But Jastrow has shown²⁾, by a pure phenomenological treatment, that a modification of the potential, a "hard core", seems to agree with facts also at high energies. Taketani has therefore proposed³⁾ to work with a phenomenological potential in the region inside and to use pseudoscalar symmetric meson potentials in the region outside. It seems to the writer that the use of mesonic potentials also inside would be a progress. To obtain a hard core or a similar modification of the meson potential for $r < \hbar/m_N c$. (m_N : mass of the nucleon) one has to use nonlinear field equations.

This may be done, as there are several indications⁴⁾ that the field equation must be nonlinear. As Pauli has shown⁵⁾ that nuclear forces can be obtained from field equations without quantization of the field, we can treat the nonlinear field as a classical one. Following the ideas of Born's nonlinear electrodynamics⁶⁾ one can define a mesonic excitation $\varphi_l (= 1 \dots 4)$ and a mesonic field f_k where

$$f_k = \varepsilon(\varphi_k) \cdot \varphi_k. \quad (1)$$

The field equation, corresponding to the second Maxwell equation should be $f_{k,1} - f_{1,k} = 0$ so that the well-known equation

$$f_{,k} = 1/\kappa \cdot A_{,k} \quad (2)$$

follows (where $\kappa = mc/\hbar$, m mesonic mass, A the pseudoscalar). For the excitation φ_k we put:

$$\varphi_{l,i} = \kappa A + Q \quad (3)$$

where Q is the nucleonic source function. (For instance, in static approximation $Q = \sigma_k \cdot \partial/\partial x_k \cdot \psi^\dagger \psi$). From (1), (2) and (3) we obtain finally with $g(\varphi_k) = \varepsilon^{-1}(\varphi_k)$

$$g(\varphi_k) \square A - \kappa^2 A + A_{,l} g_{,l} = \kappa Q \quad (4)$$

or

$$\varphi_{l,i} g_{,i} - \kappa^2 \varepsilon(\varphi_k) \cdot \varphi_k = Q_{,k}. \quad (5)$$

If ε for $g \rightarrow 1$ (for great distances-outside region) we obtain the usual linear theory. The function $g(\varphi_k)$ can be determined by experimental adaption or by theoretical investigations. (Relativistic invariance and so on). The source function Q may be the usual one or may be determined by the interaction terms of the modified spinor theory of elementary particles by the author.⁷⁾

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Angular Distribution of the Deuteron Photodisintegration at Moderate Energies

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Recent experiments on the angular distribution of the deuteron photodisintegration reveal the existence of relatively large isotropic part in the energy region 10–20 Mev¹⁾. Although the statistical errors are large at present, the ratio a/b is probably at least 0.1 near 20 Mev, where a and b are defined by

$$d\sigma/d\Omega = a + b \sin^2 \theta.$$

b is almost entirely due to the electric dipole (ED) process ${}^3S \rightarrow {}^3P$, and this term does not change appreciably if we take into account the tensor force effects either in the deuteron ground state or in the 3P -state²⁾ (central force in the 3P -state may be as-

sumed negligible), a is due solely to the magnetic dipole (MD) process ${}^3S \rightarrow {}^1S$ if we neglect the tensor force effects entirely. But when the deuteron ground state is treated exactly including the 3D -state, both the MD process ${}^3S \rightarrow {}^1S$, ${}^3D \rightarrow {}^1D$ and the ED process ${}^3D \rightarrow {}^3P$ contributes to a , even if the 3P -state is assumed to be free. The isotropic part suffers three kinds of uncertainties: (1) the potential shape within the force range has stronger influence on the MD cross section than on the ED cross section, (2) exchange moment contributes to the MD cross section and (3) interactions in the 3P -state change the ED part of it.

According to N. Austern³⁾, a/b is about 0.02 near 20 Mev for Yukawa potential including the tensor force, and weak tensor force in the 3P -state does not affect this ratio very much. We calculate the effect of the exchange moment introduced phenomenologically, which was considered less thoroughly by N. Austern³⁾. For the exchange moment operator, we assume⁴⁾,

$$\begin{aligned} M_{\text{exch}} = & \mu_N [1/2 \cdot (\sigma_N - \sigma_P) g_1 f_1(r_{NP}) \\ & - 1/2 \cdot [(\sigma_N - \sigma_P) \cdot r_{NP}] r_{NP} g_2 f_2(r_{NP})] P_{NP}^x \end{aligned}$$

where μ_N is the nuclear magneton, P_{NP}^x is the space exchange operator, g_1 and g_2 are arbitrary parameters and $f_1(r)$ and $f_2(r)$ are the radial functions. A relation between g_1 and g_2 can be obtained, for each assumed shape of f_1 and f_2 , from the anomalous magnetic moment of the triton which is estimated 0.26 n.m. This calculation necessitates the wave-function of the triton for which we make use of the wave-function of Pease and Feshbach.⁴⁾ The second term in M_{exch} gives rise to the transition ${}^3S \rightarrow {}^1D$ and ${}^3D \rightarrow {}^1S$ for the deuteron photodisintegration. The potential shape, which determines the wave-functions is assumed to be Yukawa well. For the triplet state, the set of parameters is chosen from the table of Feshbach and Schwinger,⁵⁾ which gives the most reasonable results for the deuteron ground state, the low energy nucleon scattering and the triton binding energy. Singlet effective range is assumed to be 2.5×10^{-13} cm*. Interactions in the 3P -state are neglected.

Three shapes are considered for f_1 and f_2 :

- i) exponential $f_1(r) = f_2(r) = e^{-\kappa r}$
- ii) Yukawa $f_1(r) = f_2(r) = e^{-\kappa r}/\kappa r$
- iii) Villars⁶⁾ $f_1(r) = e^{-\kappa r}$, $f_2(r) = (1 + 1/\kappa r) e^{-\kappa r}$
 $g_1 = g_2 = g$,

κ is fixed to 1.18×10^{-13} cm = triplet central force

range (small changes in α do not cause a drastic change in the result). For i) and ii), one parameter remains arbitrary and these two give almost the same results for the same value of the parameter. For iii), the anomalous magnetic moment of the triton fixes g to 9.3. Although a/b increases with increasing g_2 , g_2 for i) must be as large as 87 to make the

E_γ	exp. ($g_2=10$)	exp. ($g_2=87$)	Villars
10 Mev	0.023	0.053	0.026
20 Mev	0.027	0.100	0.033

The ratio a/b
 E_γ is the energy of the incident photon

ratio $a/b=0.1$ at 20 Mev, which seems rather unreasonable.

Thus we arrive at the conclusion, that the large isotropic part of the differential cross section at moderate energies can not be accounted for by the exchange moment alone. A singular interaction in the 3P -state may explain this phenomenon as was pointed out by N. Austern²⁾ and we are going to test the singular tensor forces such as the one obtained from the symmetrical π -meson theory.

We should like to express our sincere thanks to Prof. T. Yamanouchi for his interest to this work, and to Mr. H. Horie for his valuable advices in calculating the anomalous magnetic moment of the triton.

* When the calculation was near the end, the value $r_{0S}(\pi\pi)=2.03\pm0.23\times10^{-13}$ cm. was reported by E. M. Hafner et al. (Phys. Rev. 89 (1953), 204). This decreases the $^3S\rightarrow^1S$ cross section by about 30% at 20 Mev.

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Cloud Chamber Study of Nuclear Interactions in Lead and Carbon of Secondaries emitted in Cosmic Ray Penetrating Showers

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Nuclear events produced by shower particles of cosmic rays have been observed in a cloud chamber operated at Mt. Norikura, altitude 2,840 meters. In the chamber were mounted three lead and four carbon plates each 10 mm thick, as shown in Fig. 1, and a comparison was made for nuclear interactions originating in these two materials.

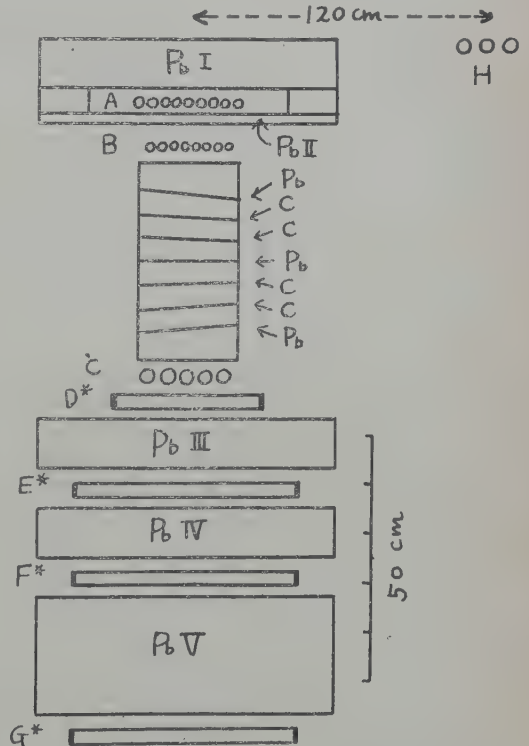


Fig. 1. Diagram of counter and cloud chamber arrangement.

A sextuple coincidence $A \geq 3 B \geq 2 C \geq 1 - II$ was required for the master pulse which triggered an expansion of the chamber. Trays D^* , E^* , F^* and G^* were used in hodoscope and the tray II with the area of about 400 cm^2 was located at a distance of 120 cm from the chamber and used as an anti-coincidence tray against air showers. Hence the penetrating showers produced in the lead above the chamber were detected and the nuclear interactions of the secondaries entering the chamber were observed.

From the analysis of about 5,000 stereo-scopic cloud chamber photographs obtained, the following results were deduced.

(i) An estimate of the fraction of π -mesons among the lightly ionizing secondaries can be made by counting the number of secondary nuclear interactions produced by ionizing and non-ionizing particles.¹⁾ Of 161 secondary nuclear events used, 49 were produced by non-ionizing secondaries and so it followed that $56 \pm 7\%$ of lightly ionizing secondaries are π -

mesons and majority of the remaining $44 \pm 7\%$ are protons.^{1), 2)}

(ii) From totals of 165 nuclear events in lead and 89 in carbon, interaction mean free paths for nuclear collisions were found to be $442 \pm 40 \text{ g cm}^{-2}$ for lead and $217 \pm 30 \text{ g cm}^{-2}$ for carbon. The effect of different detection efficiencies in the both materials was corrected. These values are obviously too large because of undetectable nuclear events, and hence, if we take the mean free path for lead to be 142 g cm^{-2} , corresponding to the geometrical cross section, and consider that the ratio of those for lead and carbon is given by the above values, the mean free path for carbon is determined to be $70 \pm 16 \text{ g cm}^{-2}$.³⁾

(iii) For lead and carbon, histograms of secondary showers with n or more shower particles *versus* n

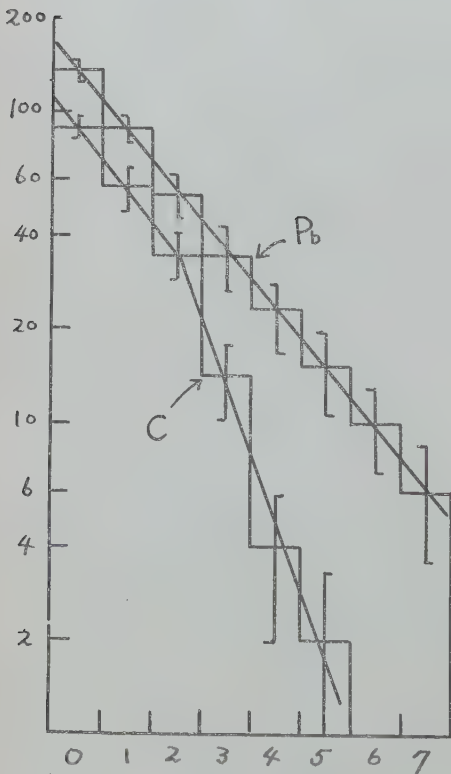


Fig. 2. Corrected integral multiplicity spectra of nuclear events in lead and carbon.

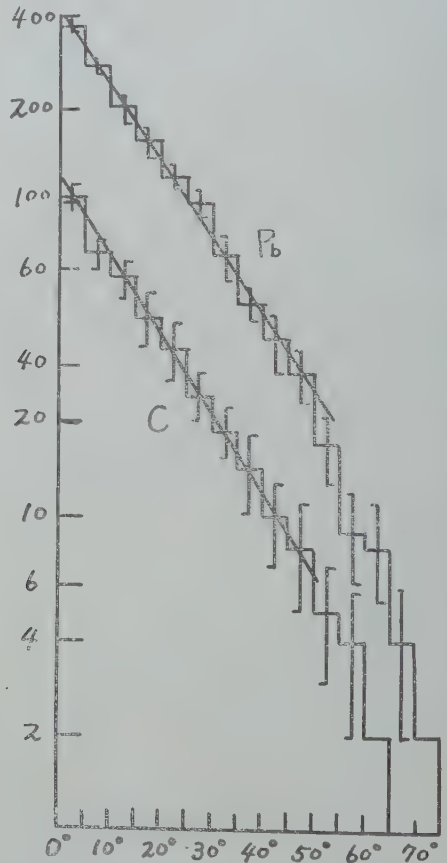


Fig. 3. Integral angular distributions of shower particles of penetrating showers in lead and carbon.

were plotted in Fig. 2. The contribution of electrons which may exist among the shower particles has been corrected statistically. The figures show a large difference between these two multiplicity spectra.⁽¹⁾ The multiplicity spectrum of lead decreases exponentially with increasing n but on the contrary that of carbon shows a knee at $n=2$ or 3. This will be explained qualitatively as follows. If a penetrating shower produced in carbon has four or more shower particles, the scattered nucleon or the created meson had to have plural collisions in the nucleus but such a particle is supposed to have rather lower energy and has a small probability of recoiling another nucleon, and hence the multiplicity spectrum decreases steeply beyond $n=2$ or 3. For penetrating showers in lead, on the contrary the energetic particle with a little change in its direction is able to make plural collisions several times in the nucleus and so the spectrum does not show a knee as in carbon.

(iv) Finally, the integrated angular distributions of shower particles with respect to the shower axes were plotted in Fig. 3. From the figures it is seen that there is no definite difference between the angular distributions of the shower particles emitted in the secondary nuclear interactions in lead and carbon.⁽⁵⁾ It thus seems plausible that a particle which was scattered or recoiled at a large angle with reference to colliding particle has not so high energy and cannot repeat nucleonic collisions within the nucleus, so that such a particles presumably does not contribute to change the angular distribution. Inasmuch as the particles scattered at large angles make no plural collisions in the nuclei, the angular distributions of the shower particles of lead and carbon will not exhibit a definite difference as has been borne out in the present experiment.

The details of the experiment will be published in *Journal of Physical Society of Japan*.

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The Absolute Intensity of Cosmic Rays at Geomagnetic Latitude 25°N

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July 23, 1953

With a view to obtaining the intensities of cosmic rays at such low geomagnetic latitude as in Japan the intensities of the hard and soft components of Cosmic Rays have been measured by a coincidence method with 6 G—M counters as telescope.

The counter telescope was used in either sextuple or double coincidence with or without lead absorber placed between counters. Apart from minor differences, the apparatus is essentially the same as that used by Greisen.⁽¹⁾ The electronic circuits employed here are of usual type. The counters were made of aluminium, the internal diameter of which was 2.8 cm and the length was 20 cm. The experiment was carried out under the thin metallic roof in a penthouse on the roof of Kobe University.

In order to obtain zenith angle distribution of the intensities, the telescope was tilted by 0°, 25°, 39°, and 45° to the south.

With the apparatus described above, the following measurements were performed.

(a) Sextuple coincidences with all lead absorber in place. The counting rate will be denoted by N . N is proportional to the number of mesons with energy above 215 Mev.

(b) Sextuple coincidences without the lead absorber; this rate will be denoted by $N+n$. The difference n between (a) and (b) is proportional to the number of electrons and slow mesons with energy above approximately 6 Mev and 23 Mev, respectively.

(c) Double coincidences between the extreme counters with the same absorber as in (b).

(d) Double coincidences between the extreme counters with the other inactive counters and all lead absorbers removed. The difference between the counting rate (d) and (c) will be denoted by n' , representing the number of electrons with energy between 2.4 Mev and 23 Mev.

These results are presented in Table I, and plotted in Fig. 1, where the values of $\log N$ and of $\log n$ are represented against the corresponding values of $\log(\epsilon/\cos \theta)$, the logarithm of the distance in g/cm² to the top of the atmosphere; ϵ is 1039 g/cm²,

Table I

θ	0°	25°	39°	45°
$N+n$	1.52 ± 0.014	1.23 ± 0.009	0.86 ± 0.009	0.71 ± 0.007
N	1.15 ± 0.011	0.93 ± 0.008	0.67 ± 0.006	0.56 ± 0.006
n	0.37 ± 0.018	0.30 ± 0.012	0.19 ± 0.011	0.15 ± 0.009
n'	0.30 ± 0.021		0.19 ± 0.019	
$n+n'$	0.67 ± 0.028		0.38 ± 0.022	

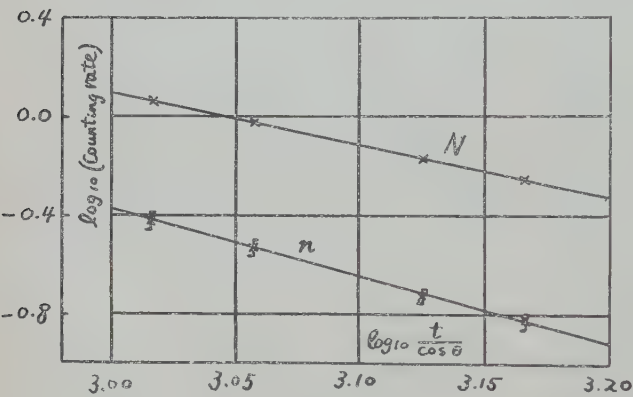


Fig. 1

computed from average atmospheric pressure during the period of our experiment. Fig. 1 shows that the lines are given by the power of $\cos \theta$ and the exponent is 2.1 for the hard component and 2.8 for the soft one. The former value is fortuitously identical with that obtained by Greisen.¹⁾

Assuming that the directional intensity $I(\theta)$ are strictly proportional to the counting rate $N(\theta)$,

$$N(\theta) = \text{const} \cdot I(\theta), \tag{1}$$

For the purpose of determining the proportionality constant, we measured the effective length of the counters and found it to be 18.8 cm, which value also is just the same as that of Greisen and Nereson.²⁾

Using this value and applying the geometrical corrections at the ends of the counters, we obtain for (1)

$$I(\theta) = 1/2.66 \cdot N(\theta). \tag{2}$$

Finally, substituting the value for $N(\theta)$ in the Table I, we have as the absolute intensity for the hard component at geomagnetic latitude 25°N ($0.72 \pm 0.01 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^{-1} \text{ sterad}^{-1}$).

In comparison with the absolute intensity obtained by Greisen³⁾ at geomagnetic latitude greater than

45°N ($0.83 \times 10^{-2} \text{ cm}^{-2} \text{ sec}^{-1} \text{ sterad}^{-1}$), ours is 13% smaller. The difference is considered to be due mainly to the latitude effect, if we assume that secular variation of the cosmic ray intensities has not occurred during the past decade. In this respect, the details will be discussed in a future publication.

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The π -nucleon Scattering and the Damping Effect*

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July 29, 1953

It is well known that the convergency of the perturbation method for the meson theory is very doubtful because of large coupling constants. With respect to this point we shall compare the covariant 4-th order calculations of the π -nucleon scatterings in the symmetrical p.s. meson theory with p.s. coupling (to be published later) and the exact ones for a special process.

Recently, D. Ito¹⁾ obtained exactly the 2-nd order self-energy type radiative corrections of the nucleon propagator $S_F(p)$ in the scalar meson theory with scalar coupling. This method is also applicable to our case: Dyson's integral equation

$$S_L(p) = S_F(p) - iS_F(p) \Sigma_2^*(p) S_L(p) \tag{1}$$

where

Table I. Matrix elements of the π^+ - and π^- -proton elastic scatterings.

$$\gamma_5 S_L \gamma_5 \propto (A \gamma k_1 + B_4 K)$$

The incident meson energy is 130 Mev. For π^+ at single 45° , but for π^- angle independent.

	β	Calcu. up to the 4-th order		Exact calcu.	
		A	B	A	B
$\pi^+ + P$	0.3	1+0.053	-0.0507	1+0.051	-0.0469
	4.2	1+0.739	-0.710	1+0.728	-0.424
$\pi^- + P$	0.3	1-0.044-i0.046	-0.09-i0.057	1-0.036-i0.031	-0.071-i0.038
	4.2	1-0.61-i0.64	-1.26-i0.79	1-0.22-i0.22	-0.181-i0.041

$$\sum_L^*(\rho) = \sum_v -i f_{\rho}^2 / \hbar c \int d^4 k \cdot 1 [(2\pi)^4 \cdot \gamma_5 [i\gamma(\rho - k) - K] \gamma_5 / [(\rho - k)^2 + K^2] [k^2 + x^2] + \delta K \quad (2)$$

can be solved exactly and we have

$$S_L(\rho) = i / [(i\gamma\rho + K_s) - \sum_L^*(\rho)], \quad (3)$$

using

$$A + ABA + ABABA + \dots = (A^{-1} - B)^{-1}. \quad (4)$$

$\sum_L^*(\rho)$ is ilnearly divergent; after mass and charge renormalizations we have

$$S_L(\rho) = i / [(i\gamma\rho + K)(1 - \beta F) - \beta G K],$$

$$\beta = \sum 1 / A \cdot f_v^2 / 8\pi^2 \hbar c,$$

$$F(\rho^2 + K^2) = \int_0^1 du (1-u) [\log A^2 / A_0^2 - 2K^2 \cdot u^2 / A_0^2],$$

$$G(\rho^2 + K^2) = \int_0^1 du \log A^2 / A_0^2, \quad (5)$$

$$A^2 = (\rho^2 + K^2)u(1-u) + K^2 u^2 + x^2(1-u),$$

$$A_0^2 = K^2 u^2 + x^2(1-u).$$

The matrix elements of the π^+ - and π^- -proton elastic scatterings are proportional to $\gamma_5 S_L(\rho_1 - k_1) \gamma_5$ and $\gamma_5 S_L(\rho_1 + k_1) \gamma_5$ respectively, where ρ_1 is the initial nucleon 4-momentum, k_1 and k_2 are the initial and final meson 4-momenta.

We tabulate $\gamma_5 S_L \gamma_5$ in Table I for $\beta=0.3$ and 4.2 (the latter corresponds to the conventional one ≈ 10) and the incident meson energy 130 Mev.

For the π^+ -scattering there is no damping effect and even for a large coupling constant the calculation up to the 4-th order is a good approximatoion of the exact one. On the other hand, for the π^- -scattering, where the repetition of the 2-nd order processes occurs, the calculation up to the 4-th order is too large in magnitude compared with the exact one and this indicates a very large damping effect. Recently, G. Wentzel²⁾ and K. A. Brueckner et al.³⁾ discussed the similar subjects from the view point of the core term effect.

The authors wish to express their appreciation to Prof. K. Nakabayashi for his valuable discussion.

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Note on the Finite Nuclear Size Effect in Beta-decay

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Rose and Holmes¹⁾ pointed out that if the finite nuclear size is taken into account the eigenfunctions of the electron on the nuclear surface take the values different from those for the case of point charge and this has a considerably large effect upon the β -decay. On the other hand, the author²⁾ pointed out recently that some arbitrariness occurs in the theory of β decay owing to the behavior of the eigenfunctions of the electron in the nucleus. These two effects are closely related. When a large cancellation of the terms in the correction factor does not occur, the above theories^{1), 2)} will be sufficient. But when

a large cancellation occurs (RaE³) is a good example), we must take care of small corrections, so that it is desirable to treat the finite nuclear size effect synthetically, avoiding particular charge distributions. We consider only the first forbidden transition, because a large cancellation mainly takes place only in this case. (In the case of the higher forbidden transition the similar argument will be possible). We use the approximation $aZ \ll 1$, since this approximation seems unexpectedly good even when $aZ \approx 0.6$ (RaE³), and it seems very difficult to treat this problem without this approximation. Of course, we assume $p \ll 1$ (p : electron momentum, ρ : nuclear radius) and neglect its second and higher powers. We have only to discuss the radial wave functions whose quantum numbers are $\kappa = \pm 1$.

We use the notation of Rose¹⁾ and Konopinski and Uhlenbeck⁵⁾ except for the suffixes of wave functions for which κ 's are applied here. First, the regular wave functions for the case of point charge are in this approximation written as (we are interested only in the region $r \lesssim O(\rho)$, therefore $r \ll 1$)

$$\begin{aligned} f_{-1}^{\text{reg}} &= \sqrt{F} \sqrt{(W+1)/W} \cdot p \{aZ/2 + (W-1)r/3\}, \\ g_{-1}^{\text{reg}} &= -\sqrt{F} \sqrt{(W+1)/W} \cdot p, \\ f_1^{\text{reg}} &= \sqrt{F} \sqrt{(W-1)/W} \cdot p, \\ g_1^{\text{reg}} &= \sqrt{F} \sqrt{(W-1)/W} \cdot p \{aZ/2 + (W+1)r/3\}. \end{aligned} \quad (1)$$

The irregular functions are

$$\begin{aligned} f_{-1}^{\text{irr}} &= -\sqrt{F'} \sqrt{(W-1)/W} \cdot p (r/\rho)^{-2}, \\ g_{-1}^{\text{irr}} &= \sqrt{F'} \sqrt{(W-1)/W} \\ &\quad \times p \{aZ/2 + (W+1)r\} (r/\rho)^{-2}, \\ f_1^{\text{irr}} &= \sqrt{F'} \sqrt{(W+1)/W} \\ &\quad \times p \{aZ/2 + (W-1)r\} (r/\rho)^{-2}, \\ g_1^{\text{irr}} &= \sqrt{F'} \sqrt{(W+1)/W} \cdot p (r/\rho)^{-2}, \end{aligned} \quad (2)$$

where

$$F' = \frac{4|\Gamma(-s + iaZ \cdot W/\rho)|^2}{\Gamma^2(1-2s)} e^{\pi \alpha Z \cdot W/\rho} (2\rho)^{-2s-2}.$$

(1) and (2) are suitably normalized at $r \rightarrow \infty$. (Of course only the terms in (1) and (2) are insufficient for $r \rightarrow \infty$). The wave functions in the nucleus (of course, regular) are easily obtained with the above approximation by the interaction method of Rose⁶⁾; except for the normalization factors, they are

$$\begin{aligned} f_{-1}^{\text{in}} &= -(W-1)/3 \cdot r + r^{-2} \int_0^r r'^2 V(r') dr', \\ g_{-1}^{\text{in}} &= 1, \quad f_1^{\text{in}} = 1, \\ g_1^{\text{in}} &= (W+1)/3 \cdot r - r^{-2} \int_0^r r'^2 V(r') dr', \end{aligned} \quad (3)$$

where $V(r)$ is the potential for the electron, and we assume that it is a central force and $\int_0^r r'^2 V(r') dr' (r < \rho)$ is of the order of aZr^2 or smaller, but no other assumption is necessary. These assumptions will be satisfied by the practical potential.

Outside of the nucleus the potential is the same as that for the case of point charge, therefore the practical wave functions are written as the linear combinations of (1) and (2),

$$f_i = a_i f_i^{\text{reg}} + b_i f_i^{\text{irr}}, \quad g_i = a_i g_i^{\text{reg}} + b_i g_i^{\text{irr}}, \quad (i=1, 2)$$

b_i 's are so small that we can normalize (4) at $r \rightarrow \infty$ in the same manner as (1) by putting $a_{-1} \approx a_1 \approx 1$. In the nucleus the normalized wave functions are

$$f_i = c_i f_i^{\text{in}}, \quad g_i = c_i g_i^{\text{in}}. \quad (5)$$

Then, we can connect (4) and (5) smoothly by putting

$$\begin{aligned} c_{-1} &= -\sqrt{F} \sqrt{(W+1)/W} \cdot p, \\ c_1 &= \sqrt{F} \sqrt{(W-1)/W} \cdot p. \end{aligned} \quad (6)$$

After all, the normalized wave functions in the nucleus are

$$\begin{aligned} f_{-1} &= \sqrt{F} \sqrt{(W+1)/W} \cdot p \{ (W-1)/3 \cdot r - r^{-2} \\ &\quad \times \int_0^r r'^2 V(r') dr' \}, \\ g_{-1} &= -\sqrt{F} \sqrt{(W+1)/W} \cdot p, \\ f_1 &= \sqrt{F} \sqrt{(W-1)/W} \cdot p, \\ g_1 &= \sqrt{F} \sqrt{(W-1)/W} \cdot p \{ (W+1)/3 \cdot r - r^{-2} \\ &\quad \times \int_0^r r'^2 V(r') dr' \}. \end{aligned} \quad (7)$$

The first forbidden correction factors in question can be written in the following form, neglecting the relatively small terms which are independent of electron potential and do not affect the following argument,

$$\begin{aligned} C &= (2F'^2)^{-1} \{ \{ A_{S-1}^2 + \{ A_{f-1}^2 + \{ B_{f-1}^2 + \{ B_{S1}^2 \\ &\quad + 2(A_{g-1})^* (\{ B_{f-1} - 2(\{ A_{f1})^* (\{ B_{g1} \}, \end{aligned} \quad (8)$$

where A is an arbitrary operator and B is an operator independent of electron energy, and the relations

between the phases of the nuclear matrix elements⁷⁾ have been used. Substituting (7) into (8), we get

$$\begin{aligned}
 C = & |A|^2 + |\int Br^{-1/2} f^2 dq| \\
 & - (\int Br)^* (\int Br^{-2} \int_0^r r'^2 l^*(r') dr') 2f^2 / (3|l|) \\
 & + |\int Br^{-2} \int_0^r r'^2 V(r') dr'|^2 \\
 & - 2(\int A)^* (\int Br) f^2 / (3W) \\
 & + 2(\int A)^* (\int Br^{-2} \int_0^r r'^2 V(r') dr'). \quad (9)
 \end{aligned}$$

If we put

$$\int Br^{-2} \int_0^r r'^2 l^*(r') dr' = -aZ/(2\rho) \cdot \int Br, \quad (10)$$

(9) can be written as

$$\begin{aligned}
 C = & |A|^2 + |\int Br|^2 \{ f^2/9 + aZ/2\rho \cdot 2f^2/3W \\
 & + (aZ/2\rho)^2 \} - 2(\int A)^* (\int Br) \{ f^2/3W + aZ/2\rho \}. \quad (11)
 \end{aligned}$$

This is the usual form of the correction factor in the case of $aZ \ll 1$. However, (10) is not always correct; the ratio of the nuclear matrix elements in both sides of (10) takes in general a value different from $-aZ/(2\rho)$. In order to take into account this fact, it is sufficient to regard $aZ/(2\rho)$ in (11) as a parameter which varies in the neighborhood of the correct value; i.e., the two finite nuclear size effects^{1), 2)} are effectively expressed by regarding $aZ/(2\rho)$ as a variable.

In the case of RaE in which aZ is not small, this theory may not be precise, however, the deviation will perhaps be not so large.³⁾ In the calculation of Reference 3 $aZ/(2\rho)$ can be cancelled in appearance assuming the point charge, and according to the above argument this indicates that the effect of the

electron potential can be cancelled in appearance. When the approximation $aZ \ll 1$ is valid and the correction factor has the form of (9), the set of the correction factors (in which parameters are nuclear matrix elements) taking into account the electron potential is completely the same as that neglecting the potential. Considering these circumstances, the "correction to the nuclear matrix element"²⁾ performed in the calculation of Reference 3 will be perhaps not correct, and VT1-0 (vector+tensor, spin change 1-0) which needed this correction becomes more unlikely, though the unlikeliness of this assumption was emphasized there. On the other hand, in the case of ST1-0 and TP0-0 which had no need of this correction, the foundation becomes firmer, eliminating the fear that the charge distribution affects the results. In conclusion, it must be noted once more that we have used the approximation $aZ \ll 1$.

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Correction to the Beta-decay Nuclear Matrix Elements

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In the traditional theory of β -decay, the radial parts of the eigenfunctions of the electron which are divided by r^l (l 's are orbital angular momenta) are put out of the integral symbols replacing r by ρ (nuclear radius), and the remaining integrals are arranged as nuclear matrix elements. In the presence of the Coulomb field, however, some of them deviate from constants appreciably in the nucleus, and it is hard to put them out of the integrals. In this paper the correction factors which include these eigenfunctions in the integral symbols are given, and it is discussed how much difference from the old theory is produced.

§ 1. Introduction

The correction factors which are important for β -decay spectra were calculated by Konopinski and Uhlenbeck¹⁾ and others. In the calculation of them the radial parts of the eigenfunctions of the electron which are divided by r^l (l 's are orbital angular momenta) are put out of the integrals symbols replacing r by ρ (nuclear radius), and the remaining integrals are arranged as nuclear matrix elements. In the presence of the Coulomb field, however, some of them deviate from constants appreciably in the nucleus, and it is hard to put them out of the integrals.

As an example, we consider the first forbidden transition of the scalar type. Generally we use the notation of Konopinski and Uhlenbeck¹⁾, but the suffices of the electron eigenfunctions f_n and g_n are not those of Rose²⁾ but quantum numbers n 's. If we write the correction factor keeping the eigenfunctions of the electron in the integral symbols,

$$\begin{aligned}
 & (1/3) K^2 (2Fp^2)^{-1} \{ |\int (\beta r g_{-1}(r))|^2 + |\int (\beta r f_1(r))|^2 \} \\
 & + (2/3) K^2 (2Fp^2)^{-1} \{ \int (\beta r f_{-1}(r)/r)^* \int (\beta r g_{-1}(r)) - \int (\beta r f_1(r))^* \int (\beta r g_1(r)/r) \} \\
 & + 2(2Fp^2)^{-1} \{ |\int (\beta r g_{-2}(r)/r)|^2 + |\int (\beta r f_2(r)/r)|^2 \} \\
 & + (2Fp^2)^{-1} \{ |\int (\beta r f_{-1}(r)/r)|^2 + |\int (\beta r g_1(r)/r)|^2 \}.
 \end{aligned} \tag{1}$$

In the derivation of expression (1), we used the relations between the phases of the nuclear matrix elements.³⁾ These relations are utilized in the following calculations frequently. We also assume the real ratios among the coupling constants of the five interaction types.⁴⁾ Expression (1) is independent of the nuclear radius ρ , but it is more familiar to rewrite it using ρ :

$$\begin{aligned}
& -\frac{1}{3} K^2 \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \left| \int \left(\beta_{\mathbf{r}} \frac{g_{-1}(r)}{g_{-1}(\rho)} \right)^2 \frac{g_{-1}(\rho)^2}{\rho^2} + \left| \int \left(\beta_{\mathbf{r}} \frac{f_1(r)}{f_1(\rho)} \right)^2 \frac{f_1(\rho)^2}{\rho^2} \right\} \right. \\
& + \frac{2}{3} K \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \int \left(\beta_{\mathbf{r}} \frac{f_{-1}(r)/r}{f_{-1}(\rho)/\rho} \right)^* \int \left(\beta_{\mathbf{r}} \frac{g_{-1}(r)}{g_{-1}(\rho)} \right) \frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho^2} \right. \\
& \quad \left. - \int \left(\beta_{\mathbf{r}} \frac{f_1(r)}{f_1(\rho)} \right)^* \int \left(\beta_{\mathbf{r}} \frac{g_1(r)/r}{g_1(\rho)/\rho} \right) \frac{f_1(\rho)g_1(\rho)}{\rho} \right\} \\
& + 2 \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \left| \int \left(\beta_{\mathbf{r}} \frac{g_2(r)/r}{g_2(\rho)/\rho} \right)^2 \frac{g_2(\rho)^2}{\rho^2} + \left| \int \left(\beta_{\mathbf{r}} \frac{f_2(r)/r}{f_2(\rho)/\rho} \right)^2 \frac{f_2(\rho)^2}{\rho^2} \right\} \right. \\
& + \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \left| \int \left(\beta_{\mathbf{r}} \frac{f_{-1}(r)/r}{f_{-1}(\rho)/\rho} \right)^2 \frac{f_{-1}(\rho)^2}{\rho^2} + \left| \int \left(\beta_{\mathbf{r}} \frac{g_1(r)/r}{g_1(\rho)/\rho} \right)^2 \frac{g_1(\rho)^2}{\rho^2} \right\}. \quad (2)
\end{aligned}$$

If we replace $f_1(r)/f_1(\rho)$ etc. in expression (2) by unity, several nuclear matrix elements are reduced to one nuclear matrix element $\{\beta_{\mathbf{r}}$, which appears in the article of Konopinski and Uhlenbeck.¹⁾

In the next section it is shown how to obtain the accurate correction factors from the old correction factors given by Konopinski and Uhlenbeck¹⁾ and Smith.⁵⁾ In § 3 the differences between the new and the old nuclear matrix elements are shown and in the last section the influence of this theory is discussed.

§ 2. New correction factors

In this section it is shown how to obtain the accurate correction factors from the old ones calculated by Konopinski and Uhlenbeck¹⁾ and Smith.⁵⁾ For brevity, we put

$$\begin{aligned}
\frac{f_k(r)/r^{k-1}}{f_k(\rho)/\rho^{k-1}} &\equiv L_k, & \frac{g_{-k}(r)/r^{k-1}}{g_{-k}(\rho)/\rho^{k-1}} &\equiv L_{-k}, \\
\frac{g_k(r)/r^k}{g_k(\rho)/\rho^k} &\equiv m_k, & \frac{f_{-k}(r)/r^k}{f_{-k}(\rho)/\rho^k} &\equiv m_{-k},
\end{aligned} \quad (3)$$

where $k > 0$. We also write L_{k-1} , M_{k-1} and N_{k-1} used by Konopinski and Uhlenbeck as L_{k-1}^+ , M_{k-1}^+ , and N_{k-1}^+ . As to the terms including L_{k-1}^\pm and M_{k-1}^\pm it is easily understood without detailed calculation that the following alteration is necessary:

$$\begin{aligned}
L_{k-1}^\pm \left\{ \int A^* \cdot \int B + \text{c.c.} \right\} &\longrightarrow \\
\left(\frac{1}{2F\hat{p}^2} \right) \left\{ \int (A L_{-k})^* \int (B L_{-k}) \frac{g_{-k}(\rho)}{\rho^{2k-2}} \pm \int (A L_k)^* \int (B L_k) \frac{f_k(\rho)}{\rho^{2k-2}} + \text{c.c.} \right\}, & \quad (4)
\end{aligned}$$

$$\begin{aligned}
M_{k-1}^\pm \left\{ \int A^* \cdot \int B + \text{c.c.} \right\} &\longrightarrow \\
\left(\frac{1}{2F\hat{p}^2} \right) \left\{ \int (A m_{-k})^* \int (B m_{-k}) \frac{f_{-k}(\rho)}{\rho^{2k}} \pm \int (A m_k)^* \int (B m_k) \frac{g_k(\rho)}{\rho^{2k}} + \text{c.c.} \right\}, & \quad (5)
\end{aligned}$$

where A and B are arbitrary operators. The terms including N_{k-1}^{\pm} is a little more complicated. First, for the square terms the alteration is

$$N_{k-1}^{\pm} \left| \int A \right|^2 \longrightarrow \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \int (Am_{-k})^* \int (Al_{-k}) \frac{f_{-k}(\rho)g_{-k}(\rho)}{\rho^{2k-1}} \right. \\ \left. \mp \int (Al_k)^* \int (Am_k) \frac{f_k(\rho)g_k(\rho)}{\rho^{2k-1}} \right\}. \quad (6)$$

Second, if one of the nuclear matrix elements is $Q_n(\alpha, \mathbf{r})$ or $Q_n(\beta\alpha, \mathbf{r})$ in Greuling's notation⁽⁶⁾ (namely $\int \alpha$, $\int \beta\alpha$, A_{ij} , A_{ij}^{\dagger} etc.), representing these operators by A , the alteration is

$$N_{k-1}^{\pm} \left\{ \int A^* \int B + \text{c.c.} \right\} \longrightarrow \left(\frac{1}{2F\hat{p}^2} \right) \left\{ \int (Al_{-k})^* \int (Bm_{-k}) \frac{f_{-k}(\rho)g_{-k}(\rho)}{\rho^{2k-1}} \right. \\ \left. \mp \int (Al_k)^* \int (Bm_k) \frac{f_k(\rho)g_k(\rho)}{\rho^{2k-1}} + \text{c.c.} \right\}. \quad (7)$$

The other terms appear in the mixed interaction type terms. In the first forbidden ST (scalar+tensor) the following term should be added to C_{ST}^1 (in the same notation and normalization as Smith⁽⁵⁾):

$$i\lambda_S\lambda_T \frac{2K}{3} \left(\frac{1}{2F\hat{p}^2} \right) \left[\left\{ \int (\beta\mathbf{r}m_{-1})^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}l_{-1}) - \int (\beta\mathbf{r}l_{-1})^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}m_{-1}) \right\} \frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho} \right. \\ \left. + \left\{ \int (\beta\mathbf{r}l_1)^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}m_1) - \int (\beta\mathbf{r}m_1)^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}l_1) \right\} \frac{f_1(\rho)g_1(\rho)}{\rho} \right]. \quad (8)$$

The following term should be added to C_{VA}^1 :

$$-i\lambda_V\lambda_A \frac{2K}{3} \left(\frac{1}{2F\hat{p}^2} \right) \left[\left\{ \int (\mathbf{r}m_{-1})^* \int (\boldsymbol{\sigma} \times \mathbf{r}l_{-1}) - \int (\mathbf{r}l_{-1})^* \int (\boldsymbol{\sigma} \times \mathbf{r}m_{-1}) \right\} \frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho} \right. \\ \left. + \left\{ \int (\mathbf{r}l_1)^* \int (\boldsymbol{\sigma} \times \mathbf{r}m_1) - \int (\mathbf{r}m_1)^* \int (\boldsymbol{\sigma} \times \mathbf{r}l_1) \right\} \frac{f_1(\rho)g_1(\rho)}{\rho} \right]. \quad (9)$$

In C_{VT}^1 the following alteration is necessary:

$$i\lambda_V\lambda_T (2K/3) N_0^{-} \left\{ \int \mathbf{r}^* \int \beta\boldsymbol{\sigma} \times \mathbf{r} - \text{c.c.} \right\} \longrightarrow \\ i\lambda_V\lambda_T \frac{2K}{3} \left(\frac{1}{2F\hat{p}^2} \right) \left[\left\{ \int (\mathbf{r}m_{-1})^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}l_{-1}) + \int (\mathbf{r}l_{-1})^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}m_{-1}) \right\} \frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho} \right. \\ \left. + \left\{ \int (\mathbf{r}l_1)^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}m_1) + \int (\mathbf{r}m_1)^* \int (\beta\boldsymbol{\sigma} \times \mathbf{r}l_1) \right\} \frac{f_1(\rho)g_1(\rho)}{\rho} \right]. \quad (10)$$

In C_{SA}^1 the alteration is

$$\begin{aligned}
& -i\lambda_s\lambda_A(2K/3)N_0^-\left\{\int\beta\mathbf{r}^*\int\boldsymbol{\sigma}\times\mathbf{r}-\text{c.c.}\right\}\longrightarrow \\
& -i\lambda_s\lambda_A\frac{2K}{3}\left(\frac{1}{2F\rho^2}\right)\left[\left\{\int\beta\mathbf{r}m_{-1}\right\}^*\int(\boldsymbol{\sigma}\times\mathbf{r}l_{-1})+\int(\beta\mathbf{r}l_{-1})^*\int(\boldsymbol{\sigma}\times\mathbf{r}m_{-1})\right]\frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho} \\
& +\left\{\int(\beta\mathbf{r}l_1)^*\int(\boldsymbol{\sigma}\times\mathbf{r}m_1)+\int(\beta\mathbf{r}m_1)^*\int(\boldsymbol{\sigma}\times\mathbf{r}l_1)\right\}\frac{f_1(\rho)g_1(\rho)}{\rho}. \quad (11)
\end{aligned}$$

In the second forbidden ST the following term should be added to C_{ST}^2 :

$$\begin{aligned}
& i\lambda_s\lambda_T\frac{K^3}{15}\left(\frac{1}{2F\rho^2}\right)\sum_{ij}\left[\left\{\int(R_{ij}^3m_{-1})^*\int(T_{ij}^3l_{-1})-\int(R_{ij}^3l_{-1})^*\int(T_{ij}^3m_{-1})\right\}\frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho}\right. \\
& \left.+\left\{\int(R_{ij}^3l_1)^*\int(T_{ij}^3m_1)-\int(R_{ij}^3m_1)^*\int(T_{ij}^3l_1)\right\}\frac{f_1(\rho)g_1(\rho)}{\rho}\right] \\
& +i\lambda_s\lambda_TK\left(\frac{1}{2F\rho^2}\right)\sum_{ij}\left[\left\{\int(R_{ij}^3m_{-2})^*\int(T_{ij}^3l_{-2})-\int(R_{ij}^3l_{-2})^*\int(T_{ij}^3m_{-2})\right\}\frac{f_{-2}(\rho)g_{-2}(\rho)}{\rho^3}\right. \\
& \left.+\left\{\int(R_{ij}^3l_2)^*\int(T_{ij}^3m_2)-\int(R_{ij}^3m_2)^*\int(T_{ij}^3l_2)\right\}\frac{f_2(\rho)g_2(\rho)}{\rho^3}\right]. \quad (12)
\end{aligned}$$

where R_{ij}^3 etc. are not matrix elements but operators. The following term should be added to C_{VA}^2 :

$$\begin{aligned}
& -i\lambda_V\lambda_A\frac{K^3}{15}\left(\frac{1}{2F\rho^2}\right)\sum_{ij}\left[\left\{\int(R_{ij}^3m_{-1})^*\int(T_{ij}^3l_{-1})-\int(R_{ij}^3l_{-1})^*\int(T_{ij}^3m_{-1})\right\}\frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho}\right. \\
& \left.+\left\{\int(R_{ij}^3l_1)^*\int(T_{ij}^3m_1)-\int(R_{ij}^3m_1)^*\int(T_{ij}^3l_1)\right\}\frac{f_1(\rho)g_1(\rho)}{\rho}\right] \\
& -i\lambda_V\lambda_AK\left(\frac{1}{2F\rho^2}\right)\sum_{ij}\left[\left\{\int(R_{ij}^3m_{-2})^*\int(T_{ij}^3l_{-2})-\int(R_{ij}^3l_{-2})^*\int(T_{ij}^3m_{-2})\right\}\frac{f_{-2}(\rho)g_{-2}(\rho)}{\rho^3}\right. \\
& \left.+\left\{\int(R_{ij}^3l_2)^*\int(T_{ij}^3m_2)-\int(R_{ij}^3m_2)^*\int(T_{ij}^3l_2)\right\}\frac{f_2(\rho)g_2(\rho)}{\rho^3}\right]. \quad (13)
\end{aligned}$$

In C_{VT}^2 the alteration is

$$\begin{aligned}
& i\lambda_V\lambda_T\left\{\sum_{ij}\int R_{ij}^*\int T_{ij}^3-\text{c.c.}\right\}\{(1/15)K^3N_0^-+KN_1^-\}\longrightarrow \\
& i\lambda_V\lambda_T\frac{K^3}{15}\left(\frac{1}{2F\rho^2}\right)\sum_{ij}\left[\left\{\int(R_{ij}^3m_1)^*\int(T_{ij}^3l_{-1})+\int(R_{ij}^3l_{-1})^*\int(T_{ij}^3m_{-1})\right\}\frac{f_{-1}(\rho)g_{-1}(\rho)}{\rho}\right. \\
& \left.+\left\{\int(R_{ij}^3l_1)^*\int(T_{ij}^3m_1)+\int(R_{ij}^3m_1)^*\int(T_{ij}^3l_1)\right\}\frac{f_1(\rho)g_1(\rho)}{\rho}\right]
\end{aligned}$$

$$\begin{aligned}
& + i\lambda_v \lambda_T K \left(-\frac{1}{2F\hat{p}^2} \right) \sum_{ij} \left[\left\{ \int (R_{ij}^3 \tilde{m}_{-2})^* \int (T_{ij}^3 l_{-2}) + \int (R_{ij}^3 l_{-2})^* \int (T_{ij}^3 \tilde{m}_{-2}) \right\} \frac{f_{-2}(\rho) g_{-2}(\rho)}{\rho^3} \right. \\
& \left. + \left\{ \int (R_{ij}^3 l_2)^* \int (T_{ij}^3 m_2) + \int (R_{ij}^3 m_2)^* \int (T_{ij}^3 l_2) \right\} \frac{f_2(\rho) g_2(\rho)}{\rho^3} \right]. \quad (14)
\end{aligned}$$

In C_{SA}^2 the alteration is

$$\begin{aligned}
& -i\lambda_s \lambda_A \left\{ \sum_{ij} \int R_{ij}^3 \int T_{ij} - \text{c.c.} \right\} \left\{ (1/15) K^3 N_0^- + K N_1^- \right\} \longrightarrow \\
& -i\lambda_s \lambda_A \frac{K^3}{15} \left(-\frac{1}{2F\hat{p}^2} \right) \sum_{ij} \left[\left\{ \int (R_{ij}^3 m_{-1})^* \int (T_{ij}^3 l_{-1}) + \int (R_{ij}^3 l_{-1})^* \int (T_{ij}^3 m_{-1}) \right\} \frac{f_{-1}(\rho) g_{-1}(\rho)}{\rho} \right. \\
& \left. + \left\{ \int (R_{ij}^3 l_1)^* \int (T_{ij}^3 m_1) + \int (R_{ij}^3 m_1)^* \int (T_{ij}^3 l_1) \right\} \frac{f_1(\rho) g_1(\rho)}{\rho} \right] \\
& -i\lambda_s \lambda_A K \left(-\frac{1}{2F\hat{p}^2} \right) \sum_{ij} \left[\left\{ \int (R_{ij}^3 m_{-2})^* \int (T_{ij}^3 l_{-2}) + \int (R_{ij}^3 l_{-2})^* \int (T_{ij}^3 m_{-2}) \right\} \frac{f_{-2}(\rho) g_{-2}(\rho)}{\rho^3} \right. \\
& \left. + \left\{ \int (R_{ij}^3 l_2)^* \int (T_{ij}^3 m_2) + \int (R_{ij}^3 m_2)^* \int (T_{ij}^3 l_2) \right\} \frac{f_2(\rho) g_2(\rho)}{\rho^3} \right]. \quad (15)
\end{aligned}$$

We do not calculate for the higher forbidden transitions than the second forbidden, as it seems useless. If we put $l_{\pm k} = m_{\pm k} = 1$, we obtain of course the old correction factors of Konopinski and Uhlenbeck and Smith.

§ 3. Nature of the new nuclear matrix elements

In this section we examine the $l_{\pm k}$ and $m_{\pm k}$ of (3) to understand the difference between the new and the old nuclear matrix elements. However, it is very troublesome to calculate for many nuclei, and moreover the potentials for an electron in nuclei are ambiguous, therefore we calculate only approximately to understand the tendencies. We assume uniform charge distribution in nuclei which is simple and likely one. Therefore, the potential is

$$V(r) = \begin{cases} -\frac{3aZ}{2\rho} + \frac{aZ}{2\rho^3} r^2 & : r < \rho, \\ -aZ/r & : r > \rho. \end{cases} \quad (16)$$

To obtain the electron eigenfunctions we use the method of Rose,⁷⁾ and in good approximation we take up the first term of the iteration, namely except for constant factors

$$\begin{aligned}
f_{-k} & \sim r^{-k-1} \int_0^r \{ V(r') - (W-1) \} r'^{2k} dr', \\
g_{-k} & \sim r^{k-1}, \\
f_k & \sim r^{k-1}, \\
g_k & \sim r^{-k-1} \int_0^r \{ V(r') - (W+1) \} r'^{2k} dr'. \quad (17)
\end{aligned}$$

With this approximation, from (3), (16) and (17)

$l_{\pm k}=1,$

$m_{\pm k}=$

$$\begin{aligned} &\left\{ \begin{aligned} &\frac{3(2k+3)}{4(k+2)} - \frac{(2k+1)}{4(k+2)} \left(\frac{r}{\rho}\right)^2 - (W\pm 1) \frac{\rho}{aZ} \frac{(2k+1)(2k+3)}{8(k+2)^2} \left\{1 - \left(\frac{r}{\rho}\right)^2\right\} : r < \rho, \\ &\frac{(2k+1)(2k+3)}{4k(k+2)} \frac{\rho}{r} - \frac{3}{4k(k+2)} \left(\frac{\rho}{r}\right)^{2k+1} \\ &+ (W\pm 1) \frac{\rho}{aZ} \left\{ \frac{2k+3}{2(k+2)} - \frac{(2k+1)(2k+3)^2}{8k(k+2)^2} \frac{\rho}{r} + \frac{3(2k+3)}{8k(k+2)^2} \left(\frac{\rho}{r}\right)^{2k+1} \right\} : r > \rho, \end{aligned} \right. \end{aligned} \tag{18}$$

where the higher power terms of $(W\pm 1)\rho/(aZ)$ are neglected, because in the ordinary energy range and not too small Z value $(W\pm 1)\rho/(aZ) \ll 1$. Therefore, in the expression (18) for m_{\pm} the first two terms are main, and they are independent of Z and W . They are indicated in Fig. 1. The deviations of $m_{\pm k}$'s from constants in the nucleus are mainly r^2 types. Therefore, one might assert that it should belong to the $n+2$ -th forbidden transition, where n -th forbidden is now discussed. However, these r^2 type terms are too

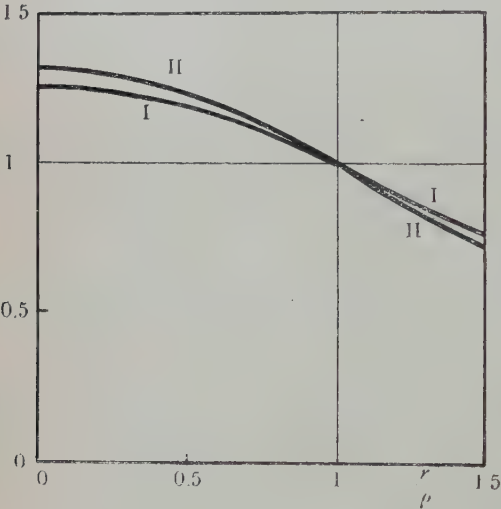


Fig. 1. Ordinate

$$\begin{aligned} &\left\{ \begin{aligned} &\frac{3(2k+3)}{4(k+2)} - \frac{(2k+1)}{4(k+2)} \left(\frac{r}{\rho}\right)^2 : r < \rho, \\ &\frac{(2k+1)(2k+3)}{4k(k+2)} \frac{\rho}{r} - \frac{3}{4k(k+2)} \left(\frac{\rho}{r}\right)^{2k+1} : r > \rho. \end{aligned} \right. \\ &\text{I, } k=1, \quad \text{II, } k=2. \end{aligned}$$

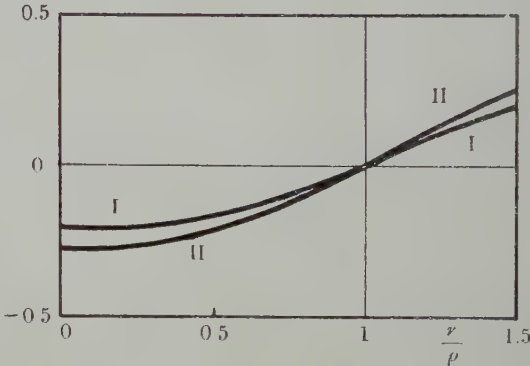


Fig. 2. Ordinate:

$$\begin{aligned} &\left\{ \begin{aligned} &-\frac{(2k+1)(2k+3)}{8(k+2)^2} \left\{1 - \left(\frac{r}{\rho}\right)^2\right\} : r < \rho, \\ &\frac{2k+3}{2(k+2)} - \frac{(2k+1)(2k+3)^2}{8k(k+2)^2} \frac{\rho}{r} \\ &\quad + \frac{3(2k+3)}{8k(k+2)^2} \left(\frac{\rho}{r}\right)^{2k+1} : r > \rho. \end{aligned} \right. \\ &\text{I, } k=1, \quad \text{II, } k=2. \end{aligned}$$

large to class in the $n+2$ -th forbidden. The third terms of (18) show the energy dependence; except for the factor $(W+1)\rho/(uZ)$ they are independent of Z and W , and shown in Fig. 2. The second term of the Rose's iteration method⁷⁾ is nearly proportional to Z^2 , but it is only 0.1 even when $Z=90$.

In the present situation nuclear matrix elements can not be calculated exactly, and in many cases they are treated as unknown parameters.* Under these circumstances, the above calculation will be sufficient to give the rough estimate of the differences between these parameters and their energy dependences. According to the above calculation, a nuclear matrix element including $I_{\pm k}$ is almost equal to the corresponding old one, but one including $m_{\pm k}$ may differ appreciably, especially when the radial part of the nuclear wave function has the node. Besides, a nuclear matrix element including $m_{\pm k}$ may have energy dependence of the relative magnitude of about $0.2\rho/(uZ)$ per mc^2 , then they will appear only when very large cancellation occurs.

§ 4. Discussion

This theory hardly affects the allowed transition and the forbidden transition whose spin change is one larger than the order of forbiddenness, for they have only nuclear matrix elements including $I_{\pm k}$ and $I_{\pm k} \approx 1$. In the other cases this theory may have influence. In the present situation nuclear matrix elements are treated as unknown parameters in many cases, and the above theory increases the number of these parameters. Therefore this theory increases the possibility of explanation. The example will be shown in the subsequent paper concerning the β -spectrum of RaE. Moreover, some deviations from the relations of Ahrens and Feenberg⁸⁾ and Yamada⁹⁾ may appear from this origin, when these relations are applied to the nuclear matrix elements including $m_{\pm k}$.

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* Though we have used the word "unknown" parameters, they are not quite arbitrary. The orders of their magnitudes are limited by the ft-values.

Theoretical Reinvestigation of the β -spectrum of RaE

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The β -spectrum of RaE is reinvestigated taking into account the finite de Broglie wave length effect pointed out by Rose et al. and the correction to the nuclear matrix elements pointed out by the author, and it is shown that the conclusion of Petschek and Marshak that only the assumption of tensor+pseudoscalar, spin change 0-0, parity change yes is able to explain the experimental results is incorrect. Taking into account the above two effects, many other assumptions can explain the experiments. The results are as follows: TP0-0 (tensor+pseudoscalar and spin change 0-0), ST1-0 and VA1-0 fit the experiments only with the finite de Broglie wave length effect, A0-0 (including AP0-0) and VT1-0 requires both effects, but VT1-0 seems too artificial. Other cases can not fit the experiments.

§ 1. Introduction and summary

It has been well known long since that the β -spectrum of RaE deviates from the allowed shape, but there had been no definite theory to explain it. By the Mayer's nuclear shell model it is made clear that this transition is parity change yes, and last year Petschek and Marshak¹⁾ concluded that only the mixture of tensor and pseudoscalar interaction and spin change 0-0 (of course parity change yes) is able to explain the experiments. Some time after, however, Ahrens, Feenberg and Primakoff²⁾ insisted using their method³⁾ to evaluate the ratios between nuclear matrix elements that the nuclear matrix element $\langle \beta \gamma_5 \rangle$ in pseudoscalar type is very small and the large ratio of the coupling constants $|G''/G'| \approx 133$ is required to fit the Petschek's and Marshak's interpretation. However, this large ratio causes many awkward affairs,* and recently Ruderman⁶⁾ has shown that if the nuclear force includes a large $\beta \gamma_5$ type one the relation of Ahrens, Feenberg and Primakoff²⁾ becomes invalid and the magnitude of $\langle \beta \gamma_5 \rangle$ is of the order of other momentum type nuclear matrix elements ($\langle \alpha \rangle$, $\langle \beta \alpha \rangle$ and $\langle \gamma_5 \rangle$). According to his theory it is sufficient to take the coupling constant G_p to the same order as G_T to fit the result of Petschek and Marshak¹⁾. On the other hand, if the nuclear force is entirely due to the pseudovector coupling of π -mesons, the evaluation of Ahrens, Feenberg and Primakoff²⁾ is valid, and a large pseudoscalar coupling constant is required. Therefore, if the foundation of the article of Petschek and Marshak were reliable and if it becomes clear that a

* First, as pointed out by Ahrens, Feenberg and Primakoff²⁾ a new nuclear matrix element should be added to the old one, $\langle \beta \gamma_5 \rangle$. The effect of this new nuclear matrix element to the β -spectrum of RaE will be examined by Takebe.⁴⁾ Moreover, this large coupling constant destroys the allowed shape of high energy β -spectra.⁵⁾ Also Cf. reference 6.

large pseudoscalar coupling constant is unreasonable, we should arrive at the conclusion that there would exist a $\beta\gamma_5$ type nuclear force and perhaps π -mesons would interact with nucleons with pseudoscalar coupling.[†]

In the theory of β -spectra the deviation from the allowed shape is represented by the correction factor which was calculated by Konopinski and Uhlenbeck⁷⁾ and others. They calculated the factors arising from the eigenfunctions of the electron by developing them into the power series in $p\rho$ (p : electron momentum, ρ : nuclear radius) up to the first terms which do not vanish when $Z=0$. However, it was pointed out by Rose, Perry and Dismuke⁸⁾ that when p and Z are large the above approximation is insufficient, and the effect of the higher power terms of $p\rho$ is considerably large. This effect is called "finite de Broglie wave length effect" (abbreviated as F. deB. W. L. E. hereafter). Even if p is not so large, this effect may become important when large cancellation takes place. Recently the author⁹⁾ pointed out that the method of Konopinski and Uhlenbeck⁷⁾ and others to treat the nuclear matrix elements is rough and some alteration is necessary in a forbidden transition whose spin change is not larger than the order of forbiddenness. As far as nuclear matrix elements are treated as unknown parameters, this theory increases the number of these parameters, and so increases adjustability of the theory. This theory is called C. N. M. E. (correction to the nuclear matrix elements) hereafter.

As these new facts have been known, the theoretical reinvestigation becomes necessary. In fact, the conclusion of Petschek and Marshak is upset as shown in the following sections. The results are as follows: TP0-0 (abbreviation of tensor + pseudoscalar, spin change 1-0 and of course parity change yes), ST1-0 and VA1-0 fit the experiments only with F. deB. W. L. E., A0-0 (including AP0-0) and VT1-0 require both F. deB. W. L. E. and C. N. M. E., but VT1-0 seems too artificial. Other cases can not fit the experiments.

§ 2. Calculation I

In this section the theory is advanced taking into account only F. deB. W. L. E. and C. N. M. E. and neglecting all other effects. In § 3 other effects are considered, but they do not alter the general tendencies. We use the notation of Konopinski and Uhlenbeck⁷⁾ and Smith.¹⁰⁾ First, we calculate the first three terms of I_0 , N_0 , I_0^- and N_0^- developed into the power series in $p\rho$. These power series converge considerably well, but there are cases in which the first two terms vanish in the linear combination of them, and so the first three terms are necessary. They are

$$L_0 = (1+s)/2 - aZ\rho/(2s+1) \cdot \{(2s+3)W + s/W\} \\ + \rho^2/(2s+1)^2 \cdot (-2p^2 - sp^2 + 7a^2Z^2 + 8sa^2Z^2 + 8a^2Z^2p^2 + 4sa^2Z^2p^2), \quad (1)$$

$$N_0 = -aZ/(2\rho) - 1/(2s+1) \cdot (sp^2/W - 2a^2Z^2W) \\ - aZ\rho/\{(2s+1)^2(s+1)\} \cdot (-6p^2 - 5sp^2 + 3a^2Z^2 + 4sa^2Z^2W^2 + 8a^2Z^2p^2), \quad (2)$$

† Of course the mixture of both Ps and Pv couplings can be admitted.

$$L_0^- = s(1+s)/(2W) - aZ\rho(1+s) \\ - \rho^2/\{(2s+1)^2 W\} \cdot (p^2 + 2sp^2 - 8a^2 Z^2 - 7sa^2 Z^2 - 9a^2 Z^2 p^2 - 8sa^2 Z^2 p^2 \\ + 4a^4 Z^4 W^2), \quad (3)$$

$$N_0^- = -saZ/(2\rho W) + a^2 Z^2 \\ + \rho/\{(2s+1)^2 W\} \cdot (aZp^2 + saZp^2 - 3a^3 Z^3 - 4sa^3 Z^3 W^2 - 4a^3 Z^3 p^2). \quad (4)$$

The first three terms of the power series are sufficient to take into account the F. deB. W. L. E. in the energy range of RaE. This fact has been assured by the more accurate calculation at $p=3mc$. The most important foundation of the following argument is almost complete vanishing of the quantity $L_0 M_0 - N_0^2$, which guarantees the large cancellations that appear in the following calculations. Namely,

$$(L_0 M_0 - N_0^2)/N_0^2 = (f_1 f_{-1} + g_1 g_{-1})^2 / (f_{-1} g_{-1} - f_1 g_1)^2 \lesssim 0.0005,$$

where suffices of f_1 etc. are the quantum numbers x 's. The above quantity does not depend on the details of the wave functions, because it has the square form. If we use (1) and (2) we obtain

$$L_0 M_0 - N_0^2 = s^2 / (2s+1)^2 \cdot (1 - s^2 / W^2),$$

where the higher power terms in $p\rho$ are neglected. The error of this equation is very small and quite negligible. As the wave functions with the higher orbital angular momenta which have the higher power terms in $p\rho$ are neglected in the current analysis of β -decay, it is consistent to neglect the higher power terms in L_0 , M_0 , and N_0 . The precision of the table of Rose et al.⁽⁸⁾ is somewhat insufficient for the following calculations.

First, we treat in detail ST1-0 which is the most interesting now,⁽¹¹⁾⁽¹²⁾ and then give the results of the other cases briefly. We use the relations between the phases of the nuclear matrix elements⁽¹³⁾ and take the coupling constants as real.⁽¹⁴⁾

la) ST1-0

The correction factor not including C. N. M. E. is

$$G_s^2 |\{\beta \mathbf{r}\}|^2 \{ (1/3) K^2 L_0 + (2/3) K N_0 + M_0 + 2L_1 \} \\ + G_T^2 [|\{\beta \mathbf{a}\}|^2 L_0 + |\{\beta \mathbf{a} \times \mathbf{r}\}|^2 \{ (1/6) K^2 L_0 - (2/3) K N_0 + M_0 + (1/2) L_1 \} \\ - \{ \{\beta \mathbf{a}^* \} \{\beta \boldsymbol{\sigma} \times \mathbf{r} + \text{c.c.}\} \} \{ (1/3) K L_0 - N_0 \}] \\ + G_N G_T [\{ \{\beta \mathbf{r}^* \cdot i\} \{\beta \boldsymbol{\sigma} \times \mathbf{r} + \text{c.c.}\} \} (L_1 - M_0) - \{ \{\beta \mathbf{r}^* \cdot i\} \{\beta \mathbf{a} + \text{c.c.}\} \} \{ (1/3) K L_0 + N_0 \}]. \quad (5)$$

To expression (5) we apply C. N. M. E.⁽⁹⁾ in which as a suitable approximation we put as follows:

$$\frac{f_k(r)/r^{k-1}}{f_k(\rho)/\rho^{k-1}} = \frac{g_{-k}(r)/r^{k-1}}{g_{-k}(\rho)/\rho^{k-1}} \equiv l_k, \\ \frac{f_{-k}(r)/r^k}{f_{-k}(\rho)/\rho^k} = \frac{g_k(r)/r^k}{g_k(\rho)/\rho^k} \equiv m_k. \quad (6)$$

The correction factor including C. N. M. E. is

$$\begin{aligned}
 C = & G_s^2 \left[(1/3) K^2 L_0 \{ (\beta \mathbf{r} l_1) \}^2 + (2/3) K N_0 \{ (\beta \mathbf{r} l_1) \}^* \{ (\beta \mathbf{r} m_1) \} + M_0 \{ (\beta \mathbf{r} m_1) \}^2 \right. \\
 & + 2 L_1 \{ (\beta \mathbf{r} l_2) \}^2 + G_T^2 \left[L_0 \{ (\beta \mathbf{a} l_1) \}^2 + (1/6) K^2 L_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \}^2 \right. \\
 & - (2/3) K N_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} + M_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \}^2 \\
 & \quad \left. + (1/2) L_1 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_2) \}^2 \right. \\
 & - (2/3) K L_0 \{ (\beta \mathbf{a} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \} + 2 N_0 \{ (\beta \mathbf{a} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} \\
 & + 2 G_s G_T \left[L_1 \{ (\beta \mathbf{r} l_2) \}^* \cdot i \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_2) \} - M_0 \{ (\beta \mathbf{r} m_1) \}^* \cdot i \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} \right. \\
 & + (1/3) K N_0 \{ (\beta \mathbf{r} m_1) \}^* \cdot i \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \} - \{ (\beta \mathbf{r} l_1) \}^* \cdot i \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} \\
 & \left. \left. - (1/3) K L_0 \{ (\beta \mathbf{r} l_1) \}^* \cdot i \{ (\beta \mathbf{a} l_1) \} - N_0 \{ (\beta \mathbf{r} m_1) \}^* \cdot i \{ (\beta \mathbf{a} l_1) \} \right] \right]. \quad (7)
 \end{aligned}$$

For simplicity, utilizing the adjustability of nuclear matrix elements we put*

$$\begin{aligned}
 G_s \{ (\beta \mathbf{r} l_i) \} &= (i/2) G_T \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_i) \}, \\
 G_s \{ (\beta \mathbf{r} m_i) \} &= - (i/2) G_T \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_i) \}. \quad (8)
 \end{aligned}$$

Then expression (7) becomes

$$\begin{aligned}
 C = & G_T^2 \left[L_0 \{ (\beta \mathbf{a} l_1) \}^2 - (1/3) K L_0 \{ (\beta \mathbf{a} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \} \right. \\
 & + 3 N_0 \{ (\beta \mathbf{a} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} + (1/4) K^2 L_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \}^2 \\
 & \left. - (1/2) K N_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \}^* \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} + (9/4) M_0 \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \}^2 \right]. \quad (9)
 \end{aligned}$$

We put

$$\{ (\beta \mathbf{a} l_1) \} : \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} l_1) \} : \{ (\beta \boldsymbol{\sigma} \times \mathbf{r} m_1) \} = X : Y : 1, \quad (10)$$

then X and Y are real numbers¹³⁾ and except for the energy independent factor the correction factor becomes

$$C \sim L_0 X^2 - (1/3) K L_0 X Y + 3 N_0 X + (1/4) K^2 L_0 Y^2 - (1/2) K N_0 Y + (9/4) M_0. \quad (11)$$

According to the relations of Ahrens and Feenberg³⁾ X is of the order of $aZ/(2\rho)$ and much larger than unity. The largest term in (11) is of order $(aZ/2\rho)^2$ and energy independent.^{††} Therefore, in order to obtain a strongly energy dependent correction factor necessary for RaE, we must cancel this large energy independent term. To do this we put

$$X = \frac{3}{1+s} \frac{aZ}{2\rho} + x, \quad (12)$$

and we regard the magnitude of x as of the order of unity. Then not only the ρ^{-2} term but also ρ^{-1} term vanishes. Replacing $Y \rightarrow y$, using (1) and (2), neglecting the higher

* Relations (8) will be not always necessary to explain the β -spectrum of RaE.

† We regard the relations of Ahrens and Feenberg³⁾ as statistical ones giving only order of magnitude in each case.

†† Small energy dependence allowed by C. N. M. E.⁹⁾ is neglected in this section. It is discussed in § 3 only qualitatively.

power terms of $p\rho$ and omitting the energy independent factor $(1+s)/2$, (11) becomes

$$C = x^2 - \frac{1}{3} Kxy - \frac{6}{(1+s)(2s+1)} \frac{p^2 + a^2 Z^2}{W} x + \frac{1}{4} K^2 y^2 \\ + \frac{1}{(1+s)(2s+1)} \frac{K(p^2 + a^2 Z^2)}{W} y + \frac{9}{(1+s)^2 (2s+1)^2} (p^2 + a^2 Z^2). \quad (13)$$

If we use the formulae of Konopinski and Uhlenbeck⁷⁾ and Smith,¹⁰⁾ ρ^{-1} term can not be cancelled contrary to ρ^{-2} term.[†] This remaining ρ^{-1} term is (in the same normalization as (13))

$$\frac{9a^3 Z^3}{2(1+s)^3 (2s+1)\rho} \left\{ (2s+3)W + \frac{s}{W} \right\}. \quad (14)$$

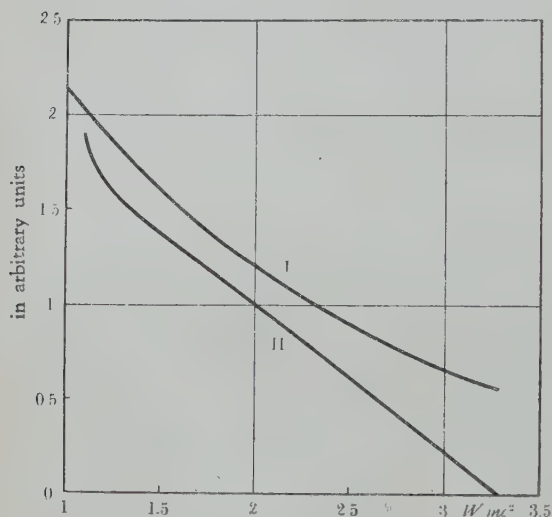


Fig. 1. ST1-0, Correction factor (I) and Kurie plot (II). For the detail, see the text.

As $s \approx 0.79$, (14) is almost proportional to W . The correction factor necessary for RaE is shown in Fig. 1, and (14) has completely opposite nature. Therefore, (14) has obstructed to obtain the desired correction factor.

If we put $y=1$ in (13), we obtain the correction factor not including C. N. M. E. In contrast with the Petschek's and Marshak's case¹⁾, very strongly energy dependent correction factor can be obtained in this case. The largest ratio of the value of the correction factor at $W=1.2mc^2$ to its value at $W=3mc^2$ attainable is about 10, and it is much larger than the necessary one.

Therefore, in this case the desired correction factor can be obtained with

$y=1$, and so other values of y need not be considered. We use the experimental data in the article of Wu,¹⁵⁾ and as the maximum energy we take $3.29mc^2$ indicated there for the

† It can be understood from the following consideration that ρ^{-1} term should vanish when ρ^{-2} term vanishes. ρ^{-1} term is an odd function of Z . A correction factor for positron emission is obtained by reversing the sign of Z . Therefore, if ρ^{-2} term vanished and ρ^{-1} term remained as a main term, the correction factor could become negative, but it is positive definite by its definition. The following consideration is also useful. A correction factor is obtained by squaring the matrix elements and summing them. When the ρ^{-2} term in the correction factor vanishes, the ρ^{-1} terms in the matrix elements vanish. Therefore, the ρ^{-1} term in the correction factor never remain. There are approximate expressions of correction factors for the case of $aZ \ll 1$ given by Konopinski and Uhlenbeck⁷⁾ and Smith.¹⁰⁾ If this approximate formula is used, the correction factor never becomes negative. As far as F. deB. W. L. E. is not included, the approximate formula is superior to the one before approximation. In fact, these approximate formulas are unexpectedly good even if $aZ \approx 0.6$ as RaE.

moment. When $y=1$, $x=3.5$ gives the suitable correction factor; this correction factor and the Kurie plot are shown in Fig. 1. The Kurie plot is straight above $W=1.2mc^2$. We do not consider below this energy because of the difficulties of both theory (especially the screening effect of the atomic electrons) and experiment.

It is necessary to show how much cancellation gives the correction factor as in Fig. 1. For this purpose we have only to know the absolute values of the nuclear matrix elements to fit the decay life, and they are conveniently represented by the corrected $f_c t$ -values (indicated as $f_c t$ -values hereafter); for example, the $f_c t$ -value for $\int (\beta \sigma \times r l_1)$ is defined by

$$f_c t = (2\pi^3/G_T^2) \cdot \log 2 / |\int (\beta \sigma \times r l_1)|^2.$$

With the above ratio of the nuclear matrix elements this $f_c t$ -value for $\int (\beta \sigma \times r l_1)$ is 5.9×10^8 . It is inferred that $\int (\beta \sigma \times r l_1)$ is of the order of B_{ij}^3 , so that the standard value of the above $f_c t$ -value will be $10^8 \sim 10^9$. The above value 5.9×10^8 is suitable. If cancellation were small, the $f_c t$ -value would be $10^{10} \sim 10^{11}$. When cancellation becomes larger, the $f_c t$ -value becomes smaller than 10^8 , and in this case other small effects have possibility to disturb the result. If the $f_c t$ -value is of the order 5×10^8 as in ST1-0, the result is stable as shown in § 3. In each following case we shall show the $f_c t$ -value for the G-T type and coordinate type nuclear matrix element (in the first forbidden it includes a σ and a r).

1b) VA1-0

The mixture of vector and pseudovector differs from the mixture of scalar and tensor by the Dirac matrix β , and this effect is represented by the change of the sign of K (neutrino energy). Corresponding to (8) we put

$$\begin{aligned} G_A \int (\sigma \times r l_i) &= 2i G_V \int (r l_i), \\ G_A \int (\sigma \times r m_i) &= 2i G_V \int (r m_i), \end{aligned} \quad (15)$$

and corresponding to (10) we put

$$\int (a l_1) : i \int (r l_1) : i \int (r m_1) = X : (1/2) Y : (1/2), \quad (16)$$

then we get as the correction factor the expression (11) with the reversed sign of K (namely with the reversed sign of Y). In the same way as ST1-0 we get the correction factor (13) with reversed sign of K (namely with the reversed sign of y). If we put $y=-1$ (corresponding to the omission of C. N. M. E.) and $x=4.2$ in (13), we get the correction factor and the Kurie plot almost the same as in Fig. 1. The $f_c t$ -value for $\int \sigma \times r$ (the ones for $\int (\sigma \times r l_1)$ and $\int (\sigma \times r m_1)$ are the same) is 1.25×10^9 .

1c) VT1-0

This case is considerably complicated because of the large number of the nuclear matrix elements. We put

$$\begin{aligned} \int (a l_1) : i \int (r l_1) : i \int (r l_2) : i \int (r m_1) &= X : Y : Y : 1, \\ \int (\beta a l_1) : \int (\beta \sigma \times r l_1) : \int (\beta \sigma \times r l_2) : i \int (\beta \sigma \times r m_1) &= X' : Y' : Y' : 1, \\ G_V i \int (r m_1) : G_T \int (\beta \sigma \times r m_1) &= 1 : \xi. \end{aligned} \quad (17)$$

To cancel the large energy independent term we put

$$\begin{aligned} X &= 2/(1+s) \cdot (uZ/2\rho) + x, & Y &= y, \\ X' &= 2/(1+s) \cdot (uZ/2\rho) + x', & Y' &= y', \end{aligned}$$

(18)

then using (1)~(4) the correction factor becomes

$$\begin{aligned} C \sim & x^2 - \frac{2}{3} Kxy - \frac{4}{(1+s)(2s+1)} \frac{p^2 + u^2 Z^2}{W} x + \left(\frac{1}{3} K^2 + \frac{2A}{9} p^2 \right) y^2 \\ & + \frac{4K}{3(1+s)(2s+1)} \frac{p^2 + u^2 Z^2}{W} y + \frac{4}{(1+s)^2(2s+1)^2} (p^2 + u^2 Z^2) \\ & + \xi^2 \left[x'^2 - \frac{2}{3} Kx'y' - \frac{4}{(1+s)(2s+1)} \frac{p^2 + u^2 Z^2}{W} x' + \left(\frac{1}{6} K^2 + \frac{A}{18} p^2 \right) y'^2 \right. \\ & \left. + \frac{4K}{3(1+s)(2s+1)} \frac{p^2 + u^2 Z^2}{W} y' + \frac{4}{(1+s)^2(2s+1)^2} (p^2 + u^2 Z^2) \right] \\ & + \xi \left[\frac{2s}{W} xx' - \frac{2sK}{3W} (xy' + x'y) + \frac{s_1 A p^2}{9W} yy' - \frac{8s}{(1+s)^2(2s+1)^2} \frac{p^2 + u^2 Z^2}{W} \right], \end{aligned}$$

(19)

where A was used by Davidson¹⁶⁾:

$$A = \frac{s_1 + 2}{2(s+1)} \frac{F_1}{F}.$$

(20)

If we put $y=y'=1$ corresponding to the omission of C. N. M. E., the required correction factor can not be obtained. However, if we put, for example, $y=y'=0.65$, $s=3$, $x'=1.8$ and $\xi=-5$, we get the correction factor and the Kurie plot shown in Fig. 2. This

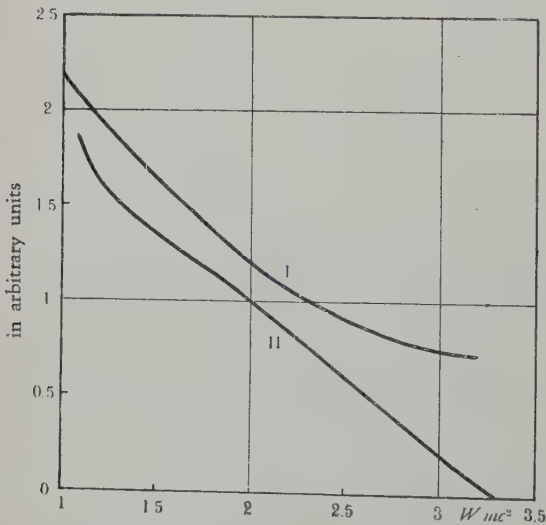


Fig. 2. VT1-0, Correction factor (I) and Kurie plot (II).

Kurie plot deviates from the straight line a little on account of the p^2 -term, but this small deviation can be remedied by the small shift (about $-0.01mc^2$) of the maximum energy. According to reference 9 the values $y=y'=0.65$ seems a little too small, but not necessarily unreasonable. However, the restriction to the values of many parameters is very stringent and seems unlikely. The f_0^e -value for $\int (\beta \sigma \times r l_1)$ is 1.1×10^8 and the one for $\int (\beta \sigma \times r m_1)$ is 4.6×10^7 , and the cancellation is considerably large.

1d) SA1-0

In this case the large energy independent terms can not be cancelled by each

other as in the preceding three cases. Therefore, to get the desired correction factor we have to take an unnatural ratio of the nuclear matrix elements. Neglecting the scalar type and putting

$$\int (\sigma \times r l_1) : \int (\sigma \times r l_2) : \int (\sigma \times r m_1) = 1 : 1 : \rho / (aZ) \cdot x, \quad (21)$$

we must consider that x is of the order of unity. If we put $x = -0.55$, we get the correction factor and the Kurie plot similar to those of Fig. 2, but this large C. N. M. E. is unlikely. The f_c^t -value for $\int (\sigma \times r l_1)$ is 8.3×10^7 . Inclusion of the scalar type does not change the main tendencies.

2a) TP0-0

We treat only the case in which $\int \beta \gamma_5$ is not so small as to invalidate the formulae of Konopinski and Uhlenbeck⁷⁾ and Smith¹⁰⁾ (Cf. §1). We put

$$G_{T2} \int (\beta \gamma_5 l_1) : G_T \int (\beta \sigma \cdot r l_1) : G_T \int (\beta \sigma \cdot r m_1) = X : Y : 1. \quad (22)$$

To remove the large energy independent term we put

$$X = -\frac{2}{1+s} \frac{aZ}{2\rho} + x, \quad Y = y, \quad (23)$$

then the correction factor becomes

$$C \sim x^2 - \frac{2}{3} Kxy + \frac{4}{(1+s)(2s+1)} \frac{p^2 + a^2 Z^2}{W} x + \frac{1}{9} K^2 y^2 - \frac{4K}{3(1+s)(2s+1)} \frac{p^2 + a^2 Z^2}{W} y + \frac{4}{(1+s)^2(2s+1)^2} (p^2 + a^2 Z^2). \quad (24)$$

If we put $y=1$ (corresponding to the neglect of C. N. M. E.) and $x = -3.1$ we get the correction factor and the Kurie plot similar to those of Fig. 1. The f_c^t -value for $\int (\beta \sigma \cdot r)$ is 8.6×10^8 .

2b) A0-0

The relation between A0-0 and TP0-0 is similar to that between VA1-0 and ST1-0.

Putting

$$i \int (\gamma_5 l_1) : \int (\sigma \cdot r l_1) : \int (\sigma \cdot r m_1) = X : Y : 1, \quad (25)$$

and using (23) we get as the correction factor the expression (24) with the reversed sign of K (namely with the reversed sign of y). In this case, as in the case of VT1-0, the desired correction factor can not be obtained with $y = -1$ corresponding to the neglect of C. N. M. E. However, $y = -0.8$ and $x = -1.7$ supply the correction factor and the Kurie plot similar to those of Fig. 1. The f_c^t -value for $\int (\sigma \cdot r l_1)$ is 8.3×10^7 and the one for $\int (\sigma \cdot r m_1)$ is 5.3×10^7 , and the cancellation is considerably large.

2c) AP0-0

There is no large interference between pseudovector and pseudoscalar interaction, so that if $\int \beta \gamma_5$ in pseudoscalar type is large (of the ordinary order of $\int \gamma_5$), the desired correction factor can not be obtained because of the large energy independent term of $\int \beta \gamma_5$.

However, if we take $\beta\gamma_5$ a little smaller, there is a case in which the desired correction factor can be obtained without C. N. M. E. Putting

$$iG_A\{\gamma_5 : G_A\{\sigma \cdot r : iG_P\{\beta\gamma_5 = -\frac{2}{1+s} \frac{uZ}{2\rho} + x : 1 : x', \quad (26)$$

$x = -1.65$ and $x' = 0.29$ is this case, and the correction factor and the Kurie plot are similar to those of Fig. 1. However, the $f_c t$ -value for $\{(\sigma \cdot r)\}$ is 1.5×10^7 and too small.

2d) T0-0

In this case, as in the case of SA1-0, very large C. N. M. E. is required. Putting

$$\{(\beta\sigma \cdot r l_1) : \{(\beta\sigma \cdot r m_1) = 1 : \rho/(uZ) \cdot x, \quad (27)$$

$x = -1.25$ gives the correction factor and the Kurie plot similar to those of Fig. 1, but this case is very unlikely as SA1-0. The $f_c t$ -value for $\{(\beta\sigma r l_1)\}$ is 1.4×10^8 .

We have examined all the linear combinations of the interaction types allowed by the Fierz condition, assuming the spin of RaE as 1 or 0. (Of course we take the spin of the final state as zero.) The existence of both Fermi and G-T type interaction is almost certain, so that we do not consider the single type. Of course the assumption of spin 2 of RaE does not fit the experiments.

§ 3. Calculation II and discussion

In this section we consider the effects not included in § 2. However, it is very laborious to calculate for all the cases, and moreover there is ambiguity such as charge distribution in a nucleus, so that we confine our object to ST1-0 which is the most interesting one.¹¹⁾¹²⁾ It will be shown that for ST1-0 the effects considered in this section will not change the general tendencies. From the calculation for ST1-0 it will be conjectured that these effects will not strongly affect the most other cases.

First we consider the third forbidden transition; this third forbidden is not the ordinary one but the one which has the same selection rule as the first forbidden, and is neglected in most cases. The $f_c t$ -value of the ordinary third forbidden transition is of the order 10^{16} , so that it seems quite negligible at first sight. But this $f_c t$ -value is for the coordinate type nuclear matrix element, and for the velocity type nuclear matrix element it will be of the order 10^{13} for RaE, and the uncorrected $f t$ -value may be of the order 10^{10} . The order of the power of $\hbar\rho$ in the largest term of the third forbidden is the same as (13). Therefore, one might be anxious about the neglect of this term.

For ST1-0 the additional expression to the correction factor (5) arising from the largest part of the third forbidden and its interference with the first forbidden is

$$\begin{aligned} C' = & G_T^2 \{ |\beta a r|^2 \{ (1/6) K^2 M_0 + (1/2) M_1 \} \\ & + G_T^2 \{ \beta a r^{2*} \{ \beta \sigma \times r + \text{c.c.} \} \{ - (1/6) K^2 N_0^* + (1/3) K M_0 - (1/2) N_1 \} \\ & + G_T^2 \{ \beta a r^{2*} \{ \beta a + \text{c.c.} \} (1/3) K N_0 + G_S G_T \{ i \{ \beta a r^{2*} \{ \beta r + \text{c.c.} \} \\ & \{ (1/3) K M_0 + N_1 \} \end{aligned}$$

$$\begin{aligned}
& + G_T^2 \{ |\mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r})|^2 \{ (1/2) K^2 M_0 + (9/2) M_1 \} \\
& + G_T^2 \{ \mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r}) * \{ \beta \boldsymbol{\sigma} \times \mathbf{r} + \text{c.c.} \} \{ (1/6) K^2 N_0 - (2/3) K M_0 + (3/2) N_1 \} \\
& + G_T^2 \{ \mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r}) * \{ \beta \mathbf{a} + \text{c.c.} \} \{ - (2/3) K N_0 \} \\
& + G_S G_T \{ i \mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r}) * \{ \beta \mathbf{r} + \text{c.c.} \} \{ - (1/3) K^2 N_0 - (2/3) K M_0 - 3 N_1 \} , \quad (28)
\end{aligned}$$

in which C. N. M. E. is neglected. We put (8), (10), (12) and

$$\{ \beta \mathbf{a} \mathbf{r}^2 : \mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r}) : \{ \beta \boldsymbol{\sigma} \times \mathbf{r} = u \rho : v \rho : 1, \quad u = O(1), \quad v = O(1), \quad (29)$$

then with the same normalization as (13), (28) becomes

$$\begin{aligned}
C' \sim & \frac{a^2 Z^2}{6(1+s)^2} K^2 u^2 + \frac{a^2 Z^2}{18(2+s_1)^2} A p^2 u^2 + \frac{aZ}{3(1+s)} K^2 u - \frac{2aZ}{3(1+s)} K u x \\
& + \frac{a^2 Z^2}{2(1+s)^2} K^2 v^2 + \frac{a^2 Z^2}{2(2+s_1)^2} A p^2 v^2 + \frac{4aZ}{3(1+s)} K v x. \quad (30)
\end{aligned}$$

Corresponding to the omission of C. N. M. E*, we put

$$\{ \beta \mathbf{a} \mathbf{r}^2 = \rho^2 \{ \beta \mathbf{a}, \text{ namely } u = \frac{3aZ}{2(1+s)} + x \rho. \quad (31)$$

It seems $\{ \mathbf{r}(\beta \mathbf{a} \cdot \mathbf{r})$ is an independent nuclear matrix element, but we put $v=u$ for the moment. Then the pure third forbidden terms in (30), namely u^2 and v^2 terms, are quite negligible, but owing to the interference terms the same value of parameter $x=3.5$ as in § 2 causes a little deviation from the experiments. $x=3.7$ produces the correction factor and the Kurie plot similar to those in Fig. 1. The $f_c t$ -value for $\{ \beta \boldsymbol{\sigma} \times \mathbf{r}$ increases to 7.7×10^8 .

Next, we consider the finite nuclear size correction¹⁷⁾ (abbreviated as F. N. S. C.). Owing to the finite nuclear size, electron's wave functions are changed from those in the pure Coulomb field, and putting (8) the correction factor for ST1-0 becomes

$$\begin{aligned}
C \sim & (1 + \mathcal{A}^{L_0}) L_0 X^2 - (1/3) K(1 + \mathcal{A}^{L_0}) L_0 X Y + 3(1 + \mathcal{A}^{N_0}) N_0 X \\
& + (1/4) K^2(1 + \mathcal{A}^{L_0}) L_0 Y^2 - (1/2) K(1 + \mathcal{A}^{N_0}) N_0 Y + (9/4)(1 + \mathcal{A}^{M_0}) M_0, \quad (32)
\end{aligned}$$

where \mathcal{A}^{L_0} etc. represent the deviations from the case of the pure Coulomb field, and to match the notation the suffices of L_0 etc. are written as one smaller than those of Rose and Holmes.¹⁷⁾ We put

$$\frac{1 + \mathcal{A}^{L_0}}{1 + \mathcal{A}^{N_0}} X = X', \quad \frac{1 + \mathcal{A}^{L_0}}{1 + \mathcal{A}^{N_0}} Y = Y', \quad (33)$$

then the correction factor can be written as

$$C \sim \frac{(1 + \mathcal{A}^{N_0})^2}{(1 + \mathcal{A}^{L_0})} \left[L_0 X'^2 - \frac{1}{3} K L_0 X' Y' + 3 N_0 X' + \frac{1}{4} K^2 L_0 Y'^2 - \frac{1}{2} K N_0 Y' \right]$$

* The part causing C. N. M. E. has the factor whose form is approximately $\{1 - (r/\rho)^2\}$ in the integrand. Therefore, when C. N. M. E. is neglected, r may be replaced by ρ .

$$+\frac{9}{4}M_0\Big] + \frac{9}{4}\frac{(1+\mathcal{A}^{L_0})(1+\mathcal{A}^{M_0})-(1+\mathcal{A}^{N_0})^2}{(1+\mathcal{A}^{L_0})}M_0. \quad (34)$$

The last term of (34) is closely connected with the important quantity $(1+\mathcal{A}^{L_0})(1+\mathcal{A}^{M_0})I_0M_0-(1+\mathcal{A}^{N_0})^2N_0^2$, whose vanishing guarantees the large cancellation as in the case of no F. N. S. C. The last term of (34) vanishes for the pure Coulomb field and even in the case of surface charge distribution which is the oppositely extreme case it can be shown by the elementary calculation that its absolute value does not exceed $0.3 \cdot \left(\frac{(1+\mathcal{A}^{L_0})(1+\mathcal{A}^{M_0})I_0M_0-(1+\mathcal{A}^{N_0})^2N_0^2}{(1+\mathcal{A}^{N_0})^2N_0^2} \lesssim 0.0012 \right)$. This value 0.3 has only small

effect except for the cases of very large cancellation as VT1-0. In the practical case it will perhaps take the intermediate value. In fact, according to the calculation of Takebe⁴¹ who assumed a uniform charge distribution, it is of the order 0.05 and completely negligible. According to the graphs of Rose and Holmes¹⁷⁾ (for $Z=83$ instead of the rigorous one for $Z=84$), however, it becomes of the order 4 only in the range of the very small electron energy. It is probably due to the neglect of F. deB. W. L. E., or the lack of the precision of the graphs. Therefore, we use the results of Takebe.⁴¹

The factor $(1+\mathcal{A}^{N_0})^2/(1+\mathcal{A}^{L_0})$ in the beginning of (34) has only small energy dependence, and it diminishes the correction factor uniformly. The form in the square brackets of (34) is the same as (11), but owing to (33) neglect of C. N. M. E. corresponds not to $Y'=1$ but to $Y'=(1+\mathcal{A}^{L_0})/(1+\mathcal{A}^{N_0})$ and X' and Y' have small energy dependence due to $(1+\mathcal{A}^{L_0})/(1+\mathcal{A}^{N_0})$. If X' and Y' were completely energy independent, the large energy independent term could be cancelled in the same way as in § 2. Actually some remainder is left because of the small energy dependence of X' and Y' , but this remainder is not so large.

The numerical calculation is carried out, using the \mathcal{A}^{L_0} etc. calculated by Takebe⁴¹ and including the third forbidden part explained in the beginning of this section, carrying out F. N. S. C. for this part too. $X=21.07$ and $Y=1$ (not X' and Y') corresponding to the neglect of C. N. M. E. produce the correction factor and the Kurie plot almost the same as those of Fig. 1. Namely F. N. S. C. is almost completely compensated by the alteration of the ratio of the nuclear matrix elements. The $f_c t$ -value for $\beta\sigma \times \nu$ is 6.4×10^8 in this case.

Even if the graphs of Rose and Holmes¹⁷⁾ are used, we obtain the similar result except for the decrease of the $f_c t$ -value to about a half, provided that the last term of (34) is neglected. However, if we apply the values obtained from their graphs to this term, the Kurie plot deviates from a straight line a little, although this deviation is mostly remedied by the small shift of the maximum energy as was done by Petschek and Marshak¹⁾. (In this case the suitable maximum energy is $3.26mc^2$)

In the cases of VA1-0 and TP0-0, the cancellations are of the order of that of ST1-0, and so F. N. S. C. will be perhaps compensated by the change of the ratios of the nuclear matrix elements. On the other hand, when the cancellation is very large as in VT1-0 and A0-0, some effects may remain.

Until now, we have omitted the energy dependence of the nuclear matrix elements (including the difference between $\{Am_1$ and $\{Am_{-1}$ in which A is an arbitrary operator. Cf. Reference 9). However, according to Reference 9, the x in (12), (18), (21), (23), (26) and (27) may have an energy dependence of the order of $0.2/mc^2$. The part causing this energy dependence has the similar form to the part causing C. N. M. E., so that it is consistent to put this energy dependence to zero when C. N. M. E. is not performed. Therefore, no change is necessary in ST1-0, but in VT1-0, which is the secondly interesting case and in which C. N. M. E. is carried out in § 2, it is inconsistent to neglect this energy dependence, and the analysis of § 2 is insufficient. However, this problem is closely connected with the charge distribution in the nucleus which is ambiguous, and if we want to take into account the third forbidden effect, the number of the independent parameters amounts to nine, and the analysis becomes very difficult. As it does not seem so necessary to analyze the VT1-0 case thoroughly in the present situation, let us content ourselves with the following qualitative conclusion: "In the VT1-0 case the number of the parameters is so large that we can fit the experiments, and strictly speaking this case can not be excluded. But this method is very artificial and only in the very small part of the parameter space the correction factor has the large energy dependence necessary to fit the experiments, and so VT1-0 is unlikely".

We have seen the magnitude of the cancellation by the $f_c t$ -values. It is interesting to examine the $f_c t$ -values for $\{A(I_1 - m_1)$ (A is an arbitrary operator) in connection with C. N. M. E. If we take the G - T type and coordinate type nuclear matrix elements as in § 2, they are 3.75×10^8 in VT1-0, 8×10^7 in SA1-0, 1.3×10^8 in T0-0, and infinity in the case of no C. N. M. E. Comparing with reference 9, it seems that the standard values of these $f_c t$ -values are larger than 2×10^6 . Therefore, only A0-0 is admissible, and VT1-0 seems very strained interpretation, and SA1-0 and T0-0 are excluded completely.

The screening effect of the atomic electrons is somewhat like F. N. S. C. When cancellation is small, this effect is much smaller than F. N. S. C. above $H^* = 1.2mc^2$. Therefore, it will be perhaps small and negligible even when cancellation is large. However, this effect may contribute advantageously to fit the experiments, for this effect is very small at $H^* = 3mc^2$ (near the maximum energy) and relatively large in the low energy range.

In the above discussion we have assumed that the β -spectrum of RaE is simple, but there is an opinion that it is complex (Cf. Reference 1). Although this opinion is unlikely, it seems difficult to exclude it completely in the present situation.

In conclusion, RaE does not offer the evidence that the pseudoscalar type exists in the β -decay interaction. Recently it has become very probable that G - T type interaction is tensor by the experiments of electron-neutrino angular correlation.¹¹⁾ Therefore, if the spin of RaE is one, ST1-0 is probable, and if it is zero, TP0-0 is probable. It is desirable to know the spin of RaE. Only when we know that it is zero, the possibility to conclude that the interaction between the nucleon and the π -meson in pseudoscalar type will arise by the detailed analysis of the case of large pseudoscalar coupling constant

in β -decay⁽¹⁾⁽⁵⁾ (Cf. § 1). After all, the admissible β -decay interaction types are ST, STP, VT and VTP in the present situation: ST and STP is the most probable, VTP is probable next to them (The conclusion of Mahmoud and Konopinski⁽¹²⁾ that the vector and the tensor interactions are incompatible with each other seems not decisive but only probable), VT is the most improbable.

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Note added in proof :

(1) Concerning the C. N. M. E. and F. N. S. C. some progress has been made very recently. (M. Yamada, *Prog. Theor. Phys.* in this issue.) This will suggest that the above two effects have little influence to obtain a strongly energy dependent correction factor.

(2) L_0 , N_0 , L_0^- and N_0^- used in reference 1 are correct only up to the second terms of (1)~(4). (Private communication from Dr. A. G. Petschek.)

On the Nuclear Saturation

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Tamm-Dancoff method is applied to the bound system of many particles, on the view point of the meson theory. The obtained results seriously modify the usual method in which the many body problem is treated with the simple addition of the two body potential. It is shown that this modification has a possibility to give the nuclear saturation.

§ 1. Introduction

The usual attempts to resolve the difficulty on the nuclear saturation have been developed in the following two ways:

(i) the modification of the two-body potential (or, two body interaction), or (ii) taking account of the interaction other than (i). On the point (i), it is well known that Majorana force leads to the nuclear saturation. However, Majorana force is not strong enough¹⁾, as indicated in the experiment of the low energy nucleon-nucleon scattering. The hard core²⁾ suggested in the high energy proton-proton scattering experiment cannot resolve the problem³⁾, because it is far less effective to the neutron-proton force due to its short range. The situation remains unchanged, even if the tensor force is included⁴⁾. These attempts have been all based on the assumption that the nuclear force effective in the nucleus is the simple addition of the two-body force. In this paper, a criticism against these usual methods will be given on the view point of the method developed by Tamm and Dancoff⁵⁾ in the meson theory.

Before entering this problem, we shall discuss briefly on the point (ii). It has already suggested⁶⁾ that the many body force** will play an important roll in the nucleus, while the concrete analysis has been left unchallenged. Another interesting trial is the introduction of the non-linear interaction suggested from the meson theory⁷⁾. This interaction makes the nuclear force less effective in higher nuclear density, and gives the binding energy proportional to the mass number A . This view point, however, has the defect⁸⁾ as follows; if the nuclear force is damped in the region of high nuclear density, the force acting on the nucleon which traverse through the heavy nucleus would be quite different from the one given in the lighter nucleus (the most simple case is two-body

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**) It contributes to the order of magnitude (v/c) where v and c are the velocity of the nucleons and the light, respectively.

collision), and the high energy experiment of the nucleon-nucleus collision indicates no such effect*.

Now, we shall discuss on the usual treatment with the two-body potential. The two-body potential, if it is caused by the mediation of the meson field, is obtained necessarily in the "adiabatic" approximation. In this approximation, two nucleons do not change their states during the exchange of meson. And when one applies the potential thus obtained to the many nucleon system, one takes only the simple addition of the two-body potential among the possible nucleon pairs.

Against such a treatment of the bound system, another method with non-adiabatic approximation has been proposed from the view point of the meson theory^{5,1)}. According to this method, the interaction among nucleons are inseparably combined with the motion of nucleons in the equation of motion of the bound system, and this effect of the motion of nucleons during the meson exchange leads to the considerable modification of the usual treatment with two body potential. Although, in the deuteron problem, the modification does not result in any large contribution, it gives a serious effect in the problem of the heavy nucleus.

When Tamm-Dancoff method is applied to the heavy nucleus problem, it is found that the above mentioned effect has a possibility to give the saturation property of the nucleus, and that the saturation property is not necessarily to be caused by the character of the elementary interaction itself.

§ 2. The derivation of the equation

In order to compare our discussion with the usual treatment of the many nucleon system with the two-body potential, we shall apply Tamm-Dancoff method in such approximation that the states of the meson number only 0 and 1 are taken into account. Although this approximation is quite insufficient, our discussion rather concentrates on modifying the usual treatment with the two-body potential by the use of Tamm-Dancoff method.

Let Ψ be the state of the system of the nucleon number A , and let $\Psi(0)$ and $\Psi(1)$ be the substates of the meson number 0 and 1, respectively. Then we can express

$$\Psi = \Psi(0) + \sum_k \Psi_k(1). \quad (1)$$

Here, k is the wave vector of the meson. The total Hamiltonian becomes as

$$H_A = \sum_{i=1}^A \frac{\hat{p}_i^2}{2M} + H_M + H', \quad H' = \sum_{j=1}^A H(j) + h.c. \quad (2)$$

H_M is the Hamiltonian of the free meson field, and H' is the interaction of the nucleon with the meson field, including not only various coordinates of the nucleon but also the

*) The π -nucleus potential is attractive as indicated in the experiment of the π -meson scattering by carbon nucleus⁹⁾. This fact gives the energy denominator $1/\omega_k - V$ ($V > 0$) instead of $1/\omega_k$ (ω_k is the meson energy), in the calculation of the nuclear potential in the second order approximation, and is effective to increase the potential strength, in contrast with the nonlinear interaction.

operator of the meson emission and absorption. Through our discussion, the nucleon field is not quantized.

We take Schrödinger equation of the system as

$$H_A \Psi = E \Psi. \quad (3)$$

And we separate this equation into those of substates of meson number 0 and 1, that is,

$$\left(\sum_{i=1}^A \frac{p_i^2}{2M} - E \right) \Psi(0) = - \sum_k \sum_{i=1}^A H_k(j) \Psi_k(1) \quad (4)$$

and

$$\left(\sum_{i=1}^A \frac{p_i^2}{2M} + \omega_k - E \right) \Psi_k(1) = - \sum_{j=1}^A \bar{H}_k(j) \Psi(0). \quad (5)$$

ω_k and H_k are the energy and the interaction part of the meson with the wave vector \mathbf{k} , respectively. These equations are derived by the scalar product of the equation (3) with the eigenfunction of the meson field of the meson number 0 and 1. \bar{H} indicate the Hermitian conjugate of H . In the equations (4) and (5), it should be noted that the operators appeared are all symmetric on the coordinates of each nucleon, and that one of $\Psi(0)$ and $\Psi(1)$ is necessarily antisymmetric, when the other is antisymmetric on the nucleon coordinate. Thus we shall not take explicitly into account Pauli principle in the following discussion.

Solving eq. (5) in form and substituting its solution into eq. (4), we find

$$\left(\sum_{i=1}^A \frac{p_i^2}{2M} - E \right) \Psi(0) = \sum_k \sum_{i=1}^A H_k(i) \left(1 / \sum_j \frac{p_j^2}{2M} + \omega_k - E \right) \bar{H}_k(l) \cdot \Psi(0). \quad (6)$$

In order to rewrite eq. (6), we make use of the relation

$$1 / \sum \frac{p_j^2}{2M} + \omega_k - E = 1 / \omega_k + (1 / \omega_k) \left(- \sum \frac{p_j^2}{2M} + E \right) \left(1 / \sum \frac{p_j^2}{2M} + \omega_k - E \right). \quad (7)$$

And noticing that $H_k(l)$ depends on the space coordinate of the nucleon in such a manner as $\exp(i\mathbf{k} \cdot \mathbf{r}_l)$, we can obtain the following equation:

$$\begin{aligned} \sum_j \frac{p_j^2}{2M} \cdot \sum_l \bar{H}_k(l) &= \sum_l \bar{H}_k(l) \cdot \sum_j \frac{p_j^2}{2M} - (k^2 / M) \sum_l \bar{H}_k(l) \\ &\quad + (1 / M) \sum_l \bar{H}_k(l) (\mathbf{p}_l \cdot \mathbf{k}). \end{aligned}$$

Using the above equation and eq. (7), we can express eq. (6) as

$$\begin{aligned} \left(\sum_{j=1}^A \frac{p_j^2}{2M} - E \right) \Psi(0) &= \sum_i \sum_k (1 / \omega_k) H_k(i) \bar{H}_k(l) \cdot \Psi(0) \\ &\quad - \sum_{i,l} \sum_k (1 / \omega_k) H_k(i) \left(1 / \sum \frac{p_j^2}{2M} + \omega_k - E \right) \bar{H}_k(l) \cdot \left(\sum_{j=1}^A \frac{p_j^2}{2M} - E \right) \cdot \Psi(0) \\ &\quad - \sum_k (1 / 2M) \sum_{i,l} (1 / \omega_k) H_k(i) \left[2(\mathbf{k} \cdot \mathbf{p}_l) - k^2 / \sum \frac{p_j^2}{2M} + \omega_k - E \right] \bar{H}_k(l) \cdot \Psi(0). \end{aligned} \quad (8)$$

The third term in the right hand of eq. (8) can be negligible compared with the first term, if the nuclear force is not interested in such a short range as $r \sim 1/M$, that is, $|\mathbf{k}| \sim M$. The operators $II_k(l)$ and $1/(\sum_j p_j^2/2M + \omega_k - E)$ can be interchanged each other in the same approximation as above.

Thus, we can obtain the following equation:

$$\left[1 + \sum_k \sum_{i,l} (1/\omega_k) H_k(i) \bar{H}_k(l) / \left(\sum \frac{p_j^2}{2M} + \bar{\omega}_k - E \right) \right] \left(\sum \frac{p_j^2}{2M} - E \right) \Psi(0). \\ = \sum_k \sum_{i,l} (1/\omega_k) H_k(i) \cdot \bar{H}_k(l) \Psi(0). \quad (9)$$

In the derivation of eq. (9), ω_k in $(\sum p_j^2/2M + \omega_k - E)$ is replaced by the mean value $\bar{\omega}_k$ when the summation \sum_k is taken, while this procedure is not essential in the following discussion. As is well known,

$$\sum_{i \neq l} H_k(i) \bar{H}_k(l) / \omega_k = \sum_{i \neq l} V(i, l) \quad (10)$$

is the quantity which corresponds to the usual two-body potential. In eq. (10) and the following, self-energy parts ($i=l$) are omitted.

Now, in the eq. (9), if we neglect the term,

$$\sum_{i \neq l} V(i, l) / \left(\sum \frac{p_j^2}{2M} + \bar{\omega}_k - E \right), \quad (11)$$

we can obtain the equation which coincide with the usual Schrödinger equation derived in the adiabatic approximation. In the deuteron problem, the expectation value of each term in eq. (11) would become as

$$\left\langle \sum_{j=1}^2 \frac{p_j^2}{2M} \right\rangle^* \sim \langle V(1,2) \rangle \sim |E| \sim 2-3 \text{ Mev.}$$

Accordingly, we get

$$\langle V(1,2) \rangle / \left(\sum \frac{p_j^2}{2M} - E + \omega_k \right) \sim \langle V \rangle / \omega_k \simeq 1/40 \ll 1, \quad (12)$$

and we obtain the validity of the usual treatment *in the resultant*. From our view point we can see that the usual treatment or the derivation of eq. (12) is based on the following two assumptions;

$$(i) \quad \left\langle \sum \frac{p_j^2}{2M} - E \right\rangle \ll \bar{\omega}_k$$

$$\text{and } (ii) \quad \langle V \rangle \ll \bar{\omega}_k.$$

Now, returning to the eq. (5), we find that the quantity

$$1 / \left(\sum \frac{p_j^2}{2M} + \omega_k - E \right)$$

*) The effect of the C. M. S. motion leads to only the addition of the constant in E and our estimation remains unchanged.

is the propagation function of the system during the meson exchange. The approximation (i) on this quantity leads to the assumption that the change of the system during the meson exchange is mediated by only the meson field and not by the nucleon motion. This assumption is an essential one in the adiabatic approximation. In heavy nuclei, however, (i) becomes invalid. In fact, $\langle (\sum \vec{p}^2/2M) - E \rangle^*$ would become a quantity of magnitude $\sim \text{BeV}$, and the approximation (ii) is also rejected.

From eq. (9), we obtain

$$\left(\sum \frac{\vec{p}_j^2}{2M} - E \right) \Psi(0) = \left[1 / (1 + \sum V(i, j) / [\sum \frac{\vec{p}_j^2}{2M} + \bar{\omega}_k - E]) \right] \sum V(i, j) \Psi(0) \quad (9)'$$

and from this form of the equation, we can see that the second term of the denominator in the right hand of eq. (9)' gives a damping character, and we can expect, as well, that the binding energy E of the system does not increase, in its absolute value, in such a manner as to be proportional to the square of mass number A . In the next paragraph, we shall give a crude estimation of E with a special model of the nucleus.

§ 3. Binding energy and saturation

The usual discussion of the nuclear saturation is quite qualitative because of the difficulty of the many body problem, and our following discussion has also such a weak point.

The discussion on the nuclear saturation in the treatment with the two-body potential is as the following⁹⁾. Let $\langle \hat{H} \rangle$ be the expectation value of Hamiltonian H operated on an arbitrary trial function. Then, the relation should be required as

$$\langle \hat{H} \rangle \geq E_0 \quad (13)$$

where E_0 is the lowest eigenvalue of H . If we take the state in which all nucleons in the nucleus fall into the range of the nuclear force (i.e. "collapsed state"¹⁰⁾) as a trial state, we obtain the equation¹⁰⁾

$$\langle \hat{H} \rangle = \alpha A^{5/3} - \beta A^2. \quad (14)$$

The first and the second term in the right hand are the kinetic and the potential energy of the nuclear system respectively. α and β are constants which depend on the strength of the nuclear force and its range. When the force is mainly of Wigner type, $|\langle \hat{H} \rangle|$ increases as $\propto A^2$ and it results that $\langle \hat{H} \rangle$ experimental value of \hat{H} beyond a certain mass number A . This is the well-known proof to exclude Wigner force, and the above discussion is logically strict.

On the other hand, the proof that a certain force leads to the saturation is not necessarily sufficient. For an example, in the case of Majorana force, β in (14) has a negative value, and accordingly $\langle \hat{H} \rangle$ gets a saturation character. However, the correct

*) It should be noted $E < 0$ in the bound system.

value E_0 of the binding energy does not necessarily lead to a saturation. In this sense, the positive proof on the saturation problem gives only a "necessary" condition.

Now, returning to the Hamiltonian (2), we find some difficulties to treat because E itself appears in the calculation of the expectation value $\langle H \rangle$ with the use of a trial function $\Psi(0)$. And so we take the equation (9) and investigate what value of E the eq. (9) leads by using a trial function. We shall assume $\Psi(0)$ to be a collapsed state, and estimate the expectation value of both side of the equation (9). The radial dependency of the force is not essential in our discussion and is replaced by a square well. In this approximation, the commutation of the operators $\sum p_j^2/2M$ and $\sum V(ij)$ in (9) is out of the question, because the non-commutable term is effective only in the region where the nuclear density vanishes.

Thus, we obtain the following equation on the expectation value :

$$\left[1 + \langle V \rangle \frac{1}{\langle E_{kin} \rangle + \bar{\omega}_k - E} \right] [\langle E_{kin} \rangle - E] = \langle V \rangle, \quad (15)$$

where $\langle E_{kin} \rangle$ and $\langle V \rangle$ are the expectation values of the kinetic and the potential energy respectively. Solving eq. (15) on E , we take a physically meaningful answer and get

$$E = \langle E_{kin} \rangle + (\bar{\omega}_k/2) - (\bar{\omega}_k/2) (1 + [4\langle V \rangle / \bar{\omega}_k])^{1/2}. \quad (16)$$

If we make use of the approximation (ii) in the preceding paragraph, we find eq. (16) takes a usual form of the binding energy as

$$E = \langle E_{kin} \rangle - \langle V \rangle, \quad (17)$$

where, as is well known, $\langle E_{kin} \rangle$ and $\langle V \rangle$ have the dependency on A such as $A^{5/3}$ and A^2 , respectively. Accordingly, eq. (16) takes such a form as

$$E = aA^{5/3} - \beta A, \quad (18)$$

beyond a certain mass number A , and it is quite clear from eq. (18) that $|\langle E \rangle|$ does not increase as $\propto A^2$. We should emphasize, here, that the above obtained character on the saturation does not depend on any property of V , that is, its potential shape, exchange character and its strength. If the forces among the nucleons in the nucleus are mainly caused by the mediation of the meson, the mentioned fact means that the nuclear saturation depend neither on any character of the interaction of the meson with the nucleon nor on any type of the meson. As is seen from eq. (18), the kinetic energy increases more rapidly than the potential energy with A , in the collapsed state. Accordingly, in the actual nuclei, the nuclear radius will be lengthened so as to make the kinetic energy small. However, the precise estimation of the nuclear radius cannot be given in this paper because of the crude analysis.

§ 4. Nucleon-nucleus collision

Comparing the operator appearing in the right hand side of eq. (9)' with the usual two body potential, we find a definite difference. This suggests that the force acting on

a nucleon traversing a nucleus would be quite different from the usual one. To investigate the problem in this respect will be interesting, from the view point of the objection^{*)} proposed against the nonlinear interaction. Though we are not able to find a definite answer, we shall give a brief discussion on this subject.

The Hamiltonian of the system composed of a nucleus with mass number A and a incident nucleon, is

$$H = H_A + p_0^2/2M + H(0), \quad (19)$$

where H_A is given by the eq. (2) and the index 0 indicates the quantity of the incident nucleon. The equation of motion is derived in same way as in § 2, and the resultant form can be expressed as

$$\left(\frac{p_0^2}{2M} + K_A - E\right)\Psi = \sum_{i,j=0}^A V(ij) \cdot \bar{\omega} \left(\frac{p_0^2}{2M} + K_A + \bar{\omega}_k - E\right) \cdot \Psi. \quad (20)$$

In order to treat the eq. (20) in the perturbation theory, we separate the operator $\sum_{i,j=0}^A V(ij)$ into the following two terms,

$$V = \sum_{i,j=1}^A V(ij) + \sum_{j=1}^A V(0,j) = V_A + V_0,$$

and assume

$$K_A = \sum_{j=1}^A p_j^2/2M, \quad \sum_{i,j=1}^A V(ij) \gg p_0^2/2M, \quad \sum_{j=1}^A V(0,j). \quad (21)$$

Let $\chi(E_0) \cdot \phi(E_A^0)$ be the state of the system in the case where the incident nucleon is infinitely apart from the nucleus, then we get

$$(H_A - E_A^0)\phi(E_A^0) = [\bar{\omega}V_A/(K_A - E_A^0 + \bar{\omega})]\phi(E_A^0),$$

$$\left(\frac{p_0^2}{2M} - E_0\right)\chi(E_0) = 0; \quad E = E_A + E_0, \quad (22)$$

where $\chi(E_0)$ and $\phi(E_A^0)$ are the eigenfunctions of the incident nucleon and the nucleus with the energy indicated, respectively.

We take the form of the correct solution Ψ as

$$\Psi = \chi(E_0)\phi(E_A^0) + \sum \chi(E_i)\phi(E_A^i)^*. \quad (23)$$

Substituting eq. (23) into eq. (20) and using eqs. (22) and (21), we obtain in the first approximation

$$\begin{aligned} & [(\frac{p_0^2}{2M} - E_0) \sum \chi(E_i)\phi(E_A^i)] \\ &= [(\bar{\omega}V_0)/(K_A - E_A^0 + \bar{\omega})](\chi(E_{A_0}) + \sum \chi(E_i)\phi(E_A^i)) \\ &+ [\bar{\omega}V/(K_A - E_A^0 + \bar{\omega})] \cdot \sum \chi(E_i)\phi(E_A^i) - (K_A - E_A^0) \sum \chi(E_i)\phi(E_A^i) \\ &- [\{(\frac{p_0^2}{2M} - E_0)\bar{\omega}V/(K_A - E_A^0 + \bar{\omega})\} \sum \chi(E_i)\phi(E_A^i)]. \end{aligned} \quad (24)$$

Noticing that the eq. (22) holds also for $\chi(E_i)$ and $\phi(E_A^i)$ where E_0 and E_A^0 are

*) We are not concerned with the problem of the nuclear disintegration.

replaced by E_i and E_A^i , and that $E_0 + E_A^0 = E_i + E_A^i$, we get in the same approximation,

$$\begin{aligned} & \sum_i [(\dot{p}_0^2/2M - E_i)[1 + \bar{\omega}V/(K_A - E_A^0 + \bar{\omega})^2]\chi(E_i)\phi(E_A^i) \\ &= [(1 - (K_A - E_A^0)\bar{\omega}/(K_A - E_A^0 + \bar{\omega})^2)V_0[\chi(E_0)\phi(E_0) + \sum\chi(E_i)\phi(E_A^i)]]. \end{aligned} \quad (25)$$

In the derivation of this equation, we assume the excitation energy of nucleus to be small compared with the binding energy itself, that is,

$$|E_A^0 - E_A^i| \ll |E_A^0|, |E_A^i|. \quad (26)$$

In order to represent eq. (25) by only the wave function of the incident nucleon, we note the orthogonality of the wave functions, $\phi(E_i)$'s. $\phi(E_i)$'s used here, are the solutions in which the meson number is 0. Accordingly, with only these solutions we can not obtain the orthogonality $(\phi(E_A^i), \phi(E_A^j)) = \delta_{ij}$, while including the solutions of the meson number 1 (indicated by the prime), the desired results are given as

$$(\phi(E_A^i), \phi(E_A^j)) + (\phi'(E_A^i), \phi'(E_A^j)) = \delta_{ij}. \quad (27)$$

Taking account of eq. (5), we can rewrite eq. (27) as

$$\begin{aligned} & (\phi(E_A^i), 1 + \sum_k \sum_m \frac{H_k(l)\bar{H}_k(m)}{(K_A - E_A^i + \omega_k)(K_A - E_A^j + \omega_k)}\phi(E_A^j)) \\ &= (\phi(E_A^i), [1 + \omega_k \cdot V/(K_A - E_A^i + \bar{\omega}_k)(K_A - E_A^j + \bar{\omega}_k)]\phi(E_A^j)), \end{aligned} \quad (28)$$

where self-action parts ($l=m$) are neglected, and $\bar{\omega}_k$ in the summation \sum_k is replaced by its mean value $\bar{\omega}_k$. This $\bar{\omega}_k$ is not necessarily identical with the previously used. However, the difference might be not serious because the term in the denominator $(K - E + \omega_k)$ does not change appreciably in our approximation where the inner structure ($\omega_k \sim M$) of the nuclear force is not concerned.

Using the orthogonality relation (28) and (25), we obtain the followings,

$$\begin{aligned} & (\dot{p}_0^2/2M - E_0 - v_{00})\chi'(E_0)^* = \sum\chi(E_i)v_{0i}, \\ & (\dot{p}_0^2/2M - E_i - v_{ii})\chi(E_i) = \chi(E_0)v_{i0} + \sum_{i \neq j}\chi(E_j)v_{ij}, \end{aligned} \quad (29)$$

where

$$v = V_0 - \bar{\omega}V_0V/(K_A - E_A^0 + \bar{\omega})^2, \quad (30)$$

$$v_{ij} = (\phi(E_A^i), v\phi(E_A^j)). \quad (31)$$

The first term in eq. (30) indicates the force acting between the incident nucleon and each nucleon in the nucleus, and the second is the correlation effect in the nucleus caused by the first term. If we take $\chi(E_0)$ and only one $\chi(E_i)$ in eq. (29), we find that eq. (29) gives the resonant scattering in form¹¹⁾.

Our obtained results, as is clear from its resultant form, does not present any devia-

*) $\chi'(E_0)$ is $\chi(E_0)$ plus the scattered wave with the energy E_0 .

tion from the usual treatment of the nucleon-nucleus collision, though the concrete character of the interaction is too complicated to analyse.

§ 5. Concluding remarks

We have investigated the nuclear saturation problem by using Tamm-Dancoff method from the view point of the meson theory. From our view point, it is quite clear that the usual treatment of the saturation with the two-body potential is based on the essentially insufficient assumption. If the main part of the nuclear force is mediated by the meson exchange, the adiabatic approximation is to be an essential defect in the treatment of the heavy nucleus. As is indicated from our study, the saturation property is just derived from the inclusion of the effect caused by the change of the nuclear system during the meson exchange. And it should also be noted that the derived property of the saturation does not depend on any type of the elementary interaction of the meson field with the nucleon field.

Now, we should discuss the reservations which have been assumed in our investigation.

(i) We have restricted the meson number less than one. The modification due to the inclusion of the states with the meson number > 1 , should be the such replacement of the propagation function in the right hand of the eq. (6) as

$$\left(\sum \frac{p_j^2}{2M} + \bar{\omega}_k - E \right)^{-1} \longrightarrow 1/F(E, \sum \frac{p_j^2}{2M}, V, V', \dots, \bar{\omega}, \bar{\omega}', \dots)$$

where V' , V'' , ... would be the many body force. That is, the damping factor itself is remained, while its expression is modified. However, we cannot say any, whether our obtained results on the saturation property is actually changed or not, if the higher meson number is taken into account.

(ii) Our study is based on the non-relativistic approximation, where the effect of the nucleon pair is completely neglected. In this respect, we should remark that our obtained property of the saturation is not dependent of any relativistic effect of the nucleon, e.g. the recoil effect.

(iii) The self-energy problem is also neglected, though this effect can be important in such a dense matter as a nucleus¹⁹⁾.

(iv) The assumed interaction of the meson and the nucleon is linear in the meson wave function. On the other hand, the interaction quadratic in the meson field is suggested from the pseudoscalar meson theory¹³⁾. If we take only such a interaction, we are able to treat the problem in a similar manner to § 2, using $\Psi(0)$ and $\Psi(2)$ instead of $\Psi(1)$. It result in only an unessential modification in which the shape of V' and $\bar{\omega}_k$ is to be replaced by certain corrected terms.

(v) From our analysis, we have been not able to obtain the quantitative estimation of the nuclear radius. This problem is one of the most important in the saturation problem, that is, the saturation of nuclear density. To go into this problem, we find our equation (9)' is too complicated, and unfortunately we have not yet found any powerful means.

In conclusion, we wish to express our sincere thanks to Prof. S. Sakata for his kind interest in this work, and we also thank Dr. O. Hara and Mr. E. Yamada for their helpful discussions.

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On the Interaction of Electrons with Lattice Vibrations

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The present paper treats the interaction of electrons with lattice vibrations, using the intermediate coupling method developed by Lee, Low and Pines. The energy of the system is calculated to the first approximation. The expression for the energy is equivalent to that obtained by Fröhlich.

§ 1. Introduction

The interaction energy between electrons which is given rise to by the virtual emission and absorption of phonons has so far been calculated by perturbation theory. However the perturbation calculation will not give a good approximation owing to the large value of the interaction parameter. Tomonaga¹⁾ introduced a variational method, so called "intermediate coupling method" in treating meson-nucleon problem. It is difficult to apply this method in its original form to the many-electron system. Recently Lee, Low and Pines²⁾ have developed a variational technique, which is closely related to Tomonaga's method. They have calculated the self-energy of an electron in a polar crystal and have obtained a better result than that obtained by the perturbation calculation. Their method is equivalent to a canonical transformation, which we generalize and apply to the many-electron system.

Fröhlich has carried out a renormalization calculation. He has applied a canonical transformation to the Hamiltonian: this transformation leads to a renormalization of the phonon velocity and of the interaction parameter. He has thus removed the objection of Wentzel against the use of large interaction parameter.

It is the object of the present paper to elucidate the relation between Fröhlich's renormalization calculation and Lee-Low-Pines method.

§ 2. Résumé of Lee-Low-Pines method

We consider one electron interacting with phonons. Let the Hamiltonian be

$$H = \frac{p^2}{2m} + \sum_w \hbar \omega_w b_w^\dagger b_w + \sum_w (\gamma_w b_w e^{i\mathbf{r} \cdot \mathbf{x}} + \gamma_w^* b_w^\dagger e^{-i\mathbf{r} \cdot \mathbf{x}}), \quad (1)$$

where b_w and b_w^\dagger are phonon destruction and creation operators. Lee, Low and Pines have transformed to a representation in which the Hamiltonian does not contain the electron coordinates. Then they have used a variational technique. We shall show that

the same result is obtained without eliminating the electron coordinates by modifying the canonical transformation they have used.

We choose for the trial wave function

$$\phi = U\phi_0, \quad (2)$$

where ϕ_0 is the eigenstate of the unperturbed Hamiltonian with no phonons present:

$$b_w\phi_0 = 0, \quad (\phi_0, \phi_0) = 1. \quad (3)$$

U is a unitary operator and is given by

$$U = \exp \left\{ \sum (b_w^* e^{-iwx} f(w) - b_w e^{iwx} f^*(w)) \right\} = e^S, \quad (4)$$

where $f(w)$ will subsequently be chosen to minimize the energy. We find

$$\begin{aligned} U^{-1}b_w U &= b_w + f(w)e^{-iwx}, \\ U^{-1}b_w^* U &= b_w^* + f^*(w)e^{iwx}, \\ U^{-1}pU &= p - \sum_w \hbar w (b_w^* e^{-iwx} f(w) + b_w e^{iwx} f^*(w) + |f(w)|^2). \end{aligned} \quad (5)$$

Using (3) and (5), we get

$$\begin{aligned} E = (\phi, H\phi) &= (\phi_0, U^{-1}HU\phi_0) \\ &= \frac{p^2}{2m} + \sum_w (\gamma_w f(w) + \gamma_w^* f^*(w)) + \frac{\hbar^2}{2m} \left\{ \sum_w |f(w)|^2 w^2 \right. \\ &\quad \left. + \sum_w |f(w)|^2 \left(\hbar w s - \frac{\hbar w \cdot p}{m} + \frac{\hbar^2 w^2}{2m} \right) \right\}, \end{aligned} \quad (6)$$

which is identical with the expression obtained by Lee, Low and Pines. Eq. (6) is also obtained by expanding $U^{-1}HU$ to the fourth order in S :

$$\begin{aligned} E = (\phi_0, \{ H - [S, H] + \frac{1}{2}[S, [S, H]] - \frac{1}{6}[S, [S, [S, H]]] \\ + \frac{1}{24}[S, [S, [S, [S, H]]]] \} \phi_0). \end{aligned}$$

Eq. (6) is minimized by setting

$$\frac{\delta E}{\delta f(w)} = \frac{\delta E}{\delta f^*(w)} = 0. \quad (7)$$

§ 3. Many-electron system

We adopt the Hamiltonian and notation of Fröhlich,³⁾ in which

$$\begin{aligned} H &= H_0 + H_1 + H_2, \\ H_0 &= \sum_k \epsilon_k a_k^* a_k + \sum_w \hbar w s \left(b_w^* b_w + \frac{1}{2} \right), \\ H_1 &= i \sum_w D_w (b_w a_k^* a_{k-w} - b_w^* a_{k-w}^* a_k), \\ H_2 &= \sum \hbar w (s' - s) \left(b_w^* b_w + \frac{1}{2} \right), \end{aligned}$$

where α_k and α_k^* are electron destruction and creation operators, and $D_w^2 = \frac{4E'z}{3nV} \hbar \tau v s'$.

The renormalized velocity s will be determined below.

We choose for our trial wave function

$$\psi = U\psi_0, \quad (9)$$

where ψ_0 is the eigenstate of the unperturbed Hamiltonian. Specifically, ψ_0 is defined by

$$\begin{aligned} (\psi_0, \psi_0) &= 1, \quad (\psi_0, b_w \psi_0) = (\psi_0, b_w^* \psi_0) = (\psi_0, \alpha_k \psi_0) = (\psi_0, \alpha_k^* \psi_0) = 0, \\ (\psi_0, b_v^* b_w \psi_0) &= \delta_{v,w} N_w, \quad (\psi_0, b_v b_w^* \psi_0) = \delta_{v,w} (1 + N_w), \\ (\psi_0, \alpha_k^* \alpha_q \psi_0) &= \delta_{k,q} n_k, \quad (\psi_0, \alpha_q \alpha_k^* \psi_0) = \delta_{k,q} (1 - n_k), \end{aligned} \quad (10)$$

where N_w and n_k^* are the occupation numbers of phonons and electrons. The unitary operator U is given by

$$U = \exp\left\{\sum (b_w^* \alpha_{k-w}^* \alpha_k f(\mathbf{k}, \mathbf{w}) - b_w \alpha_k^* \alpha_{k-w} f(\mathbf{k}, \mathbf{w}))\right\} = e^S. \quad (11)$$

With the help of commutation relations $[b_v, b_w^*] = \delta_{v,w}$, $\alpha_k^* \alpha_q + \alpha_q \alpha_k^* = \delta_{k,q}$, we find

$$\begin{aligned} U^{-1} H U &= H_0 + H_1 + H_2 - [S, H_0] - [S, H_1] + \frac{1}{2} [S, [S, H_0]] + \dots \\ &= \sum_k \epsilon_k \alpha_k^* \alpha_k + \sum_k \hbar \tau v s \left(b_w^* b_w + \frac{1}{2} \right) + \sum_{w,k} b_w \{ i D_w + (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s) \\ &\quad \times f^*(\mathbf{k}, \mathbf{w}) \} \alpha_k^* \alpha_{k-w} + \sum_{w,k} b_w^* \{ -i D_w + (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s) f(\mathbf{k}, \mathbf{w}) \} \alpha_{k-w}^* \alpha_k \\ &\quad + i \sum_{w,k,q} D_w (\alpha_{k-w}^* \alpha_k \alpha_q^* \alpha_{q-w} f(\mathbf{k}, \mathbf{w}) - \alpha_k^* \alpha_{k-w} \alpha_{q-w}^* \alpha_q f^*(\mathbf{k}, \mathbf{w})) \\ &\quad - i \sum_{v,w,k,q} D_v \{ b_v b_w^* (\delta_{k,q} \alpha_{k-w}^* \alpha_{q-v} - \delta_{k-w,q-v} \alpha_q^* \alpha_k) f(\mathbf{k}, \mathbf{w}) + b_v^* b_w (\delta_{k-w,q-v} \alpha_k^* \alpha_q \\ &\quad - \delta_{k,q} \alpha_{q-v}^* \alpha_{k-w}) f^*(\mathbf{k}, \mathbf{w}) - b_v^* b_w^* (\delta_{k,q-v} \alpha_{k-w}^* \alpha_q - \delta_{k-w,q} \alpha_{q-v}^* \alpha_k) f(\mathbf{k}, \mathbf{w}) \\ &\quad - b_v b_w (\delta_{k-w,q} \alpha_k^* \alpha_{q-v} - \delta_{k,q-v} \alpha_q^* \alpha_{k-w}) f^*(\mathbf{k}, \mathbf{w}) \} \\ &\quad + \frac{1}{2} \sum_{v,w,k,q} (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s) (\alpha_{q-w}^* \alpha_q \alpha_k^* \alpha_{k-w} f(\mathbf{q}, \mathbf{w}) f^*(\mathbf{k}, \mathbf{w}) \\ &\quad + \alpha_q^* \alpha_{q-w} \alpha_{k-w}^* \alpha_k f^*(\mathbf{q}, \mathbf{w}) f(\mathbf{k}, \mathbf{w})) \\ &\quad - \frac{1}{2} \sum_{v,w,k,q} (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s) \{ b_v b_w^* (\delta_{q,k} \alpha_{q-v}^* \alpha_{k-w} - \delta_{q-v,k-w} \alpha_k^* \alpha_q) f(\mathbf{q}, \mathbf{v}) f^*(\mathbf{k}, \mathbf{w}) \\ &\quad + b_w^* b_v (\delta_{k,q} \alpha_{k-w}^* \alpha_{q-v} - \delta_{k-w,q-v} \alpha_q^* \alpha_k) f^*(\mathbf{q}, \mathbf{v}) f(\mathbf{k}, \mathbf{w}) \\ &\quad + b_w^* b_v^* (\delta_{q,k-w} \alpha_{q-v}^* \alpha_k - \delta_{q-v,k} \alpha_{k-w}^* \alpha_q) f(\mathbf{k}, \mathbf{w}) f(\mathbf{q}, \mathbf{v}) \\ &\quad + b_w b_v (\delta_{q,k-w} \alpha_k^* \alpha_{q-v} - \delta_{q-v,k} \alpha_q^* \alpha_{k-w}) f^*(\mathbf{k}, \mathbf{w}) f^*(\mathbf{q}, \mathbf{v}) \} \\ &\quad + \sum_{w,v} \hbar \tau v (s' - s) (b_w^* b_w + 1/2) + \dots \end{aligned} \quad (12)$$

* N_w and n_k should be considered to be parameters, which are to be determined to make the energy as low as possible.

Using eq. (10), we see that to the first approximation

$$\begin{aligned}
 E = & \sum_k \epsilon_k n_k + \sum_w \hbar \tau v s (N_w + 1/2) - i \sum_{w,k} D_w \{ n_k (1 - n_{k-w}) + (n_k - n_{k-w}) N_w \} \\
 & \times (f^*(\mathbf{k}, \mathbf{w}) - f(\mathbf{k}, \mathbf{w})) + \sum_{w,k} (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s) \{ n_k (1 - n_{k-w}) + (n_k - n_{k-w}) N_w \} \\
 & \times |f(\mathbf{k}, \mathbf{w})|^2 + \sum_w \hbar \tau v (s' - s) (N_w + 1/2). \quad (13)
 \end{aligned}$$

We minimize (13) by setting

$$\delta E / \delta f(\mathbf{k}, \mathbf{w}) = \delta E / \delta f^*(\mathbf{k}, \mathbf{w}) = 0.$$

We get

$$f(\mathbf{k}, \mathbf{w}) = i D_w / (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s), \quad f^*(\mathbf{k}, \mathbf{w}) = -i D_w (\epsilon_{k-w} - \epsilon_k + \hbar \tau v s). \quad (14)$$

Inserting (14) into (13), we obtain the energy of the ground state of our system:

$$\begin{aligned}
 E = & \sum_k \epsilon_k n_k + \sum_w \hbar \tau v s \left(N_w + \frac{1}{2} \right) - \sum_{w,k} \frac{D_w^2 n_k (1 - n_{k-w})}{\epsilon_{k-w} - \epsilon_k + \hbar \tau v s} \\
 & - \sum_{w,k} \frac{D_w^2 (n_k - n_{k-w})}{\epsilon_{k-w} - \epsilon_k + \hbar \tau v s} N_w + \sum_w \hbar \tau v (s' - s) \left(N_w + \frac{1}{2} \right),
 \end{aligned}$$

which can be rewritten as

$$\begin{aligned}
 E = & \sum_k \left\{ \epsilon_k + \frac{4F\zeta}{3nV} \sum_w \frac{(\hbar \tau v s)^2}{(\epsilon_{k-w} - \epsilon_k)^2 - (\hbar \tau v s)^2} \right\} n_k + \sum_w \hbar \tau v s \left(N_w + \frac{1}{2} \right) \\
 & - \frac{4F\zeta}{3nV} \sum_{k,w} \frac{(\hbar \tau v s)^2}{(\epsilon_{k-w} - \epsilon_k)^2 - (\hbar \tau v s)^2} n_k n_{k-w}, \quad (15)
 \end{aligned}$$

where $F's/s = F$. The second term in the curly bracket in (15) is the self-energy of an electron and the last term is the interaction energy between electrons. The denominators in (15) will vanish for certain values of k and w . Principal values are to be used to prevent the sums from becoming infinite. The equation determining the renormalized velocity becomes

$$\hbar \tau v \delta s = \hbar \tau v (s - s') = - \frac{4F\zeta}{3nV} \sum_k \frac{n_k - n_{k-w}}{\epsilon_{k-w} - \epsilon_k + \hbar \tau v s} \hbar \tau v s. \quad (16)$$

We will have the second approximation by our method, if we evaluate the terms up to $1/24 [S, [S, [S, H]]]$ and determine $f(\mathbf{k}, \mathbf{w})$ to minimize the energy.

The author wishes to express his thanks to Messrs. S. Koide and T. Noguchi for their discussions.

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On the Elastic Frequency Distribution Function of Simple Crystals

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A close examination of characters of the approximations of Montroll and Houston is presented. It is recommended that these two approximations should be used each other complementarily. We adopt the Born-Karman model and examine mainly a square lattice, a simple cubic lattice and a face-centered cubic lattice.

§ 1. Introduction

The knowledges of frequency spectrums of simple crystals are important for the quantum theory of matter. And it is difficult to get a complete and exact answer to this problem. If we adopt the Born-Karman model, the problem to be solved here is essentially to get complete eigenvalue spectrums of some eigenvalue problems.

However, if we side-step our purely mathematical and academic interests to this topic, we can now get necessary informations through the analytically compact procedures with some reasonable amount of labor. The tools to meet this purpose have already been supplied, mainly, by the works of Montroll⁽¹⁾⁽²⁾⁽³⁾ and Houston⁽⁴⁾.

In this paper, we attempt a close examination of characters of the approximations due to these authors and put them into the forms ready to practical applications. Then the detailed application to a face-centered cubic lattice is presented in comparison with the Leighton's results⁽⁵⁾.

§ 2. Program of discussions

Imagine a typical frequency spectrum with two peaks familiar to us since Blackman. Now, it is to be noted that the complete and exact form of the curve is not always required. Let us consider the specific heat for example. We define a continuous function $g(\nu)$ in such a manner that $g(\nu)d\nu$ is the number of normal modes in the interval $\langle \nu, \nu + d\nu \rangle$ at ν . In terms of $g(\nu)$, we can write the heat capacity at constant volume C_v ;

$$C_v = k \int_0^{\nu_L} g(\nu) (\hbar\nu/kT)^2 \exp(\hbar\nu/kT) d\nu / [\exp(\hbar\nu/kT) - 1]^2, \quad (1)$$

where ν_L is the largest frequency.

a. *Low temperature.* Expanding $g(\nu)$ in a Taylor series in ν , we obtain the following formula⁽⁵⁾;

$$g(\nu) = \sum_n a_n \nu^n,$$

$$C_v = k \sum_n a_n \left(\frac{kT}{h} \right)^{n+1} \int_0^{(h\nu/kT)} \frac{x^{n+2} e^x}{(e^x - 1)^2} dx. \quad (2)$$

If the temperature T is low enough, this series is rapidly convergent. This means that we have only to know the local behavior of $g(\nu)$ near the origin, that is, if we obtain a power series in ν which traces closely the low frequency portion of exact $g(\nu)$, it will be sufficient. For this purpose, the method of Houston offers us excellent approximate formulas for usual cases. These will be discussed in detail later.

b. *High temperature.* Here exists the well-known moment expansion due to Thirring⁽⁶⁾ and Montroll⁽²⁾;

$$C_v/mk = 1 - \sum_{n=1}^{\infty} (-1)^n B_n (1-2n) (h/kT)^{2n} \mu_{2n} / (2n)!, \quad (3)$$

$$\mu_n = (1/m) \int_0^{\nu_f} \nu^n g(\nu) d\nu, \quad (4)$$

where m is the number of degrees of freedom of the system in question and μ_n is n -th moment of $g(\nu)$ and the B_n 's are the Bernoulli numbers:

$$B_1 = 1/6; \quad B_2 = 1/30; \quad B_3 = 1/42; \dots$$

In this case, we only need the first few moments of $g(\nu)$. It is not necessary to know the actual form of $g(\nu)$ itself. The moments of $g(\nu)$ are obtained directly from the Born-Karman matrix.

c. *Intermediate temperature.* Here we seem to encounter some difficulties. We have not at hand a suitable compact expansion formula such as (2), (3). However, there is good reason to expect that the Montroll's polynomial approximation for $g(\nu)$ supplies satisfactory answers in this case. This will be shown later.

We have been concerned with specific heat so far, but in some other cases the number and the location of peaks of spectrum are also required. The exact answer is difficult. However, examining carefully the topological nature of equi-frequency surfaces in Brillouin zone, we can get useful informations about the number and the location of the peaks. An illustrating example will be presented in the case of a simple cubic lattice.

§ 3. On the method of Houston

To fix the idea, we take a simple cubic lattice as an illustrating example. In this case, the characteristic determinant is¹⁾,

$$\begin{vmatrix} a_{12} - 2f^2 & b_{12} & b_{13} \\ b_{21} & a_{22} - 2f^2 & a_{23} \\ b_{31} & b_{32} & a_{33} - 2f^2 \end{vmatrix} = 0,$$

$$a_{11} = (1 - 2\tau)(1 - c_1) + \tau(2 - c_1 c_2 - c_1 c_3), \quad (5)$$

$$a_{22} = (1 - 2\tau)(1 - c_2) + \tau(2 - c_2c_3 - c_2c_1),$$

$$a_{33} = (1 - 2\tau)(1 - c_3) + \tau(2 - c_3c_1 - c_3c_2),$$

$$b_{ij} = \tau s_i s_j,$$

$$c_i = \cos \varphi_i, \quad s_i = \sin \varphi_i, \quad \varphi_i = 2\pi a_i / N, \quad (-1/2N \leq a_i \leq 1/2N),$$

$$\tau = 8\gamma / 4\pi^2 M \nu_L^2, \quad \nu_L^2 = 4(u + 4\gamma) / 4\pi^2 M, \quad f = \nu / \nu_L.$$

Solving eq. (5) with regard to f , we obtain, in principle at least, the formulas,

$$f = f_i = F_i(\varphi_1, \varphi_2, \varphi_3), \quad (i = 1, 2, 3). \quad (6)$$

Subscripts i denote the three branches.

We can obtain the frequency distribution function starting from these formulas. The orthodox procedure was described neatly in a paper of Montroll⁽³⁾. Its reformulation in three dimensional case is evident. The fraction of all frequencies less than $\nu = f\nu_L$ in one branch is given by

$$N_i(\nu) = (1/V) \iiint_{F_i(\varphi_1, \varphi_2, \varphi_3) \leq f} d\varphi_1 d\varphi_2 d\varphi_3, \quad (7)$$

where the integration extends over the region interior to the surface $f = F_i(\varphi_1, \varphi_2, \varphi_3) =$ constant and V is the volume of the first Brillouin zone of the reciprocal lattice defined by eq. (5). $V = 8\pi^3$ for our simple cubic lattice. The frequency distribution $g(\nu)$, which is defined so that $g(\nu)d\nu$ is the fraction of frequencies in the interval $(\nu, \nu + d\nu)$, is given for each branch by the equation

$$g_i(\nu) = \frac{\partial}{\partial(\nu_L f)} N_i(\nu) = \frac{1}{V \nu_L} \frac{\partial}{\partial f} \iiint_{F_i(\varphi_1, \varphi_2, \varphi_3) \leq f} d\varphi_1 d\varphi_2 d\varphi_3. \quad (8)$$

The total frequency distribution function is given by the equation,

$$g(f\nu_L) = (1/3) \{g_1(f\nu_L) + g_2(f\nu_L) + g_3(f\nu_L)\}. \quad (9)$$

This is normalized to unity:

$$\int_0^{\nu_L} g(f\nu_L) d\nu = 1. \quad (10)$$

Let us apply eq. (8) to a trivial case, $\tau = 0$, in (5). Here we obtain

$$f_i^2 = (1/2)(1 - c_i) = (1/2)(1 - \cos \varphi_i), \quad (i = 1, 2, 3),$$

and then

$$\begin{aligned} g_1(f\nu_L) &= \frac{8}{8\pi^3 \nu_L} \frac{\partial}{\partial f} \int_0^\pi d\varphi_2 \int_0^\pi d\varphi_3 \int_0^{\cos^{-1}(1-2f)} d\varphi_1 \\ &= (1/\pi \nu_L) \partial/\partial f \cos^{-1}(1-2f^2) = (2/\pi \nu_L) (1-f^2)^{-1/2}, \end{aligned}$$

and using (9),

$$\nu_L g(f\nu_L) = (2/\pi) (1-f^2)^{-1/2}.$$

Unfortunately, the complicated topological nature of equi-frequency surfaces of Brillouin zone prevents us from applying this procedure directly to the general and physically interesting cases. We must resort to some approximation techniques at this point.

Introducing a polar coordinate (r, θ, φ) in rectangular $(\varphi_1, \varphi_2, \varphi_3)$ -space, we may rewrite the formula (7):

$$N_i(\nu) = (1/V) \iiint_{r_i(r, \theta, \varphi) \leq f} r^2 dr \sin \theta d\theta d\varphi. \quad (7')$$

Since we can express, in principle, r in terms of θ, φ, f ;

$$r = \chi_i(\theta, \varphi, f), \quad (11)$$

then,

$$\begin{aligned} N_i(\nu) &= (1/3V) \iiint r^3 \sin \theta d\theta d\varphi \\ &= (1/3V) \int_0^{2\pi} \int_0^\pi [\chi_i(\theta, \varphi, f)]^3 \sin \theta d\theta d\varphi, \end{aligned} \quad (12)$$

$$g_i(f\nu_L) = \frac{1}{3V\nu_L} \frac{\partial}{\partial f} \int_0^{2\pi} \int_0^\pi [\chi_i(\theta, \varphi, f)]^3 \sin \theta d\theta d\varphi. \quad (13)$$

It is very natural to expect that if we get an approximate expression of $\chi_i(\theta, \varphi, f)$, borrowing the original idea of Houston, we should be able to reach a reasonable approximation through the formulas (12), (13). Actually, Nakamura⁷⁾ treated a two dimensional square lattice very successfully along this line.

$\chi_i(\theta, \varphi, f)$ may be expressed as a series of functions each multiplied by a harmonics with suitable symmetry, $K_i(\theta, \varphi)$:

$$\begin{aligned} \chi_i(\theta, \varphi, f) &= \sum_l K_l(\theta, \varphi) I_{il}(f) = \sum_l c_l H_l(\theta, \varphi) I_{il}(f) \\ &= \sum_l H_l(\theta, \varphi) S_{il}(f), \end{aligned} \quad (14)$$

where $K_l = c_l H_l$, $S_l = c_l I_l$, $\{c_l\}$ being normalization constants. We cut the series, retaining only first few terms. Here lies our approximation.

$$\chi_i(\theta, \varphi, f) \cong \sum_{l=0}^{R-1} H_l(\theta_s, \varphi_s) S_{il}(f), \quad (15)$$

where subscripts s designate the different directions along which the secular equation is solved.

Using $\{S_{il}(f)\}$ which are obtained by solving the set of equations (15), we get approximate formulas for $\chi_i(\theta, \varphi, f)$:

$$\chi_i(\theta, \varphi, f) \cong \sum_{l=0}^{R-1} H_l(\theta, \varphi) S_{il}(f), \quad (i=1, 2, 3). \quad (16)$$

Putting these into (13) and performing the integral, we obtain the final result.

In case of our simple cubic lattice, if we take the origin at the center of first Brillouin zone, $\{K_i\}$ are so called cubic harmonics. First three harmonics are

$$\left. \begin{aligned} K_0 &= c_0 H_0 = c_0, \\ K_1 &= c_1 H_1 = c_1 \{P_4(\cos \theta) + (1/168)P_4^4(\cos \theta) \cos 4\varphi\}, \\ K_2 &= c_2 H_2 = c_2 \{P_6(\cos \theta) - (1/360)P_6^4(\cos \theta) \cos 4\varphi\}. \end{aligned} \right\} \quad (17)$$

If we normalize $\{K_i\}$ by the equations,

$$\int_0^{2\pi} \int_0^\pi K_i K_j \sin \theta d\theta d\varphi = \delta_{ij},$$

c_0, c_1, c_2 take the values, $1/(4\pi)^{1/2}$, $(3 \cdot 7)^{1/2}/2(4\pi)^{1/2}$, $(2 \cdot 13)^{1/2}/4(4\pi)^{1/2}$, respectively. However, as far as our calculation is concerned, the explicit values of these normalization constants are not necessary.

The eq. (8) is now,

$$N_i(f) = (1/3V) \int_0^{2\pi} \int_0^\pi \left[\sum_{l=0}^2 H_l(\theta, \varphi) S_{il}(f) \right]^3 \sin \theta d\theta d\varphi. \quad (18)$$

The integration is to be performed over φ first. For the integration over θ , the following formula (Gaunt) is useful;

$$\begin{aligned} & \int_{-1}^{+1} P_l^{n+w}(x) P_m^v(x) P_n^w(x) dx \\ &= (-1)^{s-m-w} \frac{(2s-2n)! s!}{(s-l)!(s-m)!(s-n)!(2s+1)!} \cdot \frac{2(m+v)! (n+w)!}{(m-v)!} \times \\ & \times \sum_i (-1)^i \frac{(l+v+w+i)! (m+n-v-w-i)!}{(l-v-w-i)! (m-n+v+w+i)! (n-w-i)! i!}, \quad (l+m+n=2s). \end{aligned} \quad (19)$$

The result is

$$\begin{aligned} N_i(f\nu_L) &= (4\pi/3V) [S_{i0}^3(f) + (48/1001)S_{i1}^3(f) - (1280/46189)S_{i2}^3(f)] \\ &+ (\pi/V) [(4/21)S_{i0}(f)S_{i1}^2(f) + (8/13)S_{i0}(f) + \\ &+ (320/3003)S_{i1}^2(f)S_{i2}(f) - (361/2431)S_{i1}(f)S_{i2}^2(f)], \end{aligned} \quad (20)$$

$$g_i(f\nu_L) = \frac{1}{\nu_L} \frac{d}{df} N_i(f\nu_L), \quad (21)$$

where,

$$\begin{aligned} S_{i0}(f\nu_L) &= \frac{1}{35} \left[10\chi_i(o, o, f) + 16\chi_i\left(\frac{\pi}{4}, o, f\right) + 9\chi_i\left(\cos^{-1} \frac{1}{\sqrt{3}}, \frac{\pi}{4}, f\right) \right], \\ S_{i1}(f\nu_L) &= \frac{1}{55} \left[35\chi_i(o, o, f) - 8\chi_i\left(\frac{\pi}{4}, o, f\right) - 27\chi_i\left(\cos^{-1} \frac{1}{\sqrt{3}}, \frac{\pi}{4}, f\right) \right], \\ S_{i2}(f\nu_L) &= \frac{1}{77} \left[6\chi_i(o, o, f) - 24\chi_i\left(\frac{\pi}{4}, o, f\right) + 18\chi_i\left(\cos^{-1} \frac{1}{\sqrt{3}}, \frac{\pi}{4}, f\right) \right]. \end{aligned} \quad (22)$$

This result is rather complicated against our expectation. It is desirable to test the efficiency of this approximate formula comparing it with an exact answer. Unfortunately we are lacking, up to now, in such a standard expression to be referred to in three dimensional case.

In a square lattice, the Houston's formula and the one corresponding to (21) happen to be identical in the first approximation. In the next approximation, they are not identical and so we can examine their efficiencies referring to the exact result given by Montroll⁽³⁾:

$$\nu_L g_{\text{exact}}(f) = 6\pi^{-1}f(1 + 3/2 \cdot f^2 + 45/16 \cdot f^4 + 27 \cdot 7/32 \cdot f^6 + \dots). \quad (23)$$

The first approximation;

$$\nu_L g_H(f) = \nu_L g_N(f) = 6\pi^{-1}f(1 + 3/2 \cdot f^2 + 3f^4 + 27 \cdot 9/35 \cdot f^6 + \dots). \quad (24)$$

The second approximation;

$$\nu_L g_H(f) = 6\pi^{-1}f[1 + 3/2 \cdot f^2 + 45/16 \cdot f^4 + 3/16 \cdot 157/5 \cdot f^6 + \dots], \quad (25)$$

$$\nu_L g_N(f) = 6\pi^{-1}f[1 + 3/2 \cdot f^2 + 1/16 \cdot (45 + 3/8)f^2 + 3/16 \cdot (157/5 + 27/40)f^6 + \dots]. \quad (26)$$

Here $g_H(f)$ is a formula obtained by the method of Houston and $g_N(f)$ is a formula corresponding to the expression (21).

This result is not encouraging. The only merit of our procedure presented above might be that we can obtain Houston's formula with correctly adjusted numerical coefficients*:

$$\begin{aligned} \nu_L g_H(f\nu_L) = \frac{4\pi}{105V} \sum_{i=1}^3 \left[10\chi_i^2(o, o, f) \frac{d\chi_i(o, o, f)}{df} + 16\chi_i^2\left(\frac{\pi}{4}, o, f\right) \frac{d\chi_i(\pi/4, o, f)}{df} \right. \\ \left. + 9\chi_i^2\left(\cos^{-1} \frac{1}{\sqrt{3}}, \frac{\pi}{4}, f\right) \frac{d\chi_i(\cos^{-1} 1/\sqrt{3}, \pi/4, f)}{df} \right]. \quad (27) \end{aligned}$$

This formula is expected to offer an excellent approximation for low frequency region, as observed in a square lattice. Let us consider formulas (12), (13) again here. It is easily inferred that these formulas offer a reliable answer only when the surface $f=\text{constant}$ in question forms a closed surface around the origin we chose. If the surface of first Brillouin zone cuts the equi-frequency surface, we can not expect a good result. For this reason, it is recommended that we use (27) only for low frequency region. In this region, the curve given by (27) will trace very closely the exact curve. However, if the position of the first peak is very close to the origin, the formula (27) will not be usable in safety. This is the case when $\tau \rightarrow 0$ in our simple cubic lattice.

Incidentally, since we have decided to use (27) only for low frequency region, we do not need its full expression, that is, we only need its Taylor series at the origin. This situation facilitates to proceed to higher order approximation of Houston's method. The higher order approximations in Houston's method mean the increase of directions along which eq. (5) is to be solved. Solving eq. (5), we obtain first,

$$f^2 = \mathcal{F}(\theta, \varphi, r). \quad (28)$$

* The correction of numerical errors in Houston's original formula was first made by Mr. Komoda in his private letter to Mr. Nakamura. We are indebted to Mr. Nakamura for this information.

If we want the full expression of (27), the above equation must be solved with regard to r in the form,

$$r = \chi(\theta_s, \varphi_s, f). \quad (29)$$

This task is not always very easy, if not impossible, when we try to solve along several directions. However, if we restrict ourselves to obtain the Taylor expansion in f , we only need $\{(d^n r / d f^n)_{f=0}\}$ which can be calculated from (28) directly. This simple manipulation has actually been used in obtaining the second approximation for a square lattice (25), (26) and will be used for a face-centered cubic lattice.

§ 4. On the method of Montroll

It is possible to give several modifications of Montroll's original formulation. Montroll obtained an even power series approximation. We present here an odd power series approximation⁹⁾. This will serve for our better understanding of Montroll's method. Thus, noticing that the Born-Karman matrix M gives us the eigenvalues of $\eta = \nu^2$, not of ν itself, it is very natural to attempt to calculate at first the distribution function $D(\eta)$ for the eigenvalue η of M in the region $0 \leq \eta \leq \eta_L$. Then we can transfer from $D(\eta)$ to $g(\nu)$, our desired frequency distribution, by a change of variable, and by noticing the relation $D(\eta) d\eta = g(\nu) d\nu$. In this manner, we can actually obtain an odd function approximation for $g(\nu)$.

a. *Expansion of $D(\eta)$ by Legendre polynomials.*

In the following analysis, we are concerned with the Born-Karman matrix M itself. Denoting the i -th eigenvalue of M by $\eta_i = \nu_i^2$, then

$$M\phi_i = \eta_i \phi_i, \quad (30)$$

and in general

$$M^k \phi_i = \eta_i^k \phi_i, \quad (31)$$

and

$$\text{Trace } M^k = \sum_i \eta_i^k. \quad (32)$$

Since the total number of eigenvalues is equal to the order of the matrix, we can now define the distribution function $D(\eta)$ for the distribution of eigenvalue η as follows;

$$\int_{\eta_s}^{\eta_L} D(\eta) d\eta = m, \quad (33)$$

where m denotes the order of the matrix.

The k -th moment of $D(\eta)$ is given by

$$(\mu_k)_\eta = \frac{1}{m} \int_{\eta_s}^{\eta_L} D(\eta) \eta^k d\eta = \frac{\text{Trace } M^k}{m}, \quad (34)$$

where η_s, η_L are the smallest and largest eigenvalue respectively ($0 \leq \eta_s \leq \eta_L$).

Suppose the distribution function is expanded as a linear combination of Legendre

polynomials ;

$$D(\eta) = \sum_{n=0}^{\infty} \alpha_n P_n[(2\eta - \eta_L - \eta_s)/(\eta_L - \eta_s)], \quad (35)$$

where

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (36)$$

Putting $x = (2\eta - \eta_L - \eta_s)/(\eta_L - \eta_s)$, the coefficients $\{\alpha_n\}$ are

$$\alpha_n = \frac{(2n+1)}{2} \int_{-1}^1 D\left(\frac{(1+x)\eta_L + (1-x)\eta_s}{2}\right) P_n(x) dx. \quad (37)$$

Without the loss of generality, we can assume η_s to be equal to zero. Then,

$$\eta_s = 0, \quad \eta/\eta_L = \xi, \quad x = 2(\eta/\eta_L) - 1 = 2\xi - 1, \quad (38)$$

and then (35), (37) are

$$D(\eta_L \xi) = \sum_{n=0}^{\infty} \alpha_n P_n(2\xi - 1), \quad (39)$$

$$\alpha_n = (2n+1) \int_0^1 D(\eta_L \xi) P_n(2\xi - 1) d\xi. \quad (40)$$

Further, using the Rodrigues formula (36), it is easy to show that

$$P_n(x) = \sum_{r=0}^n \frac{(-1)^{n-r}}{(r!)^2} \frac{1}{(n-r)!} \left(\frac{x+1}{2}\right)^r, \quad (41)$$

then

$$P_n(2\xi - 1) = \sum_{r=0}^n \frac{(-1)^{n-r}}{(r!)^2} \frac{1}{(n-r)!} \xi^r, \quad (42)$$

and substituting (42) into (40),

$$D(\eta_L \xi) = \sum_{n=0}^{\infty} \alpha_n \sum_{r=0}^n \frac{(-1)^{n-r}}{(r!)^2} \frac{1}{(n-r)!} \xi^r. \quad (43)$$

Therefore, if we define new coefficients $\{b_r\}$ as follows,

$$D(\eta_L \xi) = \sum_{r=0}^{\infty} b_r \xi^r, \quad (44)$$

then

$$b_r = \sum_{n=r}^{\infty} \frac{(-1)^{n-r}}{(r!)^2} \frac{1}{(n-r)!} \alpha_n, \quad (45)$$

and

$$\alpha_n = (2n+1) \sum_{r=0}^n \frac{(-1)^{n-r}}{(r!)^2} \frac{1}{(n-r)!} \int_0^1 D(\eta_L \xi) \xi^r d\xi. \quad (46)$$

From (34),

$$(\mu_k)_\eta = \frac{1}{m} \int_0^{\eta_L} D(\eta) \eta^k d\eta = \frac{\eta_L^{k+1}}{m} \int_0^1 D(\eta_L \xi) \xi^k d\xi = \frac{\text{Trace } \mathbf{M}^k}{m}, \quad (47)$$

then, for the later convenience, we define μ_k as follows,

$$\int_0^1 D(\eta_L \xi) \xi^k d\xi = (\mu_k)_\xi = \mu_k. \quad (48)$$

In this definition,

$$\mu_k = (\mu_k)_\xi = \frac{m}{\eta_L^{k+1}} (\mu_k)_\eta = \frac{\text{Trace } \mathbf{M}^k}{\eta_L^{k+1}}, \quad (49)$$

therefore, we obtain for a_n ,

$$a_n = (2n+1) \sum_{r=0}^n (-1)^{n-r} \frac{1}{(r!)^2} \frac{(n+r)!}{(n-r)!} \mu_r. \quad (50)$$

From (45) and (50),

$$b_r = \sum_{n=r}^{\infty} (-1)^{n-r} \frac{1}{(r!)^2} \frac{(n+r)!}{(n-r)!} (2n+1) \sum_{s=0}^n (-1)^{n-s} \frac{1}{(s!)^2} \frac{(n+s)!}{(n-s)!} \mu_s. \quad (51)$$

Therefore, if we introduce new coefficients C_r ;

$$b_r = \sum C_{rs} \mu_s, \quad (52)$$

then finally,

$$\begin{aligned} r < s: \quad C_{rs} &= \frac{1}{(r!)^2} \frac{1}{(s!)^2} (-1)^{r+s} \sum_{m=s}^{\infty} (2m+1) \frac{(m+r)!}{(m-r)!} \frac{(m+s)!}{(m-s)!}, \\ r \geq s: \quad C_{rs} &= \frac{1}{(r!)^2} \frac{1}{(s!)^2} (-1)^{r-s} \sum_{m=r}^{\infty} (2m+1) \frac{(m+r)!}{(m-r)!} \frac{(m+s)!}{(m-s)!}. \end{aligned} \quad (53)$$

Furthermore, if we cut the summation in (53) at k -th term, we can perform the summation actually. The result is

$$C_{rs} = (-1)^{r+s} \frac{1}{(r!)^2 (s!)^2} \frac{1}{(r+s+1)} \frac{(k+r+1)! (k+s+1)!}{(k-r)! (k-s)!}. \quad (54)$$

Once $\{C_{rs}\}$ are obtained, b_r can be expressed as a linear combination of μ_s 's. Then if we know the numerical values of μ_s 's for each case, we shall be able to reach the approximate expansion of $D(\eta)$ by means of (44).

Above results are general ones and applicable to any lattice type and dimension in principle.

b. Power series approximation by direct application of the method of least squares.

Let us suppose to try to approximate $D(\eta)$ by a function $U(a_0, a_1, \dots, a_k; \eta) \equiv U(a; \eta)$ of a set of parameters $\{a_n\}$ chosen in such a manner that

$$I(a_0, a_1, \dots, a_k) \equiv I(a) = \int_0^{\eta_L} [D(\eta) - U(a; \eta)]^2 d\eta, \quad (55)$$

to be a minimum. According to the calculus of variations, the necessary condition for $I(a)$ to be a minimum is

$$\partial I(a)/\partial a_n = 0; \quad (n=0, 1, \dots, k). \quad (56)$$

Now, if we take the following power series as $U(a; \eta)$,

$$U(a; \eta) = \sum_{n=0}^k a_n (\eta/\eta_k) = \sum_{n=0}^k a_n \xi^n, \quad (57)$$

then, from (56), we finally reach a set of equations,

$$\mu_n - \sum_{m=0}^n a_m \frac{1}{[(m+n)+1]} = 0, \quad (n=0, 1, \dots, k). \quad (58)$$

These are the $k+1$ equations determining $\{a_m\}$. Explicitly,

$$\left. \begin{aligned} a_0/1 + a_1/2 + \dots + a_k/(k+1) &= \mu_0, \\ a_0/2 + a_1/3 + \dots + a_k/(k+2) &= \mu_1, \\ a_0/(k+1) + a_1/(k+2) + \dots + a_k/(2k+1) &= \mu_k. \end{aligned} \right\} \quad (59)$$

If we define the following three matrices,

$$A(k) = \begin{bmatrix} 1, & 1/2, & 1/3, & \dots, & 1/(k+1) \\ 1/2, & 1/3, & \dots, & & 1/(k+2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/(k+1), & \dots, & & & 1/(2k+1) \end{bmatrix}, \quad (60)$$

$$a(k) = (a_0, a_1, a_2, \dots, a_k), \quad (61)$$

$$\mu(k) = (\mu_0, \mu_1, \mu_2, \dots, \mu_k), \quad (62)$$

the above set of linear equations (58) becomes the matrix equation

$$a(k) A(k) = \mu(k). \quad (63)$$

Now suppose $A^{-1}(k)$ is the inverse of $A(k)$. Then,

$$a(k) = \mu(k) \cdot A^{-1}(k). \quad (64)$$

Here we must notice that $(a(k))_i = a_{i-1}$ and $(\mu(k))_i = \mu_{i-1}$. Our next task is to find $A^{-1}(k)$ and it can be obtained in a general form. Denoting the (i, j) element of $A^{-1}(k)$ as a_{ij}^* , then

$$\begin{aligned} a_{ij}^*(k) &= a_{ji}^*(k) = \frac{(-1)^{i+j}}{[(i+j)-1]! \{(i-1)!\}^2 \{(j-1)!\}^2 (k-i+1)!(k-j+1)!}, \\ &\quad (i=1, 2, \dots, k+1), \quad (j=1, 2, \dots, k+1), \quad (k=0, 1, 2, \dots). \end{aligned} \quad (65)$$

Therefore from (64),

$$\begin{aligned} (a(k))_i &= \sum_{j=1}^{k+1} (\mu(k))_j a_{ji}^*(k) \\ &= \frac{(i+k)!}{\{(i-1)!\}^2 (k-i+1)!} \sum_{j=1}^{k+1} \frac{(-1)^{i+j} (j+k)! (\mu(k))_j}{[(i+j)-1]! \{(j-1)!\}^2 (k-j+1)!}. \end{aligned} \quad (66)$$

Accordingly we obtain,

$$a_i = \sum_{j=0}^k \frac{(-1)^{i+j}}{(i!)^2 (j!)^2} \frac{1}{(i+j+1)} \frac{(i+k+1)! (j+k+1)!}{(k-i)! (k-j)!} \mu_j. \quad (67)$$

Comparing this result with (54), one sees that these two methods, described above, give exactly the same result—the same approximation. This is, of course, the direct consequence from the mathematical character of Legendre polynomials.

c. Relation between $D(\eta)$ and $g(\nu)$.

Once we obtain $D(\eta)$, using the results of preceding sub-sections, it is easy to derive the frequency distribution $g(\nu)$ through the change of variable $\eta \rightarrow \nu$. Now we see

$$D(\eta) d\eta = 2\nu D(\nu^2) d\nu = g(\nu) d\nu, \quad (68)$$

then

$$g(\nu) = 2\nu D(\nu^2).$$

More explicitly, from the definition of μ^i and the formula (49), it is written as

$$(\nu_L/m) g(\nu) = \sum_{n=0}^k 2a_n (\nu/\nu_L)^{2n+1}, \quad (70)$$

where $\{a_n\}$ are given as numerical coefficients and m is the order of Born-Karman matrix which is equal to the total number of vibrational freedoms of the system.

d. Results and discussions.

To examine the nature of our approximation, numerical calculations were performed for the square lattice for which Montroll⁽²⁾ obtained the exact distribution function.

The moments for the case $\tau=1/3$ are

$$\begin{aligned} \int_0^1 g(f) df &= 1, & \mu_{2n} &= \int_0^1 f^{2n} g(f) df, \\ \mu_0 &= 1, & \mu_2 &= 1/2, & \mu_4 &= 23/72, & \mu_6 &= 11/48, \\ \mu_8 &= 1823/10368, & \mu_{10} &= 86875/746496. \end{aligned}$$

Using these moments, we obtain the following results;

$k=2$:

$$g(f) = 7/6 \cdot f + 5f^3 - 5f^5, \quad (71)$$

$$g_M(f) = 15/512 + 1/512 \cdot (2730f^9 - 2065f^4), \quad (72)$$

$k=5$

$$g(f) = 125/108 \cdot f + 35/48 \cdot f^3 + 1715/36 \cdot f^5 - 40145/216 \cdot f^7 + 2240/9 \cdot f^9 - 2695/24 \cdot f^{11}, \quad (73)$$

$k=6$:

$$g(f) = 65471/31104 \cdot f - 202517/5184 \cdot f^3 + 1154965/2592 \cdot f^5 - 143990/81 \cdot f^7 \\ + 5590097/1728 \cdot f^9 - 2366287/864 \cdot f^{11} + 2269267/2592 \cdot f^{13}. \quad (74)$$

Here $g(f)$ denotes our result obtained from (70), and $g_M(f)$ the result of Montroll's even power series approximation. In Fig. 1 we show their curves in comparison with the exact one. The results will be self-explanatory. We recognize that the moment method is essentially an overall approximation and is not suited for a local approximation.

Now, let us return to the discussions for high and intermediate temperature in section 2. If the temperature is high enough, only the first few moments will be necessary and the calculations of them are not so tedious. But, if the temperature is such that the convergence of the series (3) is not so good, we encounter some difficulties. Direct calculation of higher moments from the Born-Karman matrix requires ever increasing labor. In this connection, it is interesting to try to use our formula, for example, (71) in purpose of easy calculation of higher moments:

$$\mu_{2n} \cong \int_0^1 f^{2n} \left[\frac{7}{6}f + 5(f^3 - f^5) \right] df = \frac{7}{12} \frac{1}{(n+1)} + \frac{5}{2} \frac{1}{(n+2)(n+3)}.$$

Using this formula, we obtain the approximate values of higher moment:

$$\mu_0 = (11/48); \quad \mu_2 = (37/210) \quad \mu_4 = (143/1008) \dots$$

For intermediate temperature, it is desirable to use the full expression for $g^r(f)$, not having recourse to expansion formulas.

§ 5. On the peaks of a frequency spectrum

Our conjectures concerning the number and the position of the peaks of a frequency spectrum are based on the examination of the topological nature of the equi-frequency surfaces in Brillouin zone.

Let us consider (5), for example, in a sectional plane $(\varphi_1 - \varphi_3)$:

$$\begin{vmatrix} (1-2\tau)(1-c_1) + \tau(2-c_1-c_1c_2) - 2f^2 & \tau s_1 s_2 & 0 \\ \tau s_1 s_2 & (1-2\tau)(1-c_2) + \tau(2-c_2-c_1c_2) - 2f^2 & 0 \\ 0 & 0 & \tau(2-c_1-c_2) - 2f^2 \end{vmatrix} = 0,$$

namely,

$$\begin{vmatrix} (1-\tau)(1-c_1) + \tau(1-c_1c_2) - 2f^2 & \tau s_1 s_2 \\ \tau s_1 s_2 & (1-\tau)(1-c_2) + \tau(1-c_1c_2) - 2f^2 \end{vmatrix} = 0, \\ \tau(2-c_1-c_2) - 2f^2 = 0.$$

It is to be noted that the above two-two determinant has just the same form as the determinant of the square lattice extensively examined by Montroll³⁾.

Accordingly, we can make full use of the results due to Montroll for the planes $(\varphi_1 - \varphi_2)$, $(\varphi_2 - \varphi_3)$, $(\varphi_3 - \varphi_1)$. Furthermore, if $\tau = 1/3$, we can factorize easily eq. (5)

also in the planes $\varphi_1 = \pm\pi$, $\varphi_2 = \pm\pi$, $\varphi_3 = \pm\pi$, that is, the surface of the first Brillouin zone. In addition to these materials, we can solve eq. (5) along several directions. We do not describe the details of the analysis of the topological nature of the equi-frequency surfaces, because a simple cubic lattice is devoid of direct physical significance. The results have already been reported in another place⁹⁾.

§ 6. Applications to a face-centered cubic lattice

In order to compare our results with the Leighton's directly, we adopt the same secular determinant for a monatomic, face-centered cubic lattice:

$$\begin{vmatrix} 2 + 2(\gamma/u) \sin^2 x & \sin x \sin y & \sin x \sin z \\ -\cos x (\cos y + \cos z) - \lambda^2 & & \\ \sin x \sin y & 2 + 2(\gamma/u) \sin^2 y & \sin y \sin z \\ & -\cos y (\cos z + \cos x) - \lambda^2 & \\ \sin x \sin z & \sin y \sin z & 2 + 2(\gamma/u) \sin^2 z \\ & & -\cos z (\cos x + \cos y) - \lambda^2 \end{vmatrix} = 0. \quad (75)$$

The quantities appearing in this determinant have the same meanings as in the Leighton's paper⁵⁾.

a. The method of Montroll

The above determinant is invariant under the transformation, $x \rightarrow \pi - x$, $y \rightarrow \pi - y$, $z \rightarrow \pi - z$. This situation is favorable to the calculation of the moments necessary to the method of Montroll. Thus, in the calculations of the moments using the determinant, we can take the interval of variable x, y, z as $-\pi \leq x, y, z \leq \pi$. This corresponds to count doubly the lattice points of the first Brillouin zone of the face-centered cubic lattice. We need further to know the maximum value of λ as a function of $x = (\gamma/u)$. We could not get a definite conclusion on this point. However, it seems almost certain that, when $x \leq 0$, the maximum value of λ , λ_m , will be equal to 2, and when $x \geq 0$, the point $x = y = z = \pi/2$ will give the value $\lambda_m = (4 + 2x)^{1/2}$. We confine ourselves to the case of $x \leq 0$ in the following discussion of the method of Montroll. Actually, the Leighton's considerations are also confined to this case. The first five moments are as follows:

$$\mu_{2n} = \lim_{(N_1, N_2, N_3 \rightarrow 0)} \frac{\text{Trace [Born-Karman matrix defined by (75)]}}{3N_1 \cdot N_2 \cdot N_3},$$

$$\mu_0 = 1, \quad \mu_2 = 2 + 2x, \quad \mu_4 = 5 + 4x + (3/2)x^2,$$

$$\mu_6 = (57/4) + 15x + 9x^2 + (5/2)x^3,$$

$$\mu_8 = (703/16) + (115/2)x + (179/4)x^2 + 20x^3 + (35/8)x^4.$$

Using these moments, we apply to the face-centered cubic lattice the existing two method, namely the even power series approximation with constant term due to Montroll

and the odd power series approximation explained in the section 4 of this paper. The results are as follows :

$$x=0.0 :$$

$$D_E(\lambda) = -0.09011 + 1.03990\lambda^2 - 0.29395\lambda^4 - 0.00543\lambda^6 + 0.00681\lambda^8,$$

$$D_0(\lambda) = -0.23315\lambda + 2.02515\lambda^3 - 1.45798\lambda^5 + 0.40749\lambda^7 - 0.04025\lambda^9.$$

$$x=-0.1 :$$

$$D_E(\lambda) = -0.11570 + 1.57028\lambda^2 - 1.11805\lambda^4 + 0.36347\lambda^6 - 0.04436\lambda^8,$$

$$D_0(\lambda) = -0.21538\lambda + 2.45972\lambda^3 - 2.11004\lambda^5 + 0.69267\lambda^7 - 0.07843\lambda^9.$$

These results are not inconsistent with the Leighton's as a whole. Compare Figs. 2,3 of this paper with Fig. 1 of the Leighton's paper. Here we can clearly observe the inadequacy of the moment method for lower frequency region. However, this weak point of the moment method may be complemented by the method of Houston which will be discussed in the following.

b. The method of Houston

As explained in the last part of the section 3, if we want to get only the Taylor series expansion at the origin $\lambda=0$, we can obtain the values of the necessary coefficient $\{(d^n r/d\lambda^n)_{\lambda=0}\}$ directly from the formulas corresponding to (28). In the case of our face-centered cubic lattice, the formulas corresponding to (28) assume the following simple form ;

$$\lambda^2 = a^2 \sin^2(r/b) - ca^2 \sin^4(r/b). \quad (76)$$

Then the Taylor series expansion of $\chi_i^2(\theta_s, \varphi_s, \lambda) \cdot d\chi_i(\theta_s, \varphi_s, \lambda)/d\lambda$ is given as follows :

$$\chi_i^2 \cdot d\chi_i/d\lambda = (b/a)^3 \{\lambda^2 + 5/6a^2 \cdot \lambda^4 + 7/360a^2 \cdot (37 + 150c + 405c^2)\lambda^6 + \dots\}. \quad (77)$$

Here, the constants a, b, c are, of course, the functions of x, θ_s and φ_s . Using (27) and (77), we obtain the formula for low frequency region in the following form.

$$D_H(\lambda) = A(x)\lambda^2 + B(x)\lambda^4 + C(x)\lambda^6 + \dots, \quad (78)$$

where $A(x), B(x)$ and $C(x)$ are certain functions of x , $-(1/4) < x$, and their expressions are given in the appendix.

In order to compare the results from (78) with the Leighton's one, we took the values of x , $x=0.0$ and $x=-0.1$. Remembering that the $G(\lambda)$ in the Leighton's paper corresponds to $3D_H(\lambda)$, we obtained the following results :

$$x=0.0 :$$

$$G(\lambda) = 0.99\lambda^2 + 0.35\lambda^4 + 0.42\lambda^6 + \dots,$$

$$3D_H(\lambda) = 1.015\lambda^2 + 0.488\lambda^4 + 0.302\lambda^6 + \dots,$$

$$x=-0.1 :$$

$$G(\lambda) = 1.32\lambda^2 + 0.40\lambda^4 + 0.36\lambda^6 + \dots,$$

$$3D_R(\lambda) = 1.567\lambda^2 + 1.351\lambda^4 + 1.727\lambda^6 + \dots$$

The curves of these formulas are contained in Figs. 2 and 3. We see that, in the case of $x = -0.1$, the discrepancy between the two results is rather remarkable, contrary to the case of $x = 0$. Which is better is, up to now, an open question, because there is no exact result to be referred to.

§ 7. Conclusion

Although we discussed rather many materials in this paper, our underlying aim is to bring the approximations based on the Born-Karman model to the domain of practical use against the prevalence of the Debye approximation. For example, the results presented in the section 6 may be used for some practical purposes.

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Appendix

The coefficients of the formula (78) are given here.

$$A(x) = \frac{1}{105\pi^2} \left\{ 72\sqrt{2} + 10(1+2x)^{-\frac{3}{2}} + 128(5+4x)^{-\frac{3}{2}} + 128(1+4x)^{-\frac{3}{2}} \right. \\ \left. + 54\sqrt{3}(1+2x)^{-\frac{3}{2}} + 27\sqrt{3}(4+2x)^{-\frac{3}{2}} \right\},$$

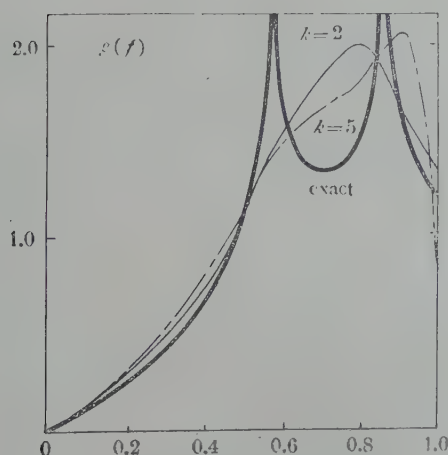


Fig. 1.

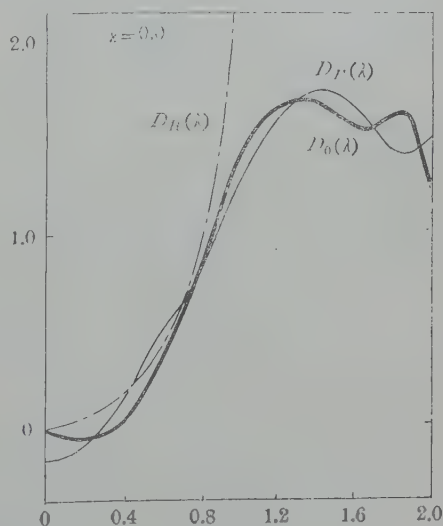


Fig. 2.

$$B(x) = \frac{1}{105\pi^2} \left\{ (70\sqrt{2}/3) + (25/12)(1+2x)^{-\frac{5}{2}} + (160/3)(5+4x)^{-\frac{5}{2}} \right. \\ \left. + (160/3)(1+4x)^{-\frac{5}{2}} + 45\sqrt{3}(1+2x)^{-\frac{5}{2}}(45\sqrt{3}/2)(4+2x)^{-\frac{5}{2}} \right\},$$

$$C(x) = \frac{1}{105\pi^2} \left\{ (259\sqrt{2}/30) + (7\sqrt{2}/576)(1+2x)^{-\frac{7}{2}}(37+150c_1+405c_1^2) \right. \\ + (28/45)(5+4x)^{-\frac{7}{2}}(37+150c_2+405c_2^2) \\ + (28/45)(1+4x)^{-\frac{7}{2}}(37+150c_3+405c_3^2) \\ \left. + (777\sqrt{3}/20)(1+2x)^{-\frac{7}{2}} + (777\sqrt{3}/40)(4+2x)^{-\frac{7}{2}} \right\},$$

where

$$c_1 = 2x/1+2x, \quad c_2 = (4+4x)/(5+4x), \quad c_3 = 4x/1+4x.$$

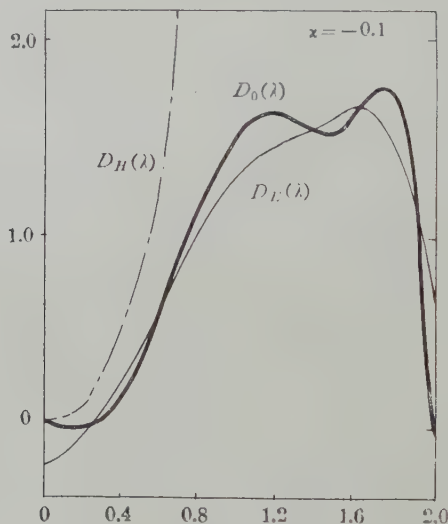


Fig. 3.

The x -dependence of the coefficients of the odd power series approximation formula $D_0(\lambda)$ for the face-centered cubic lattice are also given here.

$$D_0(\lambda) = b_0\lambda + b_1\lambda^3 + b_2\lambda^5 + b_3\lambda^7 + b_4\lambda^9,$$

$$b_0 = -\frac{955}{4096} + \frac{225}{512}x + \frac{5985}{1024}x^2 - \frac{175}{64}x^3 + \frac{11025}{2048}x^4,$$

$$b_1 = \frac{8295}{4096} - \frac{3525}{512}x - \frac{24885}{1024}x^2 + \frac{525}{64}x^3 - \frac{55125}{2048}x^4,$$

$$b_2 = -\frac{47775}{32768} + \frac{37485}{4096}x + \frac{208845}{8192}x^2 - \frac{2625}{512}x^3 + \frac{496125}{16384}x^4,$$

$$b_3 = \frac{26705}{65536} - \frac{31395}{8192}x - \frac{159075}{15384}x^2 + \frac{875}{1024}x^3 - \frac{385875}{32768}x^4,$$

$$b_4 = -\frac{21105}{524288} + \frac{33075}{65536}x + \frac{159075}{131072}x^2 + 0 + \frac{385875}{262144}x^4.$$

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Spin-spin and Spin-other-orbit Interactions

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We rewrite spin-spin and spin-other-orbit interactions in terms of the products of tensor operators and apply the method of Racah. Our procedure has some advantages from formal and practical point of view. For example, the conjugation process can be carried out only after the decomposition of spin-other-orbit operator into this form. It was confirmed that the magnetic interaction parameters are proportional to Z^3 , Z being the atomic number.

§ 1. Introduction

The spin-other-orbit and spin-spin interactions were introduced as the relativistic effects for two electron system together with some other ones. Only these two interactions, besides the individual spin-orbit interactions, contribute to the multiplet splittings and, thus, the triplet terms of the helium atom were successfully explained by taking into account these interactions¹⁾. It has also been suggested that these interactions will have an appreciable effect on the intervals of multiplets in the spectra of heavy elements.²⁾ In the recent studies of the forbidden lines, we have calculated the matrix elements of these interactions in p^n configurations for $n=2, 3$ and 4, and have been able to show that the deviations of the triplet intervals of the p^2 and p^1 configurations from Landé's rule can be accounted for and qualitative agreements between theoretical and experimental results for two doublets of p^3 configuration can be obtained.

We rewrite the spin-spin and spin-other-orbit interactions in terms of the products of tensor operators and apply the method of Racah.⁴⁾ After this procedure, matrix elements of these interactions for any l^n configurations can be obtained by means of the double-barred elements of $U^{(k)}$ and $V^{(lk)}$ which are defined in Racah's formulation. Such procedure has some advantages from formal and practical point of view, as will be shown in the following. It can be shown explicitly that for spin-spin interaction the non-vanishing elements exist only between states with the same seniority number and the values of double-barred elements are completely determined if v , S and L are given, v being the seniority number. The matrix elements for spin-other-orbit interactions are rather complicated and consist of the following three terms: (i) term of the same character as the individual spin-orbit interaction, including constant term which appears only in the coefficient of lowest order integral, (ii) term diagonal with respect to the seniority number and completely determined by v , S and L , (iii) term not diagonal with respect to v , S and L and not included in (i). The conjugation process ($l^n \rightarrow l^{2(2l+1)-n}$) can be carried out without

difficulty after this decomposition, *i.e.* it is, then, easily seen that the term (i) changes its sign and appropriate prescription must be done with respect to the constant term, while terms (ii) and (iii) are unaltered. When the matrix elements of spin-other-orbit interaction for d^n configurations ($n=2, 3$, and 4) are calculated (Table I, II and III, respectively), the elements for d^6 , d^7 and d^8 configurations can be obtained immediately by the above conjugation process.

The triplet intervals of $2p^2$ and $2p^1$ isoelectronic sequences are analysed by our formula and the numerical values for the usual spin-orbit interaction parameter ζ_p and magnetic interaction parameter M_0 are determined. M_0 is shown to be proportional to Z^3 , as is expected. M_0 for $3p^n$ configurations determined from the observed multiplet intervals for several elements become much smaller than those for $2p^n$ configurations. This holds also with respect to the magnetic interaction parameters M_0 and M_2 for $3d^n$ configurations.

§ 2. Spin-spin interaction

The spin-spin interaction between two electrons 1 and 2 is given by as^{*})

$$H_{ss} = a^2 \{ (\mathbf{s}_1 \cdot \mathbf{s}_2) - 3 (\mathbf{r}_{12} \cdot \mathbf{s}_1) (\mathbf{r}_{12} \cdot \mathbf{s}_2) / r_{12}^3 \} / r_{12}^3, \quad (1)$$

where $\mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ and a is the fine structure constant. It becomes by simple calculation

$$H_{ss} = a^2 (\mathbf{s}_1 \cdot \nabla_1) (\mathbf{s}_2 \cdot \nabla_2) \cdot 1/r_{12}, \quad (2)$$

where ∇_1 and ∇_2 operate only upon the corresponding coordinate in $1/r_{12}$. $1/r_{12}$ can be expanded into a series of scalar products of spherical harmonics, and it can easily be proved that the following formula is valid

$$\nabla_\sigma C_q^{(k)} = \sum_{KQ} (k100 | k1K0) (k1q\sigma | k1KQ) C_Q^{(K)} D^{(K)}. \quad (3)$$

Here $C_q^{(k)} = [4\pi / (2k+1)]^{1/2} \Theta(kq) \Phi(q)$ is a spherical harmonic not normalized, $(j_1 j_2 m_1 m_2 | j_1 j_2 j m)$ is the well-known Wigner coefficient, and $D^{(K)}$ are the operators upon the radial function only defined by

$$D^{(k-1)} = \partial / \partial r + (k+1)/r \quad \text{for} \quad K=k-1, \quad (4a)$$

and

$$D^{(k+1)} = \partial / \partial r - k/r \quad \text{for} \quad K=k+1. \quad (4b)$$

Thus the interaction (1) can be written as

$$H_{ss} = a^2 \sum_{kq\sigma\sigma'} \sum_{KQK'Q'} (-1)^{\sigma+\sigma'} q S_{1\sigma} S_{2\sigma'} (k100 | k1K0) (k100 | k1K'0) \\ \cdot (k1q-\sigma | k1KQ) (k1-q-\sigma' | k1K'Q') C_Q^{(K)} C_{Q'}^{(K')} (D_1^{(K)} D_2^{(K')}, a_k), \quad (5)$$

where

$$a_k = r^k / r^{k+1}. \quad (5')$$

Furthermore, making use of the relation for the Wigner coefficients⁵⁾

* We use the notations of Condon and Shortley (ref. 1) and Racah (ref. 4).

$$(l_1 l_2 m_1 m_2 | l_1 l_2 l_3 m_3) = (-1)^{l_2 + m_2} [(2l_3 + 1) / (2l_1 + 1)]^{1/2} (l_3 l_2 - m_3 m_2 | l_3 l_2 l_1 - m_1) \quad (6)$$

and defining an irreducible tensor operator of rank k $t_q^{(1K; k)}$ by

$$t_q^{(1K; k)} = (2K + 1) (1 K 0 0 | 1 K k 0) \sum_{\sigma Q} (1 K \sigma Q | 1 K k q) S_{\sigma} C_Q^{(K)}, \quad (7)$$

we get

$$H_{ss} = a^2 \sum_{kK K'} (t_1^{(1K; k)} \cdot t_2^{(1K'; k)}) (D_1^{(K)} D_2^{(K')} a_k) / (2k + 1)^2. \quad (8)$$

This equation has the desired form, for it consists of the scalar products of tensor operators which operate only upon the spin and angular parts of the individual electron wave functions, while the remaining factor depends only on their radial parts.

From (4) and (5'), we obtain for this last factor that it vanishes if $K = K' = k \pm 1$ and

$$(D_1^{(k-1)} D_2^{(k+1)} a_k) / (2k + 1)^2 = -r_1^{k-1} / r_2^{k+2} \quad (r_1 \leq r_2), \quad (9a)$$

and

$$(D_1^{(k+1)} D_2^{(k-1)} a_k) / (2k + 1)^2 = -r_2^{k-1} / r_1^{k+2} \quad (r_2 \leq r_1), \quad (9b)$$

and otherwise they vanish. For l^n configuration, which will be considered exclusively in the following, the relation

$$\begin{aligned} & a^2 \int \int_{(r_1 \leq r_2)} r_1^{k-1} / r_2^{k+2} [R_l(r_1) R_l(r_2)]^2 dr_1 dr_2 / 4 \\ & = a^2 \int \int_{(r_1 \geq r_2)} r_2^{k-1} / r_1^{k+2} [R_l(r_1) R_l(r_2)]^2 dr_1 dr_2 / 4 \quad (= M^{k-1}) \end{aligned} \quad (10)$$

is obviously valid. We denote these by M^{k-1} following Marvin.⁽⁶⁾ There are as many different integrals M^k as the numbers of the allowed value of k .

Thus, the interaction H_{ss} can be expressed for l^n configuration in terms of the equivalent operators as:

$$\begin{aligned} H_{ss} &= - \sum_k \sum_{i < j} \{ (t_i^{(1k-1; k)} \cdot t_j^{(1k+1; k)}) + (t_i^{(1k+1; k)} \cdot t_j^{(1k-1; k)}) \} M^{k-1} \\ &= - 2 \sum_k \sum_{i < j} (t_i^{(1k-1; k)} \cdot t_j^{(1k+1; k)}) M^{k-1}. \end{aligned} \quad (11)$$

Since this form is not convenient for the calculation of the matrix elements, we rewrite this in terms of operators symmetric with respect to n electrons $\mathbf{T}^{(1K; k)} = \sum_i^n t_i^{(1K; k)}$:

$$H_{ss} = - \sum_k \{ (\mathbf{T}^{(1k-1; k)} \cdot \mathbf{T}^{(1k+1; k)}) - \sum_i (t_i^{(1k-1; k)} \cdot t_i^{(1k+1; k)}) \} M^{k-1}. \quad (12)$$

It can easily be shown that the second term in the brace vanishes identically, because the tensor operator of the rank two composed of single spin operator can not exist. Hence

$$H_{ss} = - \sum_k (\mathbf{T}^{(1k-1; k)} \cdot \mathbf{T}^{(1k+1; k)}) M^{k-1}. \quad (13)$$

We shall calculate the matrix elements for this form of H_{ss} , since then, they are obtained merely by a direct application of the Racah's method of tensor operator. For the matrix elements of spin-spin interaction for l^n configuration, we have

$$(l^n aSLJM | H_{ss} | l^n a' S' L' JM) = (-1)^{S+L'+J} W(SLS'; J2) (l^n aSL || H_{ss} || l^n a' S' L'), \quad (14)$$

and

$$\begin{aligned} (l^n u S L \| H_{ss} \| l^n u' S' L') = \sum_k f_k M^{k-1} \sum_{\alpha'' S'' L''} (l^n u S L \| V^{(k-1)} \| l^n u'' S'' L'') \\ \cdot (l^n u'' S'' L'' \| V^{(k+1)} \| l^n u' S' L') W(S1 S'1; S''2) W(Lk-1 L'k+1; L''2), \end{aligned} \quad (14')$$

where

$$f_k = (-1)^k 4 [5k(k+1)(2k-1)(2k+1)(2k+3)]^{1/2} (l \| C^{(k-1)} \| l) (l \| C^{(k+1)} \| l). \quad (14'')$$

f_k does not vanish only when k is an odd integer and not be larger than $2l-1$ from the last two factors of (14''). Thus the integrals necessary for the spin-spin interaction in the l^n configurations are $M^0, M^2, \dots, M^{2l-2}$, while the coefficients of them in the matrix elements are given by (14), (14') and (14'').

If we assign each term of the configuration the "seniority number" v introduced by Racah, we can deduce some properties of double-barred elements in eq. (14'). Since k are odd integers, these elements are diagonal with respect to v and are completely determined by assigning a set of the values v, S and L irrespective of n . This rule has been given independently by Talmi⁷ for tensor force with more general radial dependence. However, our derivation for the spin-spin interaction between electrons seems more straightforward than his. Moreover, it follows that the elements are unaltered by conjugation process $\mathcal{L} \rightarrow \mathcal{R}$ (i.e. $l^n \rightarrow l^{2(2l+1)-n}$) and for the self-corresponding configurations l^{2l+1} they vanish between terms belonging to different classes.

For $k=1$, the calculations are very simple and the coefficients of M^0 in the double-barred elements are given by

$$4 [30l(l+1)(2l+1)/(2l-1)(2l+3)]^{1/2} [S(S+1)(2S+1)]^{1/2} \cdot W(S1 S'1; S1) (l^n u S L \| V^{(12)} \| l^n u' S' L').$$

The tables of the elements of $V^{(12)}$ were given by Racah⁴ for p^n and d^n configurations. For p^n configurations, where we need only the term with $k=1$, the non-vanishing double-barred elements are given by

$$(p^2 \text{ } ^3P \| H_{ss} \| p^2 \text{ } ^3P) = (p^3 \text{ } ^4S \| H_{ss} \| p^3 \text{ } ^2D) = -60 M_0,$$

where

$$M_0 = M^0/5, \quad M^0 = a^2 \int_{(r_1 \geq r_2)} 1/r_1^3 [R_p(r_1) R_p(r_2)]^2 dr_1 dr_2 / 4.$$

These agree with our previous calculation³) and, for p^2 configuration, with those of Marvin.⁶⁾

For d^n configurations, the calculations are not so simple as for p^n , since we must take into account the coefficients of M^2 as well as M^0 . The results are

$$\begin{aligned} (d^n v S L \| H_{ss} \| d^n v S' L') = 4 [3/7 \cdot S(S+1)(2S+1)]^{1/2} (d^n v S L \| 70 V^{(12)} \| d^n v S' L') \\ \cdot W(S1 S'1; S2) M_0 + 120 (21)^{1/2} \sum_{S'' L''} (d^n v S L \| V^{(12)} \| d^n v S'' L'') \\ \cdot (d^n v S'' L'' \| V^{(14)} \| d^n v S' L') W(S1 S'1; S''2) W(L2 L'4; L''2) M_2, \end{aligned}$$

* Dr. R. E. Trees has kindly informed us the proof that the matrix elements of spin-spin interaction are diagonal with respect to seniority numbers was also given by F. Innes (unpublished) independently.

where $M_0 = M^0/7$, $M_2 = M^2/49$. The tables of these matrix elements have been already given by Trees⁵⁾ in more direct way. An advantage of our method is that in order to obtain the matrix elements for l^n configurations we do not need the elements of the interaction for l^{n-1} configuration, but the elements $V^{(12)}$, ..., $V^{(12l)}$ which were necessary even for the calculation of electro-static interaction in l^n configuration.

The spin-spin interactions between an electron in incomplete shell and any inner closed shell are expected to vanish from the transformation property of the interaction operators with respect to rotation in spin space. A formal proof for this has recently been given by Elliott.⁹⁾

§ 3. Spin-other-orbit interaction

Spin-other-orbit interaction between the electrons 1 and 2 is given by

$$H_{so}' = -\omega_s^2/2r_{12}^3 \cdot \{(\mathbf{k}_{12} + 2\mathbf{k}_{21} \cdot \mathbf{s}_1) + (\mathbf{k}_{21} + 2\mathbf{k}_{12} \cdot \mathbf{s}_2)\}, \quad (15)$$

where \mathbf{k}_{12} is the relative angular momentum $\mathbf{k}_{12} = 1/i [\mathbf{r}_{12}, \mathbf{p}_1]$, and, as in the case of spin-spin interaction, we have $\mathbf{k}_{12}/r_{12}^3 = 1/i \cdot [\mathbf{r}_{12}/r_{12}^3, \mathbf{p}_1] = i [(\mathbf{r}_1/r_{12}), \mathbf{p}_1]$ (Square brackets denote vector products). For *equivalent* two electrons, calculations similar to the case of spin-spin interaction give the component of the vector \mathbf{k}_{12}/r_{12}^3 as

$$k_{12,m}/r_{12}^3 = \sum_{kKqQ} U_q^{(K;k)} C_2^{(K)}(kKqQ|kK1m) M^{k-1}, \quad (16)$$

where $U_q^{(K;k)}$ is a component of tensor of rank k with

$$\begin{aligned} (l||U^{(k-1;k)}||l) &= (-1)^{(k-1)/2} (2l+1) [(k+1)(2k-1)(2k+1)/3]^{1/2} \\ &\quad \cdot [(k-1)!(k+1)!(2l-k)!/(2l+k+1)!]^{1/2} \\ &\quad \cdot \left(l + \frac{k+1}{2}\right)! \left/ \left[\frac{k-1}{2}! \frac{k+1}{2}! \left(l - \frac{k+1}{2}\right)! \right] \right., \end{aligned} \quad (17a)$$

$$(l||U^{(k+1;k)}||l) = -[l(2l+3)/(k+1)(2k-1)]^{1/2} (l||U^{(k-1;k)}||l), \quad (17b)$$

and $U_q^{(k;k)}$ vanishes. M^{k-1} are the same integrals as those in the spin-spin interaction defined by (10). In the summation over k and K in (16), k are odd integers ($1 \leq k \leq 2l-1$) and $K = k \pm 1$ even, because otherwise $(l||C^{(K)}||l)$ and $(l||U^{(K;k)}||l)$ vanish.

Spin-other-orbit interaction operator for l^n configuration is obtained in a form similar to (11), by inserting (16) into (15) and taking into account the relations analogous to (6). This form is, however, more complicated than (11), since the summation over K enters. The procedure which derives (12) from (11) gives a convenient formula for obtaining the matrix elements. The terms corresponding to the second term in the brace of equation (12), however, do not vanish in this case and they behave like as the individual spin-orbit interaction. The double-barred matrix elements for l^n configuration of the individual spin-orbit interaction including the part of the spin-other-orbit interaction appeared just above is given by

$$\begin{aligned} (l^n a S L || H_{so}' || l^n a' S' L') &= \{ [l(l+1)(2l+1)]^{1/2} \zeta + 6(3)^{1/2} \sum_{kK} (k||U^{(K;k)}||l) \\ &\quad \cdot (l||C^{(K)}||l) W(lk1K;11) M^{k-1} \} (l^n a S L || V^{(11)} || l^n a' S' L'), \end{aligned} \quad (18)$$

where $\zeta = \alpha^2 Z/2 \cdot \int_0^\infty 1/r^3 [R_l(r)]^2 dr$ is the usual spin-orbit parameter. The matrix elements for the remaining part of spin-other-orbit interaction are also obtained as follows:

$$\begin{aligned} (\ell^n u SL \| H_{so}^{II} \| \ell^n u' S' L') = & -2(3)^{1/2} \sum_{kK} (\ell \| U^{(K); k} \| l) (\ell \| C^{(K)} \| l) \\ & \cdot \{ \sum_{\alpha' l' l'} (\ell^n u SL \| U^{(K)} \| \ell^n u' S' L') (\ell^n u' S' L' \| V^{(1k)} \| \ell^n u' S' L') \\ & \cdot W(LK L' k; L'' 1) + 2 \sum_{\alpha' l' l'} (\ell^n u SL \| U^{(k)} \| \ell^n u' S' L') \\ & \cdot (\ell^n u' S' L' \| V^{(1K)} \| \ell^n u' S' L') W(Lk L' K; L'' 1) \} M^{k-1}. \end{aligned} \quad (19)$$

The first sum in the brace of (19) with $k=1$ and $K=0$ is distinguished from the other sums, since this contains the constant matrix $U^{(0)}$, and it can easily be shown to be

$$-2n [\ell(\ell+1)(2\ell+1)]^{1/2} (\ell^n u SL \| V^{(11)} \| \ell^n u' S' L').$$

This term also has a property similar to that of the individual spin-orbit interaction, since it contains the elements of $V^{(11)}$ as a factor. Therefore, we can include it into eq. (18) and omit it from (19). Then eq. (18) becomes

$$(\ell^n u SL \| H_{so}^{II} \| \ell^n u' S' L') = \zeta' (\ell^n u SL \| [\ell(\ell+1)(2\ell+1)]^{1/2} V^{(11)} \| \ell^n u' S' L'), \quad (20)$$

where

$$\begin{aligned} \zeta' = \zeta - 2n M^0 + 6 [3/\ell(\ell+1)(2\ell+1)]^{1/2} \sum_{kK} (\ell \| C^{(K)} \| l) (\ell \| U^{(K); k} \| l) \\ \cdot W(Kl k l; l 1) M^{k-1} \end{aligned} \quad (21)$$

is a modified spin-orbit interaction parameter and it should be noticed that ζ' contains a term proportional to the number of electrons.

The matrix elements for the individual spin-orbit interaction H_{so}^I change their signs by conjugation, as are readily seen from the presence of the factor $(\ell^n u SL \| V^{(11)} \| \ell^n u' S' L')$. Evidently, the number of electrons, n in (21) should be replaced by $2(2\ell+1)-n$. The elements between terms belonging to the same class vanish for the self-corresponding $\ell^{2\ell+1}$ configuration. The matrix elements of mutual spin-orbit interaction, from which the individual spin-orbit interaction terms are subtracted, consist of two groups of terms of different character, since the second sum in the brace of (19) is diagonal to the seniority number v and is fully determined by assigning the values of $u v SL$, independent of n , what does not hold, however, for the first sum. The elements of H_{so}^{II} are unaltered by conjugation and elements between terms belonging to the different classes vanish for self-corresponding configurations. The simplest of the terms of (19) are the second one in the brace with $k=1$ and $K=0$. This is diagonal with respect to u , S , and L , and contains no summation over any intermediate states:

$$-4 [S(S+1)(2S+1)L(L+1)(2L+1)]^{1/2} M^0 \delta(u, u') \delta(S, S') \delta(L, L').$$

The matrix elements for the spin-orbit interactions are given by

$$\begin{aligned} (\ell^n u SL JM \| H_{so} \| \ell^n u' S' L' JM) = & (-1)^{S+L'-J} W(SLS' L'; J 1) \\ & \cdot (\ell^n u SL \| H_{so} \| \ell^n u' S' L'), \end{aligned} \quad (21)$$

where

$$H_{so} = H_{so}^I + H_{so}^{II}.$$

For p^n configurations, to which terms with $k=1$ and $K=0$ and 2 contribute, the matrix elements are easily obtained. Putting $M_0 = M_0^0/5$, as before, those are obtained from (20) and (19) as follows:

$$(p^n SL \| H_{so}^I \| p^n S' L') = \zeta' (p^n SL \| 6^{1/2} V^{(11)} \| p^n S' L'),$$

where

$$\zeta' = \zeta_p - 5(2n-3)M_0,$$

and

$$(p^n SL \| H_{so}^{II} \| p^n S' L') = (1) + (2) + (3),$$

where

$$(1) \quad -20 [S(S+1)(2S+1)L(L+1)(2L+1)]^{1/2} M_0 \delta(S, S') \delta(L, L'),$$

$$(2) \quad -30 [L(L+1)(2L+1)]^{1/2} (p^n SL \| 2V^{(12)} \| p^n S' L') W(L1L'2; L1) M_0,$$

$$(3) \quad -30 \sum_{L''} (p^n SL \| U^{(22)} \| p^n SL'') (p^n SL'' \| 6^{1/2} V^{(11)} \| p^n S' L') \\ \cdot W(L2L'1; L''1) M_0.$$

These results agree with those of our previous calculations³⁰ for p^n and, for p^2 configuration, also with Marvin.⁶⁾ Owing to the above mentioned prescription for conjugation, the matrix elements for p^4 configuration can be obtained at once from the corresponding elements for p^2 , while, in the previous work, they are obtained by direct calculations using the explicit wave functions for p^4 .

For d^n configurations, the calculations of matrix elements are more involved, since terms with $k=1$ and 3 appear. The matrix elements for H_{so}^I can be obtained by

$$(d^n v SL \| H_{so}^I \| d^n v' S' L') = \zeta' (d^n v SL \| (30)^{1/2} V^{(11)} \| d^n v' S' L'),$$

where

$$\zeta' = \zeta_d - 7(2n-3)M_0 + 42M_2.$$

The formulae for the matrix elements of H_{so}^{II} can be derived by (19), but we omit them here since they are rather lengthy. The results of calculations of the double-barred elements of spin-orbit interactions for d^2 , d^3 and d^4 configurations are shown in Table I, II, and III respectively.* The matrix elements of d^6 , d^7 and d^8 configurations can be obtained by conjugation process. In these tables, seniority numbers equal to n are omitted. It is interesting to note that equations (19) and (20) can be applicable formally even for $n=1$, i.e. the double-barred elements for $d_1^2 D$ which follows from (19) and (20) is $3(5)^{1/2} (\zeta' - 7M_0 - 42M_2)$ which is equal to $3(5)^{1/2} \zeta_d$ as is expected, by the definition of ζ' . Of course, this holds also for the case of single p configuration.

It has been shown by Araki¹⁰⁾ for the configuration with a single s electron outside closed shells and by Elliott⁹⁾ for that with a single electron with any azimuthal quantum number l , that the spin-other-orbit interactions between an electron in the outermost incomplete shell and inner closed shells has property similar to the individual spin-orbit interaction

* There are some misprints in the coefficients of M_2 for the non-diagonal elements of d^2 configuration in Table IV of ref. 6).

Table I. ($d^2vSL||H_{so}||d^2v'S'L'$)

vSL	$v'S'L'$	$(d^2vSL H_{so} d^2v'S'L')$
0^1S	$3P$	$3(2)^{1/2}(\zeta' + 14M_0 - 28M_2)$
$3P$	$3P$	$3(\zeta' - 77M_0 - 1050M_2)$
$3P$	$1D$	$-3/2(14)^{1/2}(\zeta' + 17M_0 - 2M_2)$
$1D$	$3F$	$6(\zeta' + 22M_0 - 117M_2)$
$3F$	$3F$	$3(14)^{1/2}(\zeta' - 47M_0 + 60M_2)$
$3F$	$1G$	$-3(3)^{1/2}(\zeta' + 29M_0 + 2M_2)$

Table II. ($d^3vSL||H_{so}||d^3v'S'L'$)

vSL	$v'S'L'$	$(d^3vSL H_{so} d^3v'S'L')$
$2P$	$2P$	$2\zeta' - 81M_0 + 1560M_2$
$2P$	$4P$	$-2(14)^{1/2}(\zeta' + 9M_0 - 111M_2)$
$2P$	1^2D	$-1/2(42)^{1/2}(\zeta' + 16M_0 + 228M_2)$
$2P$	$2D$	$9/2(2)^{1/2}(\zeta' + 8M_0 - 248M_2)$
$4P$	$4P$	$(10)^{1/2}(\zeta' - 63M_0 + 1050M_2)$
$4P$	1^2D	$-4(3)^{1/2}(\zeta' + 7M_0 - 105M_2)$
$4P$	$2D$	$720(7)^{1/2}M_2$
1^2D	1^2D	$3/2(5)^{1/2}(\zeta' - 42M_0 - 28M_2)$
1^2D	$2D$	$-1/2(105)^{1/2}(\zeta' + 12M_0 + 80M_2)$
1^2D	$2F$	$(42)^{1/2}(\zeta' + 6M_0 - 142M_2)$
1^2D	$4F$	$-(42)^{1/2}(\zeta' + 12M_0 + 80M_2)$
$2D$	$2D$	$1/2(5)^{1/2}(\zeta' + 222M_0 + 1020M_2)$
$2D$	$2F$	$(2)^{1/2}(\zeta' - 42M_0 - 1698M_2)$
$2D$	$4F$	$5(2)^{1/2}(\zeta' + 12M_0 - 228M_2)$
$2F$	$2F$	$-1/2(14)^{1/2}(\zeta' + 267M_0 + 1545M_2)$
$2F$	$4F$	$-(14)^{1/2}(\zeta' + 39M_0 + 399M_2)$
$2F$	$2G$	$-3/2(10)^{1/2}(\zeta' + 7M_0 - 25M_2)$
$4F$	$4F$	$2(35)^{1/2}(\zeta' - 93M_0 - 60M_2)$
$4F$	$2G$	$-3(10)^{1/2}(\zeta' + 19M_0 + 59M_2)$
$2G$	$2G$	$9/10(30)^{1/2}(\zeta' - 93M_0 + 285M_2)$
$2G$	$2H$	$6/5(55)^{1/2}(\zeta' + 22M_0 - 10M_2)$
$2H$	$2H$	$3/5(55)^{1/2}(\zeta' - 163M_0 + 220M_2)$

Table III. ($d^4vSL||H_{so}||d^4v'S'L'$)

vSL	$v'S'L'$	$(d^4vSL H_{so} d^4v'S'L')$
0^1S	2^3P	$(3)^{1/2}(3\zeta' + 14M_0 - 28M_2)$
0^1S	$3P$	$-4(42)^{1/2}(M_0 + 57M_2)$
1^1S	2^3P	$-(7)^{1/2}(\zeta' + 10M_0 + 188M_2)$
1^1S	$3P$	$2(2)^{1/2}(\zeta' - 8M_0 - 853M_2)$
2^3P	2^3P	$\zeta' - 231M_0 - 1694M_2$
2^3P	$3P$	$-2(14)^{1/2}(\zeta' + 6M_0 + 40M_2)$
2^3P	1^1D	$-1/2(14)^{1/2}(\zeta' + 19M_0 + 266M_2)$
2^3P	$1D$	$2(7)^{1/2}(\zeta' + 4M_0 - 34M_2)$
2^3P	$3D$	$12(7)^{1/2}(M_0 + 37M_2)$
2^3P	$5D$	$-4(5)^{1/2}(\zeta' + 7M_0 + 77M_2)$
$3P$	$3P$	$2(\zeta' - 99M_0 - 575M_2)$
$3P$	1^1D	$2(\zeta' - 8M_0 - 478M_2)$
$3P$	$1D$	$1/2(2)^{1/2}(\zeta' - 95M_0 - 3532M_2)$
$3P$	$3D$	$3/2(2)^{1/2}(3\zeta' + 23M_0 + 800M_2)$
$3P$	$5D$	$1/2(70)^{1/2}(\zeta' - 11M_0 - 424M_2)$
2^1D	$3D$	$2(10)^{1/2}(\zeta' + 2M_0 - 108M_2)$
2^1D	2^3F	$2(\zeta' + 14M_0 - 819M_2)$
2^1D	$3F$	$4(\zeta' + 2M_0 - 108M_2)$
$1D$	$3D$	$-(5)^{1/2}(\zeta' - M_0 - 174M_2)$
$1D$	2^3F	$(2)^{1/2}(\zeta' + 14M_0 + 168M_2)$
$1D$	$3F$	$-4(2)^{1/2}(\zeta' + 5M_0 + 18M_2)$
$3D$	$3D$	$-1/2(5)^{1/2}(\zeta' + 395M_0 - 822M_2)$
$3D$	$5D$	$5/2(7)^{1/2}(\zeta' + 11M_0 - 210M_2)$
$3D$	$1F$	$-2(5)^{1/2}(\zeta' + 2M_0 - 603M_2)$
$3D$	2^3F	$(2)^{1/2}(5\zeta' + 34M_0 + 348M_2)$
$3D$	$3F$	$(2)^{1/2}(\zeta' + 86M_0 + 1665M_2)$
$5D$	$5D$	$15/2(\zeta' - 133M_0 + 42M_2)$
$5D$	2^3F	$-(70)^{1/2}(\zeta' + 2M_0 - 108M_2)$
D^5	$3F$	$(70)^{1/2}(\zeta' + 14M_0 + 201M_2)$
1^1F	2^3F	$(35)^{1/2}(\zeta' + 8M_0 - 42M_2)$
$1F$	$3F$	$1/2(35)^{1/2}(\zeta' - 10M_0 + 273M_2)$
$1F$	$3G$	$-9/2(\zeta' - 12M_0 - 561M_2)$
2^3F	2^3F	$(14)^{1/2}(\zeta' - 181M_0 + 156M_2)$
2^3F	$3F$	$-(14)^{1/2}(\zeta' - 4M_0 - 330M_2)$
2^3F	1^1G	$-(3)^{1/2}(\zeta' + 63M_0 + 574M_2)$
2^3F	$1G$	$(33)^{1/2}(\zeta' - 182M_2)$
2^3F	$3G$	$3(10)^{1/2}(\zeta' + 4M_0 - 34M_2)$
3^1F	$3F$	$-1/2(14)^{1/2}(\zeta' + 386M_0 - 2055M_2)$
3^1F	1^1G	$(3)^{1/2}(5\zeta' + 24M_0 - 22M_2)$
$3F$	$1G$	$-1/2(33)^{1/2}(\zeta' + 12M_0 + 487M_2)$
$3F$	$3G$	$-3/2(10)^{1/2}(\zeta' + 16M_0 - 85M_2)$
2^1G	$3G$	$3(\zeta' + 16M_0 + 410M_2)$
2^1G	$3H$	$-(66)^{1/2}(\zeta' + 6M_0 + 40M_2)$
$1G$	$3G$	$3/2(11)^{1/2}(\zeta' + 34M_0 + 710M_2)$
$1G$	$3H$	$2(6)^{1/2}(\zeta' + 24M_0 + 181M_2)$
$3G$	$3G$	$3/10(30)^{1/2}(3\zeta' - 670M_0 - 401M_2)$
$3G$	$3I$	$6/5(55)^{1/2}(\zeta' + 10M_0 - 67M_2)$
$3I$	$3I$	$3/5(55)^{1/2}(\zeta' - 365M_0 - 92M_2)$
$3H$	$1I$	$-3/2(26)^{1/2}(\zeta' + 13M_0 + 64M_2)$

for the electron in the incomplete shell. The same holds for any incomplete shell with arbitrary numbers of electrons, as it can be shown by means of the equivalent operator method described above. Therefore, we may consider that the spin-orbit parameter ζ includes this "screening" effect, so that ζ' , defined by (21), can be interpreted as including the effect due to the spin-other-orbit interactions between electrons themselves in the incomplete shell.

§ 4. Comparison with the multiplet intervals of the $2p^n$ configurations

The matrix elements of the spin-spin and spin-other-orbit interactions for the l^n configurations with arbitrary l and n , can be obtained from eqs. (14) and (21). The parameters ζ_l , M^0, \dots, M^{2l-2} appeared in these formulas can be calculated, for example, by making use of the Hartree-Fock wave-functions, provided these are known quite accurately. It was pointed out, however, that the parameters calculated from the Hartree wave functions without exchange, give much smaller values than those expected from the observed data for the multiplet intervals.³⁾ Therefore, it seems natural to choose the value of the parameters so as to give the observed values of multiplet intervals, in the present stage of our knowledge.

One might consider that the most reliable parameters can be given by comparing the roots of the secular equation for each value of J which are obtained for the matrix elements involving the whole interactions, those of electrostatic, spin-spin and spin-orbit interactions, with observed term values. However, since some unknown factors are left even in gross structure, which have not been able to be calculated quite accurately on theoretical footing, the parameters thus determined would include errors coming from the gross structure. Therefore, we should determine ζ_l and $M^0, M^2, \dots, M^{2l-2}$ in such a way that the errors in gross structure can be eliminated. For this purpose, the second-order perturbation theory starting from the Russel-Saunders coupling seems to be the most suitable method at least for lighter elements.

The first-order term of spin-orbit interactions provides the well-known Landé interval rule and gives approximate multiplet intervals in lighter elements. The deviations from this rule are caused by first-order elements of spin-spin interaction and second-order elements of spin-orbit interactions, while the second-order elements of spin-spin interaction are practically negligible.

The triplet intervals of p^3 configuration are given by

$$^3P_2 - ^3P_1 = (\zeta' - 55M_0) - 12M_0 - (\zeta' + 19M_0)^2/2(^1D - ^3P),$$

$$^3P_1 - ^3P_0 = (\zeta' - 55M_0)/2 + 30M_0 + 2(\zeta' + 10M_0)^2/(^1S - ^3P),$$

Table IV. The values of parameters ζ' , ζ_p and M_0 for $2p^3$ configuration (in cm^{-1})

		ζ'	ζ_p	M_0
C	I	32.4	32.8	0.079
N	II	96.0	97.0	0.202
O	III	220.3	222.2	0.38
F	IV	433.0	436.0	0.61
Ne	V	783.1	788.6	1.10
Na	VI	1296	1304	1.64
Mg	VII	2041	2054	2.57
Al	VIII	3064	3080	3.67
Si	IX	4348	4368	3.93

Table V. The values of parameters ζ' , ζ_p and M_0 for $2p^4$ configuration (in cm^{-1})

		ζ'	ζ_p	M_0
O	I	139.6	146.4	0.274
F	II	308.1	320.0	0.471
Ne	III	583	606	0.92
Na	IV	1008	1039	1.22
Mg	V	1621	1667	1.85
Al	VI	2487	2556	2.75
Si	VII	3654	3743	3.57

where $\zeta' = \zeta - 5M_0$, and the corresponding expressions for p^1 configuration are obtained by reversing the signs of ζ' in these equations where $\zeta' = \zeta_p - 25M_0$. The values of ζ_p and M_0 determined by comparing these equations with the observed triplet intervals of isoelectronic sequences of $2p^2$ and $2p^1$ configurations¹¹⁾ are given in Table IV and V, respectively. The diagonal elements of the spin-spin interaction contributes to the deviation for smaller values of Z , while the effects of the non-diagonal elements of the spin-orbit interaction becomes larger than that of the diagonal elements of spin-spin interaction for larger Z values. This can be seen by the fact that ζ_p is proportional to $(Z - \sigma)^4$ as is well known, σ being the screening constant, while M_0 is proportional to $(Z - \sigma')^3$ as is shown in the following, σ' being the other screening constant. We plot the values of $\zeta_p^{1/4}$ against Z in Fig. 1. The values of $\zeta_p^{1/4}$ obtained from the single $2p$ configuration and the values of $(\zeta_p - 30M_0)^{1/4}$ from $2p^3$ are also plotted for the sake of comparison, since only the parameter $\zeta' = \zeta_p - 30M_0$ can be obtained by the observed splitting of inverted doublet of

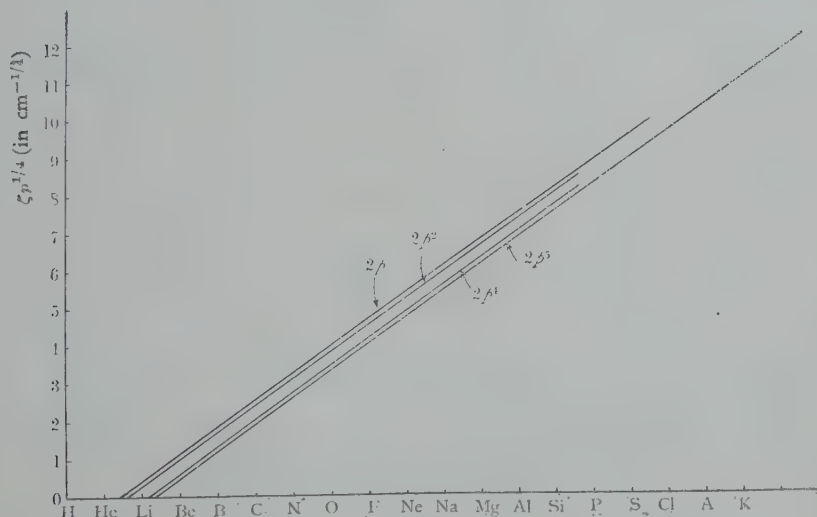


Fig. 1.

(Most of experimental values lie on each line.)

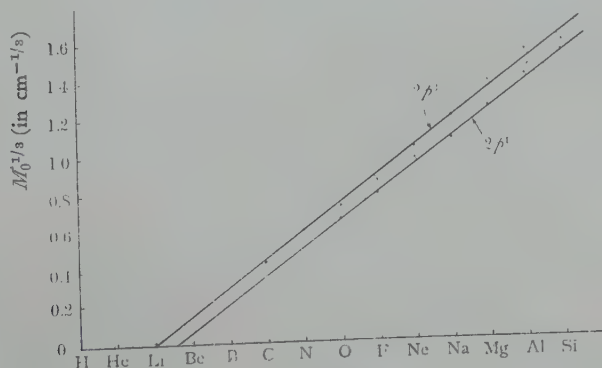


Fig. 2.

p^5 configuration, the difference between $\zeta^{1/4}$ and $\zeta_n^{1/4}$ being very small. The $\zeta_n^{1/4}-Z$ plots become parallel straight lines with slope 0.72 and the values of screening constants σ are given by 2.49, 2.66, 3.14 and 3.43 for the configurations p , p^2 , p^4 and p^5 , respectively. On the other hand, the values of M_0 for an isoelectronic sequence should be proportional to $(Z-\sigma')^3$, because the parameter M_0 represents the effects of *mutual* spin-orbit and spin-spin interactions. This has been just confirmed by plotting $M_0^{1/3}$ for $2p^2$ and $2p^4$ configurations as is seen in Fig. 2. The slopes at two straight lines are shown to be both 0.153 and the screening constants σ' are 3.08 and 3.64, respectively. These values of slopes are in agreement with 0.138 fairly well, which are calculated using the hydrogen-like $2p$ wave function. These values of σ' are nearly equal to those of the corresponding ones of electrostatic interaction parameter F_2 , while the smallness of the values of σ' for the parameter ζ_n may be accounted for by considering that ζ_n are essentially the mean value of $1/r^3$ so that the screening by other electrons have smaller influence.

The diagonal elements of the individual spin-orbit interaction vanish for p^3 configuration, so that our perturbation treatment would involve necessarily much error. However, some informations can be obtained from the two doublets of p^3 configuration by merely qualitative arguments. The two doublets of $2p^3$ isoelectronic sequence are both inverted or either of two are so, in the region of smaller Z value. This inversion can not be accounted for by the usual spin-orbit interaction, but can be interpreted only by introduction of the spin-other-orbit interaction. The doublet intervals are given by the following equations provided the perturbation treatment could be used:

$${}^2D_{5/2}-{}^2D_{3/2}=-185/2 \cdot M_0+5\zeta'^2/4({}^2P-{}^2D),$$

$${}^2P_{3/2}-{}^2P_{1/2}=-75/2 \cdot M_0+\zeta'^2\{5/4({}^2P-{}^2D)+1/({}^2P-{}^4S)\}.$$

In the right hand sides of these equations, the first terms come from the spin-other-orbit interaction and the second from the individual spin-orbit interaction. Non-diagonal elements of the spin-spin interaction are negligible. If the first term in each equation is larger in magnitude than the second, the doublet appears inverted. For larger atomic numbers, both two doublets become normal, since ζ_n increases more rapidly than M_0 , as is mentioned

Table VI. The doublet splittings of $2p^3$ configuration (in cm^{-1})

		${}^2D_{5/2}-{}^2D_{3/2}$		${}^2P_{3/2}-{}^2P_{1/2}$	
		cal.	obs.	cal.	obs.
N	I	—	-8	—	0
O	II	-28	-21.0	-8	-1.5
F	III	-40	-36	-7	0
Ne	IV	-65	-25	-3	10
Na	V	-71	-25	24	39
Mg	VI	-58	-21	104	122
Al	VII	-60	60	200	270
Si	VIII	90	280	401	580

above. For example, we can choose as the values of ζ_p and M_0 for $2p^3$ configuration, the mean values of the corresponding ones for $2p^2$ and $2p^4$, and insert into the above equations. The results are shown in Table VI. Although the agreements are not so good, the qualitative trend seems to be very promising.

§ 5. Discussion

The effects of spin-spin and spin-other-orbit interactions must be examined for heavier elements than those having $2p^n$

configurations. Among the $3p^n$ configurations, we consider, first of all, the two doublets of $3p^3$. There are no inverted doublets and the ratios of splitting of 2D to 2P are nearly equal to those of $2p^3$ doublets in larger Z region. This fact suggests that the doublet splittings of $3p^3$ configuration are almost dominated by the non-diagonal elements of the usual spin-orbit interaction, while the effect of spin-other-orbit one is very small. In the deviations from the Landé rule in the triplet intervals of $3p^2$ and $3p^4$ configurations, the situation is similar, that is, the effects of the diagonal elements of spin-spin interaction is very small in comparison with that of the non-diagonal elements of the individual spin-orbit interaction. The determination of the parameters ζ_p and M_0 similar to the preceding section also leads to the smaller values of M_0 in comparison with ζ_p . For example, the values of ζ_p and $M_0(3p)$ for P II are 157.2 and 0.062 cm^{-1} , respectively, which should be compared with the values in Table IV. The values of ζ_p for $3p^n$ ($n=2, 3$ and 4) thus determined from the multiplet intervals agree with values tabulated by Robinson and Shortley.¹²⁾ The smallness of M_0 compared with ζ_p are ascertained by using the hydrogen-like wave function and the results of calculation are given by $M_0 \sim 0.000225 Z^3 \text{ cm}^{-1}$, while for $2p^n$ configurations $M_0 \sim 0.00261 Z^3 \text{ cm}^{-1}$. The screening constants for M_0 are perhaps nearly equal to the values for F_2 of electrostatic interaction and those for ζ_{3p} are much smaller.

For $3d^n$ configurations, situations seems to be similar to in the case of $3p^n$ configurations. Since the configuration interactions with $d^{n-1}s$ etc. make the determination of the parameters more or less unreliable, any conclusion may be premature. The similar determination for Ti III $3d^2$, gives the values of parameters as 64.0, 0.177 and 0.0068 cm^{-1} for ζ_d , M_0 and M_2 , respectively. Those for $3d^n$ configurations calculated using the hydrogen-like wave function are given by

$$\zeta = 0.0147 Z^4, \quad M_0 = 0.000175 Z^3 \quad \text{and} \quad M_2 = 0.0000144 Z^3 \text{ cm}^{-1}.$$

We may say that, according to this crude estimate of the parameters, the effects of spin-spin and spin-other-orbit interactions are rather small compared with the individual spin-orbit ones even in the deviation from the interval rule, and we can not expect for heavier elements such marked contributions from them as in the helium atom and elements with $2p^n$ configurations.

In conclusion, the author wishes to express his sincere thanks to Professors T. Yamouchi and M. Kotani for their kind interests and continued encouragements throughout the course of the work. The author wishes also to thank Dr. T. Ishidzu and Mr. Y. Tanabe for their valuable discussions. This work was supported by the Grant of Aid in Fundamental Scientific Research.

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Note added in proof :

F. R. Innes (Phys. Rev. **91** (1953), 31) also pointed out that the matrix elements of spin-spin interaction are diagonal with respect to seniority numbers and independent of the numbers of electrons.

We have obtained the procedure which gives the coefficients of exchange integrals for the configurations $1^n 1'$, similar to that of the method which Racah has proposed for electrostatic interaction (ref. 4). It is one of the subjects of the forthcoming paper by the present author in cooperation with Messrs. A. Arima and Y. Tanabe.

Effect of Nucleon Excited State on Magnetic Moment Anomaly

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Approximating a nucleon in its excited state by a spinor field with higher mass, charge and spin, its effects on the anomalous magnetic moment are estimated. The result for the spin $3/2$ excited state is quadratically divergent leading to no definite conclusions. The contribution of the spin $1/2$ and isotopic spin $3/2$ excited state is positive for proton and negative for neutron if this state is of the different parity from the normal state and *vice versa* if these are of the same parity, the absolute value being larger for proton in both cases. An additional interaction, which describes the transition of a nucleon up to or down from its excited state when it interacts with an external electromagnetic field, contributes nothing in the former choice of the parities while for the latter this gives an opposite contribution to proton and neutron. The effects are, after all, not large for the reasonable choice of the coupling constant.

§ 1. Introduction

The additional magnetic moment ascribed to a nucleon in an external electromagnetic field is one of the simplest properties predicted by meson theory. The lowest order perturbation calculations, which could account for the most part of the electron anomalous moment, have been performed by several authors.¹⁾ The result that only pseudoscalar theory gives the correct signs for proton and neutron moments is very promising in connection with various other investigations. Quantitatively, however, we meet that great difficulty which is common to all nuclear phenomena. Neutron moment can be accounted for by taking $g^2/4\pi \sim 7$, while that of proton needs $g^2/4\pi \sim 52$. These large values of the coupling constant imply in themselves that the higher order corrections cannot be regarded as small. In fact Nakabayasi and Sato²⁾ have shown that the fourth order radiative corrections are quite large, the corrections exceeding the second order values in some cases. Although it may be a feature characteristic of pseudoscalar theory that the fourth order quantity is unusually larger than the second order one, one cannot help admitting that the weak coupling approach is not a good approximation.

On the other hand, we have the phenomenological approach as was done by Sachs,³⁾ according to whose conclusion the probability that the nucleon has two pions in its self-field must be considerably large. This may be the reason why the lowest order values for anomalous moments were too small. Similar analysis has been recently made by Sugawara⁴⁾ from a somewhat different standpoint. He showed that the situation is very much improved if one assumes a nucleon to be virtually in its excited state with higher spin and charge. This is an interesting conclusion to be noted since the excited states of this

nature seem to be necessary in various other problems such as pion-nucleon scattering, gamma-pion production and so on.

The estimations of excited states of this kind have been made by Minami et. al.⁽⁵⁾ for the single nucleon system and by Matsumoto et. al.⁽⁶⁾ for the nuclear force problem describing the excited states by Rarita-Schwinger field.⁽⁷⁾ The existence of such an excited state is certainly due to the strong coupling between a nucleon and its self-pion-field. Although it is a questionable procedure to treat the nucleon in excited state as if it were an elementary particle, the favorable tendency which these investigations have clarified may be regarded as a realistic foundation of this method. It would not be quite hazardous to say that this method takes at least some, if not all, of higher order corrections into account remedying the disadvantage involved in the pure weak coupling approach.

It is the purpose of the present work to follow this procedure to estimate the effects of the excited states on nucleon moment anomalies. This may be regarded as a detailed reinvestigation of Sugawara's⁽⁴⁾ analysis. For the convenience of comparison with later calculations we first briefly illustrate the main steps of the calculation of the lowest order normal contribution to anomalous moments in § 2, which is much simplified than those have been performed⁽¹⁾ if we are to calculate the magnetic moments only. The most favorable excited state, with spin and isotopic spin both $3/2$, is investigated in § 3. Since the interaction of Rarita-Schwinger field with pion field is of the second kind⁽⁸⁾ and moreover the propagation function is very singular for the high momenta, this investigation gives no definite results. In § 4 estimations are made of the effect of spin $1/2$ and isotopic spin $3/2$ excited state. This calculation is different from *normal* (without excited state) ones only in the charge dependence and a higher mass appearing in the propagation function. The effect will be seen, however, to be quite different from the normal lowest order one. An additional interaction which gives the expression for the moment, which is linear in coupling constants, is also discussed. § 5 is devoted to the summary of numerical results and discussions.

§ 2. Lowest order moment without excited state

Since this is what has been calculated by many authors previously,⁽¹⁾ only the main steps of the calculation will be briefly illustrated. The principle is to extract the expression of the form

$$-M \sum_{\mu < \nu} F_{\mu\nu}(x) \bar{\psi}(x) \sigma_{\mu\nu} \psi(x), \quad (1)$$

with $\sigma = (1/2i)[\gamma_\mu, \gamma_\nu]$, from the one nucleon portion of the effective Hamiltonian

$$H_F(x) = -\frac{1}{2} \int dx_1 dx_2 T \{ H_0^e(x), H_0^e(x_1), H_0^e(x_2) \}, \quad (2)$$

where

$$H_0^e(x) = c A_\mu^e(x) \left[-i \bar{\psi}(x) \frac{1 + \tau_3}{2} \gamma_\mu \psi(x) + \left(\varphi_1 \frac{\partial \varphi_2}{\partial x_\mu} - \varphi_2 \frac{\partial \varphi_1}{\partial x_\mu} \right) \right], \quad (3)$$

$$H_0^i(x) = ig\bar{\psi}(x)\gamma_5\tau_i\psi(x)\varphi_i(x), \quad i=1, 2, 3, \quad (4)$$

and T is Wick's chronological ordering operator.⁽⁹⁾ The first and second term of (3), substituted in to (2), gives the nucleon and meson contribution to anomalous moments respectively. These two contributions come from the processes illustrated in Fig. 1, in which (a) and (b) corresponds to the nucleon and meson contribution respectively.*



Fig. 1. Feynman diagrams of the processes contributing to the lowest order anomalous moments.

The one nucleon portion of (2) is

$$H_0^n(x) = -\frac{ie g^2}{16} A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) S_F(x-x_1) (3-\tau_3) \gamma_\mu S_F(x_2-x) \gamma_5 \psi(x_2) \times \Delta_F(x_1-x_2), \quad (5)$$

and

$$H_0^m(x) = -(ie g^2/4) A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) \gamma_5 \tau_3 S_F(x_2-x_1) \gamma_5 \psi(x_2) \times \{\Delta_F(x-x_1) \Delta_F^\mu(x-x_2) - \Delta_F(x-x_2) \Delta_F^\mu(x-x_1)\}, \quad (6)$$

where

$$S_F(x) = \frac{-2i}{(2\pi)^4} \int dk \frac{i\gamma k - m}{k^2 + m^2 - i\epsilon} e^{-ikx},$$

$$\Delta_F(x) = \frac{-2i}{(2\pi)^4} \int dk \frac{1}{k^2 + \mu^2 - i\epsilon} e^{-ikx}, \quad \Delta_F^\mu(x) = \frac{\partial}{\partial x_\mu} \Delta_F(x),$$

m and μ being nucleon and meson mass respectively.

For the calculation of the magnetic moment $A_\mu^e(x)$ is such that produces the constant magnetic field, so that one has

$$\Delta p_\mu A_\mu^e = (\Delta p)^2 A_\mu^e = 0, \quad (7)$$

where Δp_μ is the momentum difference between the initial and final nucleons. Further, in view of (1), we can drop terms of the form

* The diagram of the vacuum polarization type is not considered here, since this gives a vanishing contribution. See reference 1, especially Case's paper.

$$\bar{\psi}(x)\gamma_{\mu}\psi(x), \quad (8)$$

when they appear in the course of the calculation. With these and the equation of motion for ψ and $\bar{\psi}$ in mind calculations are exceedingly simplified.

Transforming eqs. (5) and (6) into momentum space and writing

$$\bar{\psi}_p e^{-ip'x} = \bar{\psi}(x), \quad \psi_p e^{ipx} = \psi(x), \quad (9)$$

one obtains*

$$\begin{aligned} H_0^n(x) &\approx -\frac{eg^2m}{16\pi^2} A_{\mu}^e(x) \not{p}_{\mu} \bar{\psi}(x) (3 - \tau_3) \psi(x) A_1, \\ H_0^m(x) &\approx \frac{eg^2m}{4\pi^2} A_{\mu}^e(x) \not{p}_{\mu} \bar{\psi}(x) \tau_3 \psi(x) A_2, \end{aligned} \quad (10)$$

where \not{p}_{μ} is the momentum four vector of the initial nucleon and

$$A_1 = \int_0^1 dx \frac{x^3}{m^2 x^2 + \mu^2 (1-x)} \quad \text{and} \quad A_2 = \int_0^1 dx \frac{x^2 (1-x)}{m^2 x^2 + \mu^2 (1-x)}. \quad (11)$$

Integrals (11) come from the use of the formula due to Feynman¹⁰⁾

$$\frac{1}{abc} = 2 \int_0^1 x dx \int_0^1 dy \frac{1}{[a(1-x) + bxy + cx(1-y)]^3}. \quad (12)$$

Noting the equivalence

$$A_{\mu}^e(x) \not{p}_{\mu} \bar{\psi}(x) \psi(x) \approx \frac{i}{2} A_{\mu}^e(x) \not{p}_{\mu} \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) \approx -\frac{1}{2} \sum_{\mu < \nu} F_{\mu\nu}(x) \bar{\psi}(x) \sigma_{\mu\nu} \psi(x), \quad (13)$$

eqs. (10) lead to

$$\begin{aligned} H_0^n &\approx \frac{eg^2}{32\pi^2 m} \sum_{\mu < \nu} F_{\mu\nu}(x) \bar{\psi}(x) \sigma_{\mu\nu} (3 - \tau_3) \psi(x) A_1', \\ H_0^m &\approx -\frac{eg^2}{8\pi^2 m} \sum_{\mu < \nu} F_{\mu\nu}(x) \bar{\psi}(x) \sigma_{\mu\nu} \tau_3 \psi(x) A_2', \end{aligned} \quad (14)$$

where

$$A_1' = \int_0^1 dx \frac{x^3}{x^2 + \lambda(1-x)} \quad \text{and} \quad A_2' = \int_0^1 dx \frac{x^2(1-x)}{x^2 + \lambda(1-x)}, \quad (15)$$

with $\lambda = (\mu/m)^2$. Comparing eqs. (14) with (1), one sees that the nucleon and meson contribution to anomalous moments is respectively given by

* By \approx we hereafter mean the equivalence in the calculation of moments. For other kinds of calculations such as of the electron-neutron interaction, these equivalence do not hold. Logarithmic divergencies appearing in (5) and (6) are of the form (8) playing no role in the calculation of moments.

$$\mathfrak{M}_n^0 = -\frac{g^2}{4\pi} \frac{1}{4\pi} (3 - \tau_3) A_1' \quad \text{and} \quad \mathfrak{M}_m^0 = \frac{g^2}{4\pi} \frac{1}{\pi} \tau_3 A_2', \quad (16)$$

in nuclear magneton $e/2m$.

Elementary integrals (15) can now be readily performed. Putting $\tau_3=1$ for proton and -1 for neutron, one arrives at the final results

$$\mathfrak{M}_p^0 = \frac{g^2}{4\pi} \frac{1}{\pi} \left\{ \frac{1}{4} - \frac{3}{2} \lambda + \frac{5\lambda - 3\lambda^2}{4} \ln \frac{1}{\lambda} - \frac{\lambda^{1/2}(4 - 11\lambda + 3\lambda^2)}{2(4 - \lambda)^{1/2}} \cos^{-1} \frac{\lambda^{1/2}}{2} \right\}, \quad (17)$$

$$\mathfrak{M}_n^0 = -\frac{g^2}{4\pi} \frac{1}{\pi} \left\{ 1 + \frac{\lambda}{2} \ln \frac{1}{\lambda} - \frac{\lambda^{1/2}(2 - \lambda)}{(4 - \lambda)^{1/2}} \cos^{-1} \frac{\lambda^{1/2}}{2} \right\}, \quad (18)$$

in complete agreement with previously obtained values.¹⁾ The small value of \mathfrak{M}_p^0 is due to the negative nucleon contribution in (16) and necessitates the exceedingly large value of coupling constant to fit the experimental data. This difficulty is greatly overcome if one takes the fourth order corrections into account.²⁾ However, the tendency that the proton moment is somehow too small still remains. On these points closer quantitative discussion will be made in § 5.

§ 3. Excited state with $J=3/2$, $T=3/2$

The effective Hamiltonian, of which we take the one nucleon part, is

$$H_F(x) = -\frac{1}{2} \int dx_1 dx_2 T \{ H_1^e(x), H_1^i(x_1), H_1^i(x_2) \}, \quad (19)$$

with

$$H_1^e(x) = c A_\mu^e(x) \left[-ie \bar{\Psi}_\lambda(x) Q \gamma_\mu \Psi_\lambda(x) + \left(\varphi_1 \frac{\partial \varphi_2}{\partial x_\mu} - \varphi_2 \frac{\partial \varphi_1}{\partial x_\mu} \right) \right], \quad (20)$$

and

$$H_1^i(x) = \frac{G}{\mu} \left[\bar{\Psi}_\mu(x) T_i \psi(x) + \bar{\psi}(x) T_i^* \Psi_\mu(x) \right] \frac{\partial \varphi_i(x)}{\partial x_\mu}, \quad (21)$$

where Ψ_μ is the Rarita-Schwinger's spin $3/2$ field and assumed to be of the different parity from normal nucleon field ψ .^{5,6)} T_i' and Q are extended isotopic spin operators and represented as⁶⁾

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -i & 0 \\ 0 & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & 0 \\ 0 & i \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 \\ \frac{2}{\sqrt{3}} & 0 \\ 0 & -\frac{2}{\sqrt{3}} \\ 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (22)$$

in the representation where ψ and Ψ_μ takes the forms

$$\psi = \begin{bmatrix} \psi_P \\ \psi_N \end{bmatrix}, \quad \Psi_\mu = \begin{bmatrix} \Psi_\mu^{++} \\ \Psi_\mu^+ \\ \Psi_\mu^0 \\ \Psi_\mu^- \end{bmatrix}. \quad (23)$$

$H_1^i(x)$ in (21) describes the transition of a nucleon up to or down from its excited state with spin and isotopic spin $3/2$. The second term of r. h. s. of (20) gives the meson contribution as in § 2, while the first term gives the excited nucleon contribution. The corresponding processes are schematized in Fig. 2.

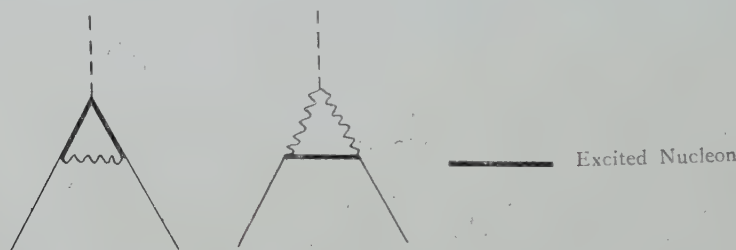


Fig. 2. Feynman diagrams of the processes giving effects of excited nucleons upon nucleon anomalous moments. (a) gives the excited nucleon contribution and (b) does meson contribution.

In evaluating the one nucleon portion of (19) we need the vacuum expectation value of the form

$$\langle T\{\Psi_\mu(x_1), \bar{\Psi}_\lambda(x_2)\} \rangle_0 = -\frac{1}{2} \mathfrak{S}_F^{\mu\nu}(x_2 - x_1). \quad (24)$$

The Green's function $\mathfrak{S}_F^{\mu\nu}$ of an excited nucleon has been calculated by Minami et. al.⁽⁵⁾ leading to the Fourier transform

$$\mathfrak{S}_F^{\mu\nu}(x) = \frac{-2i}{(2\pi)^4} \int d^4p \frac{\mathfrak{S}_F^{\mu\nu}(p)}{p^2 + M^2 - i\epsilon} e^{-ipx}, \quad (25)$$

where

$$\begin{aligned} \mathfrak{S}_F^{\mu\nu}(p) = & (i\gamma p - M) \delta_{\mu\nu} + \frac{2}{3M^2} p_\mu (i\gamma p) p_\nu \\ & - \frac{i}{3M} [\gamma_\mu (i\gamma p) p_\nu + p_\mu (i\gamma p) \gamma_\nu] - \frac{4}{3M} p_\mu p_\nu \end{aligned} \quad (26)$$

$$+ \frac{i}{M^2} \left(\frac{2}{3} \not{p}^2 + M^2 \right) \{ \not{p}_\mu \gamma_\nu + \gamma_\mu \not{p}_\nu + i \gamma_\mu (i \not{p} + M) \gamma_\nu \},$$

with M the mass of the excited nucleon.

With this preliminary the one nucleon portion of (19) gives

$$H_1^n(x) = -\frac{ieG^2}{12\mu^2} A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) \mathfrak{S}_F^{\lambda\lambda}(x-x_1) \gamma_\mu \mathfrak{S}_F^{\lambda\sigma}(x_2-x) \\ \times (3+5\tau_3) \phi(x_2) \Delta_F^{\sigma\sigma}(x_1-x_2), \quad (27)$$

for the excited nucleon contribution and

$$H_1^m(x) = \frac{ieG^2}{6\mu^2} A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) \mathfrak{S}_F^{\lambda\nu}(x_2-x_1) \tau_3 \phi(x_2) \\ \times \{ \Delta_F^{\nu\nu}(x-x_1) \Delta_F^{\lambda\mu}(x-x_1) - \Delta_F^{\lambda\nu}(x-x_1) \Delta_F^{\nu\mu}(x-x_2) \}, \quad (28)$$

for the meson contribution, where

$$\Delta_F^{\mu\nu}(x) = \frac{\partial^2}{\partial x_\mu \partial x_\nu} \Delta_F(x).$$

In eqs. (27) and (28) the disappearance of T_i 's is due to the relations

$$T_i^* Q T_i = \frac{2}{3} (3+5\tau_3) \quad \text{and} \quad T_2^* T_1^- T_1^* T_2 = \frac{4}{3} i \tau_3. \quad (29)$$

The meson contribution (28) is opposite for the proton and neutron as in the case of no excited states (eq. (10)). Thus the conclusion that the meson current contributes nothing to $\mathfrak{M}_p + \mathfrak{M}_n$ value, which has been observed by several authors,¹¹⁾ will not be modified even if one introduces excited states with isotopic spin 3/2. The remarkable difference between the present and the normal (without excited state) calculations is the charge dependence of the nucleon contribution. In the normal calculation the charge dependence is $3-\tau_3$ as given in eq. (5), while in (27) we have $3+5\tau_3$. $3-\tau_3$ is positive for both proton and neutron, this being the cause of the difficulty that the proton moment is too small as noted at the end of the foregoing section. This shortage of the proton moment may more or less be made up by (27) since $3+5\tau_3$ is of different sign for a proton and neutron and its absolute value is four times larger for the former. This may be one reason why the introduction of an excited state betters the situation.⁴⁾ To establish this statement, however, we have to perform the integration involved in (27).

As soon as we set about this work, we meet one of the most fundamental difficulties of the quantized field theory in the present form. The interaction Hamiltonian (21) is of the second kind (derivative coupling) and it is well-known that the theory is not renormalizable. Furthermore, as is readily seen from (26), the propagation function for the excited nucleon is much more singular for high momenta than that for normal nucleon. If one transforms eq. (19) into momentum space, one readily sees that the integral

contains the divergence of the fourth order. It will be seen, however, that this divergence does no harm if we confine ourselves to the calculation of the moments only because of (8). The remaining divergences, quadratic and logarithmic, give contributions to the moments but we have no satisfactory procedures to remove these infinities. Thus we have to disappointedly admit that our propitious anticipation as stated above is only a plausibility variety.

§ 4. Excited state with $J=1/2$, $T=3/2$

Let the nucleon in this excited state be represented by Ψ obeying the Dirac equation $(\gamma\partial + M)\Psi = 0$, of which the Green's function is

$$\mathfrak{S}_F(x) = \frac{-2i}{(2\pi)^4} \int d^4p \frac{i\gamma p - M}{p^2 + M^2 - i\epsilon} e^{-ipx}. \quad (30)$$

Thus the effective Hamiltonian is

$$H_F(x) = -\frac{1}{2} \int dx_1 dx_2 T \{H_2^e(x), H_2^i(x_1), H_2^i(x_2)\}, \quad (31)$$

with

$$H_2^e(x) = c A_\mu^e(x) \left[-i \Psi(x) Q \gamma_\mu \Psi(x) + \left(\varphi_1 \frac{\partial \varphi_2}{\partial x_\mu} - \varphi_2 \frac{\partial \varphi_1}{\partial x_\mu} \right) \right], \quad (32)$$

and

$$H_2^i(x) = G [\bar{\Psi}(x) T_i O \psi(x) + \bar{\psi}(x) T_i^* O \Psi(x)], \quad (33)$$

where T_i 's and Q are defined by (22) and (23), and

$$O = \begin{cases} i\gamma_5 & \text{if } \psi \text{ and } \Psi \text{ are of the same parity,} \\ 1 & \text{if } \psi \text{ and } \Psi \text{ are of the different parity.} \end{cases} \quad (34)$$

The contributing processes are illustrated in Fig. 2 (a) and (b). We will divide the present section into three. The first two are devoted to the calculation of the excited nucleon and meson contributions, while the effects of the additional interaction of the form

$$H_3^e(x) = i\epsilon A_\mu^e(x) \bar{\Psi}(x) O' C \gamma_\mu \psi(x) + \text{comp. conj.} \quad (35)$$

will be estimated in the final subsection.

4. 1. Excited nucleon contribution

The first term of r. h. s. of (32), substituted into (31), leads to the excited nucleon contribution

$$H_2^n(x) = \frac{icG^2}{12} A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) O \mathfrak{S}_F(x-x_1) (3+5\tau_3)$$

$$\times \gamma_\mu \mathcal{G}_F(x_2 - x) O\psi(x_2) \Delta_F(x_1 - x_2) \dots \quad (36)$$

Transforming into momentum space and noting the equivalence

$$\begin{aligned} & \bar{\psi}(x) O[i\gamma(p+k') - M] \gamma_\mu [i\gamma(p+k) - M] O\psi(x) \\ & \approx -2\bar{\psi}(x) [(\gamma k) - (\pm M - m)] k_\mu \psi(x). \end{aligned} \quad (37)$$

where $\psi(x)$ is defined by (9) and where upper and lower sign corresponds to $O = i\gamma_5$ and 1 respectively, one obtains

$$H_2^n(x) \approx \frac{eG^2}{12\pi^2} A_\mu^e(x) p_\mu \bar{\psi}(x) (3 + 5\tau_3) \psi(x) B_1, \quad (38)$$

where

$$B_1 = \int_0^1 dx \frac{(1-x)^2 (mx \mp M)}{(1-x)(M^2 - m^2 x) + \mu^2 x}. \quad (39)$$

If we use the equivalence (13) the excited nucleon contribution is found to be

$$\mathfrak{M}_n = \frac{G^2}{4\pi} \frac{1}{3\pi} B_1' (3 + 5\tau_3), \quad (40)$$

where

$$B_1' = \int_0^1 dx \frac{(1-x)^2 (x \mp \eta^{1/2})}{(1-x)(\eta - x) + \lambda x}, \quad (41)$$

with $\eta = (M/m)^2$.

Since $\eta \geq 1$ the elementary integral B_1' is negative for upper sign but positive for lower one. Thus \mathfrak{M}_n in (40) has the favorable sign if one assumes \mathcal{P} to be of the different parity from ψ , establishing the propitious statement in § 3 for the case of spin 1/2 excited nucleons. The case which corresponds to Sugawara's⁴⁾ intuitive analysis on excited states is naturally that of the different parity with no γ_5 , since if the interaction Hamiltonian contains γ_5 the simple model that the nucleon goes into its excited state with the emission or absorption of a pion will not be adequate because of the large matrix element between the positive and negative energy states. This conclusion will be worth noting if one remembers the fact that the different parity has been necessary to reproduce the experimental data on pion-nucleon scattering, γ -pion production,⁵⁾ nuclear forces,⁶⁾ and so on, in the case of spin 3/2 excited nucleons. As will be seen in the following subsection the same holds for the meson contribution.

The integral (41) can be readily performed to lead to the result

$$B_1' = \mp \eta^{1/2} I_0 + (1 \pm 2\eta^{1/2}) I_1 - (2 \pm \eta^{1/2}) I_2 + I_3, \quad (40)$$

where

$$\begin{aligned} I_0 &= q^{-1/2} \ln \frac{(2 - \delta - q^{1/2})(-\delta + q^{1/2})}{(2 - \delta + q^{1/2})(-\delta - q^{1/2})}, & \text{for } q > 0 \quad (M > m + \mu), \\ I_0 &= \eta^{-1/2} (\eta^{1/2} - 1)^{-1}, & \text{for } q = 0 \quad (M = m + \mu), \end{aligned} \quad (41)$$

with $\delta = \eta - \lambda + 1$, $q = \delta^2 - 4\eta$, and

$$\begin{aligned} I_1 &= \frac{1}{2} \ln \frac{\lambda}{\eta} + \frac{\delta}{2} I_0, \\ I_2 &= 1 + \frac{\delta}{2} \ln \frac{\lambda}{\eta} + \frac{\delta^2 - 2\eta}{2} I_0, \\ I_3 &= \frac{1}{2} + \delta + \frac{\delta^2 - \eta}{2} \ln \frac{\lambda}{\eta} + \frac{\delta^3 - 3\eta\delta}{2} I_0. \end{aligned} \quad (42)$$

Numerical results will be given in § 5.

4. 2. Meson contribution

The one nucleon portion of eq. (31), with $H_2^e(x)$ the second term of r. h. s. of (32), is

$$\begin{aligned} H_2^m &= -\frac{i\epsilon G^2}{6} A_\mu^e(x) \int dx_1 dx_2 \bar{\psi}(x_1) O \mathfrak{S}_F(x_2 - x_1) O \tau_3 \psi(x_2) \\ &\quad \times \{ \mathcal{A}_F(x - x_1) \mathcal{A}_F^\mu(x - x_2) - \mathcal{A}_F(x - x_2) \mathcal{A}_F^\mu(x - x_1) \}. \end{aligned} \quad (43)$$

After similar transformations we find

$$H_2^m \approx \frac{\epsilon G^2}{6\pi^2} A_\mu^e(x) \not{p}_\mu \bar{\psi}(x) \tau_3 \psi(x) B_2, \quad (44)$$

where

$$B_2 = \int_0^1 dx \frac{x(1-x)(mx \mp M)}{(1-x)(M^2 - m^2x) + \mu^2x}. \quad (45)$$

Hence the meson contribution to the anomalous moment is, in nuclear magneton,

$$\mathfrak{M}_m = \frac{G^2}{4\pi} \frac{2}{3\pi} B_2' \tau_3, \quad (46)$$

where

$$B_2' = \int_0^1 dx \frac{x(1-x)(x \mp \eta^{1/2})}{(1-x)(\eta - x) + \lambda x}. \quad (47)$$

B_2' is negative or positive according to whether we take $O = i\gamma_5$ or 1. Thus (46) is positive for proton and negative for neutron if \mathcal{V} is of the different parity from ψ , and *vice versa* if these are of the same parity.

The elementary integral (47) gives

$$B_2' = \mp \eta^{1/2} I_1 + (1 \pm \eta^{1/2}) I_2 - I_3, \quad (48)$$

I 's being defined by eqs. (41) and (42).

4. 3. Additional interaction

The estimation of the effects of excited nucleons upon γ -pion production and the Compton scattering by a nucleon⁵⁾ required to introduce an additional interaction which describes the transition of a nucleon up to or down from its excited state when it interacts with the electromagnetic field. Since we are concerned, in the present calculation, with only the weak external field, it is not probable that this transition takes place. It may be, however, of some theoretical interest to investigate the effect of this type of interaction on anomalous moments, which is the subject of this subsection.

The interaction Hamiltonian causing this transition is first assumed:

$$H_3^e(x) = i\epsilon A_\mu^e(x) \{ \bar{\Psi}(x) O' \gamma C \psi(x) + \bar{\psi}(x) O' \gamma_\mu C^* \Psi(x) \}, \quad (49)$$

where

$$O' = \begin{cases} 1 & \text{if } \psi \text{ and } \Psi \text{ are of the same parity,} \\ \gamma_5 & \text{if } \psi \text{ and } \Psi \text{ are of the different parity,} \end{cases} \quad (50)$$

and*

$$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & a \\ 0 & 0 \end{bmatrix}, \quad (51)$$

with ψ and Ψ represented as (23). The effective Hamiltonian is

$$H_F(x) = -\frac{1}{2} \int dx_1 dx_2 T \{ H_3^e(x), H_0^i(x_1), H_2^i(x_2) \}, \quad (52)$$

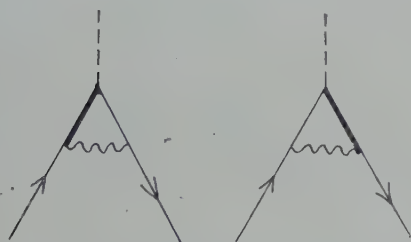


Fig. 3. Feynman diagrams of the processes in which $H_3^e(x)$ plays a role.

H_0^i and H_2^i being defined by eqs. (4) and (33), respectively. The processes we have now take into account is (a) and (b) of Fig. 3.**

The one nucleon part of (42) takes the form

$$H_3(x) = \frac{gG\epsilon}{16} A_\mu^e(x) \int dx_1 dx_2 \{ \bar{\psi}(x_1) \gamma_5 S_F(x-x_1) O' \gamma_\mu \mathfrak{S}_F(x_2-x) O \tau_i C^* T_i \psi(x_2) + \bar{\psi}(x_2) O \mathfrak{S}_F(x-x_2) O' \gamma_\mu S_F(x_1-x) \gamma_5 T_i^* C \tau_i \psi(x_1) \} A_F(x_1-x_2), \quad (53)$$

* The choice in reference 5 corresponds to $\epsilon=1.5e$, $a=-1$.

** The vacuum polarization type processes will not be taken into account since these give the vanishing contributions as in § 2.

where the first and the second term corresponds to Fig. 3 (a) and (b) respectively. If one uses the relation

$$\tau_i C^* T_i = T_i^* C \tau_i = \frac{2}{\sqrt{3}} (1-u) \tau_3, \quad (54)$$

and performs the integrations in (53) after transforming into momentum space, one finds that two contributions of the r. h. s. of (53) cancel each other for $O=1$ and $O'=\gamma_5$. For $O=i\gamma_5$ and $O'=1$, however, these two terms give the same contributions, the moment for this case being given by

$$\mathfrak{M}_a = \frac{gG}{4\pi} \frac{\epsilon}{e} \frac{1}{\sqrt{3}} \frac{1-u}{\pi} U \tau_3, \quad (55)$$

in nuclear magneton, where

$$U = \int_0^1 x^2 dx \ln \left(1 + \frac{(\eta-1)x}{\lambda(1-x) + x^2} \right). \quad (56)$$

The moment (55) is of the opposite signs for proton and neutron. Its absolute sign is, however, indefinite since this is linear in three coupling constants g , G and ϵ .

§ 5. Numerical results and discussions

Before estimating the effects of spin 1/2 isotopic spin 3/2 excited state numerically, let us first see what the situation is for the normal interaction (without excited state) only. As can be seen from Table I, the fourth order correction is quite large. The situation that the exceedingly large value for the coupling constant is required is greatly improved by taking the fourth order effect into account. Namely we can fit the proton and neutron moment by taking $g^2/4\pi \sim 7$ and $\cdot 5$ respectively.

If we postpone the estimation of effects of the additional interaction (49) for the time being, we have two parameters, $G^2/4\pi$ and η , in examining whether the effects of spin 1/2 excited states are favorable. Since the excited state with $J=1/2$, $T=3/2$ is

Table I. Contributions of normal interaction to anomalous moments to the fourth order in symmetrical $\not{p}\not{s}$ - $\not{p}\not{s}$ theory with $x=g^2/4\pi$. \mathfrak{M}_n and \mathfrak{M}_m means the nucleon and meson contribution, respectively.

	\mathfrak{M}_P		\mathfrak{M}_N	
	2nd order	4th order	2nd order	4th order
\mathfrak{M}_n	$-0.075x$	$+0.0306x^2$	$-0.151x$	$-0.0156x^2$
\mathfrak{M}_m	$+0.111x$	$+0.0025x^2$	$-0.111x$	$-0.0025x^2$
total	$+0.036x$	$+0.033x^2$	$-0.262x$	$-0.018x^2$

of no physical reality, we have no reasonable way to determine the values of these parameters. In general a larger value assumed for γ leads to a smaller value for the contribution, as can be expected intuitively. Calculations were made assuming $M=m+2\mu$ ($\gamma=1.69$), the results being given by Table II.

Table II. Contributions of the excited state with $J=1/2$, $T=3/2$. Notations are the same as in Table I except $\gamma=G^2/4\pi$.

$\bar{\Psi}\psi$	\mathfrak{M}_P		\mathfrak{M}_N	
	pseudoscalar	scalar	pseudoscalar	scalar
\mathfrak{M}_n	$+0.533\gamma$	-0.304γ	-0.133γ	$+0.076\gamma$
\mathfrak{M}_m	$+0.189\gamma$	-0.053γ	-0.189γ	$+0.053\gamma$
total	$+0.722\gamma$	-0.357γ	-0.322γ	$+0.129\gamma$

It is explicit in Table II that the effects are in a favorable direction only if $\bar{\Psi}\psi$ is pseudoscalar, or in other words, ψ and $\bar{\Psi}$ are of the different parity. The contribution is larger for proton than for neutron in absolute value. Hence one can expect that the experimental value can be reproduced by taking the same values of $G^2/4\pi$ for proton and neutron if $G^2/4\pi$ is appropriately taken.

We must note here that these discussions are only for the excited state with $J=1/2$, $T=3/2$ and must not be taken too seriously. The calculations and discussions on this excited state is of only an academic nature. Their significance is placed at the point that they may serve to anticipate the effects of the excited state of physical reality, i. e., that with $J=3/2$, which cannot be explicitly estimated in the present formalism. As we have seen in § 3 the charge dependence is the same in the effects of these two kinds of excited states. Although of other dependence, such as angular momenta and energy, we are quite ignorant, the situation seems very promising in view of the results of the case of $J=1/2$ excited state.

Finally we must mention that curious contribution given as eq. (55). As we have seen there the contribution of the interaction (49) vanishes for the favorable choice of parities, i. e. for $\bar{\Psi}\psi$ pseudoscalar. In the unfavorable case of $\bar{\Psi}\psi$ scalar (55) gives opposite contributions to proton and neutron for any values of ϵ , G , α and η , the absolute value being the same. Hence, in view of the results given in Table II, one may conclude that the additional interaction (49) does not change the situation, the favorable choice of parity remaining still favorable and the unfavorable one remaining unfavorable. It is quite impossible to anticipate the effects of (49) in the case of $J=3/2$ excited state. All that we can safely say is that \mathfrak{M}_n vanishes for $\alpha=1$, which means that the proton and neutron are coupled to the external field in a way given in (49) with the same coupling constant. Since the value $\alpha=1$ is not probable in view of the previous investigations,⁵⁾ we cannot conclude that the additional interaction (49) does not affect the anomalous moments. However, the contribution might vanish for some choice of parities for $\bar{\Psi}_\mu$ and ψ .

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Nuclear Forces in Pseudoscalar Meson Theory*

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Perturbation theoretical calculations are made of nuclear forces yielded by pseudoscalar meson theory up to the fourth order in the coupling constant, especially taking account of the recoil of the nucleons in the intermediate states. Both cases of pseudoscalar and pseudovector couplings are considered. The expressions of the nucleon-nucleon interactions obtained contain integrals of contributions of various values of nucleon momenta in the intermediate states. Expanding the integrands of these integrals in powers of μ/M , and picking up the first few terms from each of these expansions, we obtain the nuclear potentials for the case in which the nucleon recoil is neglected. These potentials are found to be in agreement with the adiabatic potentials which have been calculated by Taketani et. al. for the case of pseudovector coupling, and by Lévy and Klein for the case of pseudoscalar coupling. However, the differences between the adiabatic potentials and the potentials calculated taking account of the nucleon recoil are found to be much larger than those expected by considering them to be of the higher order in μ/M . Brief discussions are made on the qualitative properties of the nucleon-nucleon interactions calculated by us.

§ 1. Introduction

Recent analyses of the nucleon-nucleon scattering data show that the meson mass corresponding to the range of the singlet even state potential should be taken to be much larger than the observed π meson mass,²⁾ and that the angular distributions of high energy $n-p$ scattering can be explained only under the assumption that the odd state forces are very weak in comparison with the even state forces.³⁾ These facts cannot be understood by considering the second order meson potentials alone. The purpose of the present work is to calculate the fourth order nuclear potentials yielded by the pseudoscalar meson theory and to examine whether these potentials are favorable or not for the explanation of the facts above mentioned.

For the meson theory the convergence of the perturbation theory is very doubtful. For the case of nuclear forces, however, this difficulty is less serious than that for the other cases, as far as one deals with the potential at large distances, because the range of the higher order potential decreases as the order increases. Even if the expansion in powers of the coupling constant still diverges for the nuclear forces, its reliability as the asymptotic expansion will be superior to that for the other cases. For the case of pseudovector coupl-

* The preliminary report of this paper was published by K. Nakabayasi and the present author in Phys. Rev. 88.¹⁾

ing, the fourth order S matrix suffers from unrenormalizable divergences. However, for the nuclear forces, divergences appear only in the "contact terms", and do not affect the potential at finite distances. The problem of nuclear forces will therefore furnish a case favorable for the test of merits and demerits of the pseudovector and the pseudoscalar couplings.

The fourth order meson potentials have hitherto been calculated by several authors.⁴⁾⁻⁷⁾ In these calculations, however, it has been commonly assumed that the nucleons are at rest in the intermediate states, in other words, the fourth order potential calculated until now are the adiabatic ones. However, the expression of the correct fourth order potential contains integrals of contributions of various values of recoil momenta of nucleons in the intermediate states, and, as will be shown in § 5, the adiabatic potentials are obtained by carrying out the integrations after expanding the integrands in powers of μ/M , and by taking the first few terms from the resulting power series of μ/M . But the expansions of the integrands are convergent only for a restricted region of the integration variable, so the integrations after the expansions will not always give convergent series, and will give the asymptotic expansion of the correct potential, if the series is not convergent. Therefore it is possible that the adiabatic potentials are, in the strict meaning, not the terms of the lowest order in μ/M of the correct potentials, but the asymptotic forms of the latters, and errors in the adiabatic potentials should not be estimated only by considering them to be of the higher order in μ/M . In our calculation, therefore, the recoil of nucleons in the intermediate states will be carefully taken into account, and it will be shown later that differences between the adiabatic potentials and the potentials calculated taking account of the nucleon recoil are indeed very large.

We start from the Tomonaga equation for the nucleon-meson system:

$$i\partial\Psi[\sigma]/\partial\sigma(x)=H(x)\Psi[\sigma]. \quad (1)^*$$

By a canonical transformation this equation is transformed into such form that the transformed Hamiltonian does not contain the terms not causing the real processes. Expanding the transformed Hamiltonian in powers of the coupling constant, we have from (1) the equation of the form:

$$i\partial\Phi[\sigma]/\partial\sigma(x)=\{H_2(x)+H_3(x)+H_4(x)+\dots\}\Phi[\sigma]. \quad (2)$$

The nucleon-nucleon interactions are contained in H_2, H_4, \dots . We can derive the fourth order S matrix S_4 from (2) in the usual way. We then find

$$-i\int H_4(x)d^4x=S_4-\Sigma \quad (3)$$

with

$$\Sigma=(-i)^2\iint 1/2[1+\epsilon(x-x')]H_2(x)H_2(x')d^4xd^4x'. \quad (4)$$

As the direct calculation of H_4 is very difficult, we first calculate $S_4-\Sigma$, and then define H_4 so as to make (3) hold. Therefore the nucleon-nucleon interactions thus obtained

* Throughout this paper, we take $\hbar=c=1$.

are undetermined by the terms which have non-vanishing matrix elements only between states with different energies. Since such terms can be reduced to the higher order by an appropriate canonical transformation, the ambiguity of this kind is unavoidable in the effective potentials calculated by the perturbation theory.* But the matrix elements of the undetermined terms will have a factor $E_i - E_f$, where E_i and E_f are energies of initial and final states, so they can probably be neglected in the non-relativistic approximation which will be made later.

In calculating $S_4 - \Sigma$, we expand this in powers of v/c and of μ/M , assuming that $v/c \sim \mu/M$, where v is the initial or final nucleon velocity, μ is meson mass, and M is nucleon mass. From this expansion we take the first and the second terms for the case of pseudoscalar coupling. For the pseudovector coupling, however, correct adiabatic potential is not obtained by the procedure above mentioned, provided that only the leading term and the term of the relative order v/c compared to them are picked up from $S_4 - \Sigma$, because, in the expansion in powers of μ/M occurring in the process to derive the adiabatic potential, the first two terms are cancelled, and there remain the third and the following terms, to which, however, contribute the term of the relative order $(v/c)^2$ of $S_4 - \Sigma$. Therefore we must take also the term of the relative order $(v/c)^2$ of $S_1 - \Sigma$, at least for the internucleon distances where the adiabatic potential is good approximation. For the potential at such distances, there is a maximum in those values of the nucleon momenta in the intermediate states which give significant contributions to the potential, and this maximum value is much smaller than M . As a consequence of this, the nucleon momenta in the intermediate states are treated as if they are of the order of μ , in the process to derive the adiabatic potential.** (See § 5.) For the pseudovector coupling, therefore, in addition to the leading term of the relative order v/c in $S_4 - \Sigma$, we take that part of the term of the relative order $(v/c)^2$ in $S_4 - \Sigma$ which become of the lowest order in μ/M when we regard the nucleon momenta in the intermediate states to be of the order of μ ***. Then we shall have a potential which is valid no matter whether the adiabatic potential is valid or not.

On account of the assumption that $v/c \sim \mu/M$, the nucleon-nucleon interactions calculated in this paper lose their validity for small values of the inter-nucleon distance. However it is meaningless to discuss the interactions at such small distances, as far as we neglect the higher order effects and effects of the heavier mesons.

* Araki and Huzinaga attempted to avoid this ambiguity comparing the theoretical results with the experiments for the case of the second order electron-electron and proton-electron interactions transmitted by photons. They concluded that the interactions exactly calculated by the canonical transformation give the correct results. See G. Araki and S. Huzinaga, Prog. Theor. Phys. 6, 673 (1951).

** The equation $l = u^{-1}[(M/\mu)^2(1-u)^2 + u]^{1/2}$ appears in § 5, and the equation $u = M^{-1}[\sqrt{l^2 + M^2} - l]$ appears in § 6, where l is the nucleon momentum in the intermediate states. If $l \ll M$, it follows from the latter equation that $1 - u \sim l/M$. Then the former equation gives $(\mu/M)l \sim \sqrt{l^2 + \mu^2}/M$, and, in § 5, the left side of this is treated as if it is of the order of μ/M , however large values l may take. Hence $l \sim \mu$.

*** In reference 1), we have dropped the terms of the relative order $(v/c)^2$ entirely. Therefore the conclusion of reference 1) must be altered for the case of pseudovector coupling.

§ 2. Preliminaries

We first give a proof of (3) with (4) which has been expected to hold in § 1. By a unitary transformation

$$\mathcal{P}[\sigma] = U[\sigma] \phi[\sigma],$$

(1) is transformed into

$$i\delta\phi[\sigma]/\delta\sigma(x) = H'(x)\phi[\sigma] \quad (5)$$

with

$$H'(x) = U^*[\sigma] \{H(x) - i\delta/\delta\sigma(x)\} U[\sigma]. \quad (6)$$

We expand U and H' in powers of the coupling constant as follows;

$$U = 1 + U_1 + U_2 + \cdots, \\ H' = H_1 + H_2 + H_3 + \cdots.$$

Then it follows from (6) that

$$H_n = U_{n-1}^* H + U_{n-2}^* H U_1 + \cdots + U_1^* H U_{n-2} + H U_{n-1} \\ - i\delta U_n / \delta\sigma - iU_1^* \delta U_{n-1} / \delta\sigma \\ - \cdots - iU_{n-1}^* \delta U_1 / \delta\sigma. \quad (7)$$

We also have because of the unitarity of U

$$U_n + U_n^* = -U_{n-1}^* U_1 - U_{n-2}^* U_2 - \cdots - U_1^* U_{n-1}. \quad (8)$$

Now let A be a function of the field operators. We write A as sum of products in each of which every factor is either creation or annihilation operator, and all creation operators lie on the left of all annihilation operators. After this, we define the diagonal part of A , which is denoted by A_D , as the sum of the products of two or more creation operators and two or more annihilation operators in A , and define the non-diagonal part of A , which is denoted by A_{ND} , as the remaining part of A . Thus $A = A_D + A_{ND}$.

In order to determine U , we postulate that H' does not contain the terms which do not cause the real processes. This postulate is expressed by $H'_{ND} = 0$. Hence we have

$$(H'_n)_{ND} = 0, \quad n = 1, 2, 3, \cdots. \quad (9)$$

Using (7), (8) and (9), we can determine $H_1, U_1, H_2, U_2, \cdots$ successively. For example, (7) gives

$$H_1 = H - i\delta U_1 / \delta\sigma$$

But $H_{ND} = H$, hence we get from (9)

$$i\delta U_1[\sigma] / \delta\sigma(x) = H(x), \quad H_1(x) = 0.$$

Integrating the first of these equations, we have

$$U_1[\sigma] = -(i/2) \int \epsilon(\sigma, x') H(x') d^4 x'.$$

Proceeding in this way, we find for H_2 and H_4 in which we are interested that

$$H_2(x) = -(i/4) \int \epsilon(x - x') [H(x), H(x')]_D d^4 x', \quad (10)$$

$$\begin{aligned}
H_4(x) = & (i/64) \int d^4x' d^4x'' d^4x''' \\
& \times \{ \epsilon(x-x') \epsilon(x-x'') \epsilon(x-x''') ([[[H, H'], H''] + [[H, H']_{ND}, H'']_{ND} \\
& + 2(H'HH'' - HH'H'' - H''H'H')_{ND}, H'''] + 2H''[H, H']_{ND}H''') \\
& + \epsilon(x-x') \epsilon(x-x'') \epsilon(x''-x''') ([2[H, H'] - [H, H']_{ND}, [H'', H''']_{ND}] \\
& + 2H[H'', H''']_{ND}H' + 2H'[H'', H''']_{ND}H - [[H, [H'', H''']_{ND}]_{ND}, H']) \\
& + \epsilon(x-x') \epsilon(x'-x'') \epsilon(x'-x''') \\
& \times [[[H', H'']_{ND}, H''']_{ND} + 2(H''H'H''' - H'H''H''' - H'''H''H')_{ND}, H] \\
& + \epsilon(x-x') \epsilon(x'-x'') \epsilon(x''-x''') [H, [H', [H'', H''']_{ND}]_{ND}] \}, \quad (11)
\end{aligned}$$

where we have put for brevity $H(x) = H$, $H(x') = H'$, etc. Making use of the properties of the ϵ -function and the equations

$$\begin{aligned}
\int H(x) d^4x &= 0, \\
\iint \epsilon(x-x') [H(x), H(x')]_{ND} d^4x d^4x' &= 0
\end{aligned}$$

which follow from the energy-momentum conservation, it is proved without difficulty that

$$\begin{aligned}
& -i \int H_4(x) d^4x - (1/2) \iint [1 + \epsilon(x-x')] H_2(x) H_2(x') d^4x d^4x' \\
& = (-i)^4 (1/4!) \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 P(H(x_1) H(x_2) H(x_3) H(x_4)) = S_4.
\end{aligned}$$

Thus (3) with (4) has been established.

We next show how the nucleon-nucleon interactions are obtained from $S_4 - \Sigma$. We denote the nucleon field operators in the interaction representation by $\psi(x)$ and $\bar{\psi}(x)$, their positive frequency parts by $\psi^{(+)}(x)$ and $\bar{\psi}^{(+)}(x)$, and negative ones by $\psi^{(-)}(x)$ and $\bar{\psi}^{(-)}(x)$. And their Fourier transforms are defined by

$$\begin{aligned}
\psi^{(\pm)}(x) &= \int \psi^{(\pm)}(p) \exp(ip_v x_v) d^4p, \\
\bar{\psi}^{(\pm)}(x) &= \int \bar{\psi}^{(\pm)}(p) \exp(-ip_v x_v) d^4p.
\end{aligned} \quad (12)$$

Then the two-nucleon part of $S_4 - \Sigma$ is expressed by sum of the terms of the form

$$\begin{aligned}
& i \int d^4p_1 d^4q_1 d^4p_2 d^4q_2 \delta^3(p_1 + p_2 - q_1 - q_2) \\
& \times f(p_1, q_1; p_2, q_2) \bar{\psi}^{(-)}(p_1) O\psi^{(+)}(q_1) \bar{\psi}^{(-)}(p_2) O\psi^{(+)}(q_2). \quad (13)
\end{aligned}$$

In accordance with what was mentioned in § 1, we assume that the term corresponding to (13) in $\int H_4(x) d^4x$ with $x_0 = t$ is given by

$$\begin{aligned}
& - (2\pi)^{-1} \int d^4p_1 d^4q_1 d^4p_2 d^4q_2 \delta^3(p_1 + p_2 - q_1 - q_2) \\
& \times \exp[i(p_{10} + p_{20} - q_{10} - q_{20})t] f(p_1, q_1; p_2, q_2) \\
& \times \bar{\psi}^{(-)}(p_1) O\psi^{(+)}(q_1) \bar{\psi}^{(-)}(p_2) O\psi^{(+)}(q_2). \quad (14)
\end{aligned}$$

Then (3) is obviously satisfied.

Now we omit from the transformed Hamiltonian H' all its parts other than the two-nucleon part. Then (5) yields an equation describing the motion of the two-nucleon system, and this equation is written in the form

$$\begin{aligned} i\partial\psi(t, \mathbf{r}_1, \mathbf{r}_2)/\partial t \\ = [-i\alpha^{(1)}\nabla^{(1)} - i\alpha^{(2)}\nabla^{(2)} + (\beta^{(1)} + \beta^{(2)})M]\psi(t, \mathbf{r}_1, \mathbf{r}_2) \\ + \int (\mathbf{r}_1, \mathbf{r}_2 | V | \mathbf{r}_1', \mathbf{r}_2') \psi(t, \mathbf{r}_1', \mathbf{r}_2') d^3\mathbf{r}_1' d^3\mathbf{r}_2'. \end{aligned} \quad (15)$$

According to the author's previous note³⁾ on the relation between the configuration space and the second quantization formulations of the quantized field theory, the contribution of (14) to $(\mathbf{r}_1, \mathbf{r}_2 | V | \mathbf{r}_1', \mathbf{r}_2')$ is found to be

$$\begin{aligned} - (2\pi)^{-13} \int d^3\mathbf{p}_1 d^3\mathbf{q}_1 d^3\mathbf{p}_2 d^3\mathbf{q}_2 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{q}_1 - \mathbf{q}_2) \\ \times \exp[i(\mathbf{p}_1\mathbf{r}_1 + \mathbf{p}_2\mathbf{r}_2 - \mathbf{q}_1\mathbf{r}_1' - \mathbf{q}_2\mathbf{r}_2')] \\ \times [f(\mathbf{p}_1, \mathbf{q}_1; \mathbf{p}_2, \mathbf{q}_2) + f(\mathbf{p}_2, \mathbf{q}_2; \mathbf{p}_1, \mathbf{q}_1)] \\ \times A^{(1)}(\mathbf{p}_1)\beta^{(1)}O^{(1)}A^{(1)}(\mathbf{q}_1)A^{(2)}(\mathbf{p}_2)\beta^{(2)}O^{(2)}A^{(2)}(\mathbf{q}_2) \end{aligned} \quad (16)$$

with the abbreviations

$$A(\mathbf{p}) = (2E_p)^{-1}(E_p + \alpha\mathbf{p} + \beta M), \quad E_p = \sqrt{\mathbf{p}^2 + M^2},$$

where $f(\mathbf{p}_1, \mathbf{q}_1; \mathbf{p}_2, \mathbf{q}_2)$ is what is obtained by replacing $p_{10}, q_{10}, p_{20}, q_{20}$ by $E_{p_1}, E_{q_1}, E_{p_2}, E_{q_2}$ in $f(p_1, q_1; p_2, q_2)$. By virtue of the factor $\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{q}_1 - \mathbf{q}_2)$ contained in (16), we can separate from (15) an equation describing the relative motion of the nucleons. This is given by

$$\begin{aligned} i\partial\psi(t, \mathbf{r})/\partial t = [-i(\alpha^{(1)} - \alpha^{(2)})\nabla + (\beta^{(1)} + \beta^{(2)})M]\psi(t, \mathbf{r}) \\ + \int (\mathbf{r} | V | \mathbf{r}') \psi(t, \mathbf{r}') d^3\mathbf{r}', \end{aligned}$$

where

$$(\mathbf{r} | V | \mathbf{r}') = \int \left(\mathbf{R} + \frac{\mathbf{r}}{2}, \mathbf{R} - \frac{\mathbf{r}}{2} | V | \mathbf{R}' + \frac{\mathbf{r}'}{2}, \mathbf{R}' - \frac{\mathbf{r}'}{2} \right) d^3\mathbf{R}'. \quad (17)$$

Using (16) and (17), we find that the term (13) in $S_4 - \Sigma$ contributes to the nucleon-nucleon interaction $(\mathbf{r} | V | \mathbf{r}')$ the following term:

$$\begin{aligned} - (2\pi)^{-10} \int d^3\mathbf{p} d^3\mathbf{q} \exp[i(\mathbf{p}\mathbf{r} - \mathbf{q}\mathbf{r}')] \\ \times [f(\mathbf{p}, \mathbf{q}; -\mathbf{p}, -\mathbf{q}) + f(-\mathbf{p}, -\mathbf{q}; \mathbf{p}, \mathbf{q})] \\ \times A^{(1)}(\mathbf{p})\beta^{(1)}O^{(1)}A^{(1)}(\mathbf{q})A^{(2)}(-\mathbf{p})\beta^{(2)}O^{(2)}A^{(2)}(-\mathbf{q}). \end{aligned} \quad (18)$$

§ 3. Calculations of the fourth order nucleon-nucleon interactions for the case of the pseudovector coupling

In this case, the interaction Hamiltonian H in (1) is given by

$$H(x) = \sum_{\sigma=\pm}^4 i(f_\sigma/\mu) \bar{\psi}(x) \gamma_5 \gamma_\nu \tau_\sigma \psi(x) \partial \varphi_\sigma(x) / \partial x_\nu. \quad (19)$$

Since the normal dependent term has been omitted from the interaction Hamiltonian, the following replacement must be done in calculating the S matrix:

$$\left\langle P \left(\frac{\partial \varphi_0(x)}{\partial x_\mu} \frac{\partial \varphi_\tau(x')}{\partial x'_\nu} \right) \right\rangle_{\text{vac}} \rightarrow \frac{1}{2} \delta_{0\tau} \frac{\partial^2}{\partial x_\mu \partial x'_\nu} \Delta_F(x-x').$$

Hereafter we denote the two-nucleon parts of S_1 and Σ by the same notations S_2 and Σ . Then S_2 is expressed by sum of contributions of the graphs A, B, C, \dots in Fig. 1.

$$S_2 = S_A + S_B + S_C + S_D + S_E.$$

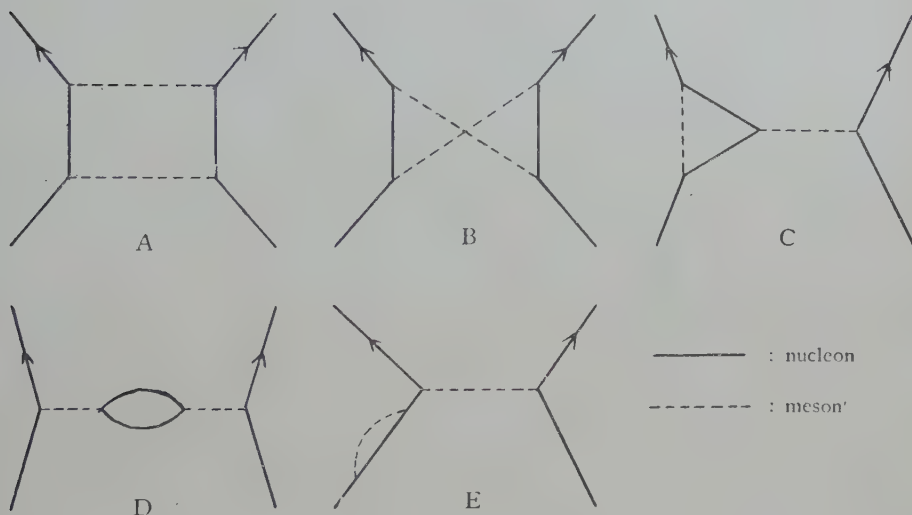


Fig. 1

Since S_E is obvious to be eliminated by the renormalization, we leave this out of consideration. Since Σ also corresponds to the graph A , we calculate S_A and $-\Sigma$ in block. As was mentioned in § 1, we expand $S_A - \Sigma$, S_B, \dots in powers of v/c and of μ/M assuming that $v/c \sim \mu/M$, and take the first few terms from each of these expansions according to the plan of approximation explained in § 1. After this the nucleon-nucleon interaction is derived from these terms by the method explained in § 2.

The result is as follows:

The fourth order nucleon-nucleon interaction V_4 is given by

$$V_4 = V_A + V_B + V_C \quad (20)$$

where V_A , V_B and V_C are the contributions of $S_A - \Sigma$, S_B and S_C respectively. S_D will be found to give no contribution.

$$V_A = \mu \tau_A \{ U_A^0(x) + U_A^3(x) (\sigma^{(1)} \cdot \sigma^{(2)}) + U_A^4(x) S_{12} + (\sigma^{(1)} \cdot \nabla) U_A^5(x) (\sigma^{(1)} \cdot \nabla) + (\sigma^{(2)} \cdot \nabla) U_A^6(x) (\sigma^{(2)} \cdot \nabla) \}, \quad (21)$$

$$\begin{aligned}
\pi U_A^c(x) = & \lambda^{-4} x^{-1} \int_0^1 du \{ 8(1-u^2)u^{-1} \\
& + 2(1-u^2)^2(1+4u^2+u^4)(1+u^2)^{-2}[2u^{-4} + (1-u)^2u^{-5}] \\
& + 8g(u)[u^{-6} + 8(1-u^2)(1+u^2)^{-1}] + (1-u^2)^4(1+u^2)^{-1}u^{-5} \\
& + (1-u)^2g(u)u^{-7}[4 + (1+u)^2 + 2(1+u)^2(1+u^2)^{-3}] \} L_0(u, x) \\
& + 4\lambda^{-3} x^{-2} \int_0^1 du [g(u)]^{1/2} L_1(u, x) \\
& - [8\lambda^{-2} x^{-2} + 12x^{-2} + (37/2)x^{-4}] K_1(2x) \\
& + [2x^{-1} - (37/2)x^{-3}] K_0(2x), \tag{22}
\end{aligned}$$

$$\pi U_A^s(x) = 4x^{-3} K_0(2x) + [(8/3)x^{-2} + 4x^{-4}] K_1(2x), \tag{23}$$

$$\pi U_A^t(x) = -4x^{-3} K_0(2x) - [(4/3)x^{-2} + 5x^{-4}] K_1(2x), \tag{24}$$

$$\pi U_B^v(x) = 4\lambda^{-3} x^{-1} \int_0^1 du u^{-1}(1-u) L_0(u, x) - 2x^{-2} K_1(2x), \tag{25}$$

$$\begin{aligned}
V_R = & \mu\tau_B \{ U_B^s(x) + U_B^s(x) (\sigma^{(1)} \cdot \sigma^{(2)}) + U_B^t(x) S_{12} \\
& + (\sigma^{(1)} \cdot \mathcal{P}) U_B^v(x) (\sigma^{(1)} \cdot \mathcal{P}) + (\sigma^{(2)} \cdot \mathcal{P}) U_B^v(x) (\sigma^{(2)} \cdot \mathcal{P}) \}, \tag{26}
\end{aligned}$$

$$\begin{aligned}
\pi U_B^c(x) = & 8\lambda^{-4} x^{-1} \int_0^1 du (1-u) u^{-1} \{ 4-u-g(u)u^{-2} \\
& + (4/3)\lambda^{-2} x^2(1-u)^4 u^{-4} + (1/3)(1-u)^2 u^{-2} \} L_0(u, x) \\
& - 4\lambda^{-5} \int_0^1 du u^{-4} \{ 4(1-u)^3 [g(u)]^{-1/2} [u^2 + (1/3)(1-u)^2] \\
& + [g(u)]^{1/2} [\lambda^2 x^{-2} u^2 (u^2 + 1-u) + (4/3)(1-u)^3] \} L_1(u, x) \\
& - [4\lambda^{-2} x^{-2} - (1/2)x^{-4}] K_1(2x) + (1/2)x^{-3} K_0(2x), \tag{27}
\end{aligned}$$

$$\begin{aligned}
\pi U_B^s(x) = & (16/3)\lambda^{-4} x^{-1} \int_0^1 du u^{-3}(1-u) g(u) L_0(u, x) \\
& + (8/3)\lambda^{-3} x^{-2} \int_0^1 du u^{-2}(1-u) [g(u)]^{1/2} L_1(u, x), \tag{28}
\end{aligned}$$

$$\begin{aligned}
\pi U_B^t(x) = & -2\lambda^{-2} x^{-3} \int_0^1 du u^{-1}(1-u) L_0(u, x) \\
& - (8/3)\lambda^{-4} x^{-1} \int_0^1 du u^{-3}(1-u) g(u) L_0(u, x) \\
& - (16/3)\lambda^{-3} x^{-2} \int_0^1 du u^{-2}(1-u) [g(u)]^{1/2} L_1(u, x), \tag{29}
\end{aligned}$$

$$\pi U_B^v(x) = 4\lambda^{-3} \int_0^1 du u^{-2}(1-u)^3 [g(u)]^{-1/2} L_1(u, x)$$

$$-2\lambda^{-2}x^{-1}\int_0^1 du u^{-1}(1-u)L_0(u, x) = x^{-2}K_1(2x), \quad (30)$$

$$V_C = -\mu\tau_C(18\pi)^{-1}\{(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + S_{12}(1+3x^{-1}+3x^{-2})\}x^{-1}e^{-x}. \quad (31)$$

In the above expressions the following abbreviations have been used :

$$\begin{aligned} \tau_A &= \sum_{\rho, \sigma=1}^4 (f_\rho^2/4\pi) (f_\sigma^2/4\pi) \tau_\rho^{(1)} \tau_\sigma^{(1)} \tau_\rho^{(2)} \tau_\sigma^{(2)}, \\ \tau_B &= \sum_{\rho, \sigma=1}^4 (f_\rho^2/4\pi) (f_\sigma^2/4\pi) \tau_\rho^{(1)} \tau_\sigma^{(1)} \tau_\rho^{(2)} \tau_\sigma^{(2)}, \\ \tau_C &= \sum_{\rho, \sigma=1}^4 (f_\rho^2/4\pi) (f_\sigma^2/4\pi) \tau_\rho^{(1)} \tau_\sigma^{(1)} \tau_\rho^{(1)} \tau_\sigma^{(2)}, \\ g(u) &= (1-u)^2 + \lambda^2 u, \quad \lambda = \mu/M, \\ L_n(u, x) &= K_n(2xu^{-1}\sqrt{g(u)}\lambda^{-1}), \quad n=0, 1, \\ x &= |x|, \quad \nabla = \partial/\partial x, \quad x = \mu r, \end{aligned}$$

r = vector from one of the nucleons to the other.

In the following, the contributions of $S_A - \Sigma$, S_B , ... will be calculated separately.

(A) Contribution of $S_A - \Sigma$

Notations : Throughout this paper, we denote the inner products $a_\mu b_\mu$ and $a_\mu a_\mu$ by (a, b) and a^2 for any four-vectors a and b . On the other hand, the following notations are used only in section (A) :

$$\begin{aligned} \langle A \rangle_1 \langle B \rangle_2 &= \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \bar{\psi}(p_1) A \tau_\rho \tau_\sigma \psi(q_1) \bar{\psi}(p_2) B \tau_\rho \tau_\sigma \psi(q_2), \\ \langle A \rangle_+ \langle B \rangle_- &= \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \bar{\psi}(\mathbf{p}, E_p) A \tau_\rho \tau_\sigma \psi(\mathbf{q}, E_q) \bar{\psi}(-\mathbf{p}, E_p) B \tau_\rho \tau_\sigma \psi(-\mathbf{q}, E_q), \\ \langle A \rangle_+^{(1)} \langle B \rangle_-^{(2)} &= \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \tau_\rho^{(1)} \tau_\sigma^{(1)} \tau_\rho^{(2)} \tau_\sigma^{(2)} A^{(1)}(\mathbf{p}) \beta^{(1)} A^{(1)}(\mathbf{q}) A^{(2)}(-\mathbf{p}) \beta^{(2)} B^{(2)} A^{(2)}(-\mathbf{q}). \end{aligned}$$

We begin with the calculation of Σ . We find that the use of the two-nucleon part of H_2 in place of H_2 itself is sufficient for calculating Σ by (4), as far as we are interested in the two-nucleon part of Σ . The two-nucleon part of $H_2(x)$ is obtained by insertion of (19) into (10), and this is found to be

$$\begin{aligned} &-(1/2)\sum_\sigma (f_\sigma^2/\mu^2) \int d^4x' \bar{A}_{\mu\nu}(x-x') \\ &\times \bar{\psi}^{(-)}(x) \gamma_5 \gamma_\mu \tau_\sigma \psi^{(+)}(x) \bar{\psi}^{(-)}(x') \gamma_5 \gamma_\nu \tau_\sigma \psi^{(+)}(x'), \end{aligned}$$

where

$$\bar{A}_{\mu\nu}(x) = \partial^\nu \bar{A}(x) / \partial x_\mu \partial x_\nu.$$

Substituting this for H_2 in (4), we have

$$\begin{aligned} \Sigma &= (1/4)\mu^{-4} \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \int d^4x d^4x' d^4x'' d^4x''' \\ &\times (1/2)[1 + \epsilon(x-x'')] \bar{A}_{\mu\nu}(x-x') \bar{A}_{\lambda\omega}(x''-x''') \\ &\times \{\bar{\psi}^{(-)}(x) \gamma_5 \gamma_\mu S^{(+)}(x-x'') \gamma_5 \gamma_\lambda \tau_\rho \tau_\sigma \psi^{(+)}(x'') \\ &\times \bar{\psi}^{(-)}(x') \gamma_5 \gamma_\nu S^{(+)}(x'-x''') \gamma_5 \gamma_\omega \tau_\rho \tau_\sigma \psi^{(+)}(x''') \end{aligned}$$

$$+ \bar{\psi}^{(-)}(x) \gamma_5 \gamma_\mu S^{(+)}(x-x''') \gamma_5 \gamma_\nu \tau_\rho \tau_\sigma \psi^{(+)}(x''') \\ \times \bar{\psi}^{(-)}(x') \gamma_5 \gamma_\nu S^{(+)}(x'-x'') \gamma_5 \gamma_\lambda \tau_\mu \tau_\sigma \psi^{(+)}(x'') \}.$$

Substituting the Fourier representations for $\bar{\psi}^{(-)}$, $\psi^{(+)}$, $S^{(+)}$ and \bar{J} , and making some rearrangement, we obtain

$$\Sigma = \Sigma_1 + \Sigma_2, \quad (32)$$

$$\begin{aligned} \Sigma_1 = & i 8 \pi (M/\mu)^4 \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) \\ & \times \int d^4 k d^4 l (1/2) [1 + \epsilon(k)] \delta(k^2 + M^2) (1/2) [1 + \epsilon(l)] \delta(l^2 + M^2) \\ & \times \delta^3(q_1 + q_2 - k - l) (k_0 + l_0 - q_{10} - q_{20} - i\epsilon)^{-1} \\ & \times \{[(l - q_2)^2 + \mu^2]^{-1} + [(k - q_1)^2 + \mu^2]^{-1}\} \\ & \times [(l - p_2)^2 + \mu^2]^{-1} \langle i(\gamma, k) + M \rangle_1 \langle i(\gamma, l) + M \rangle_2, \end{aligned} \quad (33)$$

$$\begin{aligned} \Sigma_2 = & i 2 \pi (M/\mu)^2 \mu^{-2} \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) \\ & \times \int d^4 k d^4 l (1/2) [1 + \epsilon(k)] \delta(k^2 + M^2) (1/2) [1 + \epsilon(l)] \delta(l^2 + M^2) \\ & \times \delta^3(q_1 + q_2 - k - l) [(l - p_2)^2 + \mu^2]^{-1} \\ & \times \{[(l - q_2)^2 + \mu^2]^{-1} [(k_0 + l_0 - q_{10} - q_{20}) \langle \gamma_4 (i(\gamma, k) + M) \gamma_4 \rangle_1 \\ & + 2M \langle \gamma_4 (i(\gamma, k) + M) \rangle_1 + 2M \langle i(\gamma, k) + M \rangle \gamma_4 \rangle_1] \langle i(\gamma, l) + M \rangle_2 \\ & + [(k - q_1)^2 + \mu^2]^{-1} [(k_0 + l_0 - q_{10} - q_{20}) \langle \gamma_4 (i(\gamma, l) + M) \gamma_4 \rangle_1 \\ & + 2M \langle \gamma_4 (i(\gamma, l) + M) \rangle_1 + 2M \langle i(\gamma, l) + M \rangle \gamma_4 \rangle_1] \langle i(\gamma, k) + M \rangle_2 \\ & + 2M \langle \gamma_4 (i(\gamma, k) + M) \rangle_1 \langle i(\gamma, l) + M \rangle_2 \} \}. \end{aligned} \quad (34)$$

In obtaining these, we have used the Dirac equations

$$[i(\gamma, p) + M] \psi(p) = 0, \quad \bar{\psi}(p) [i(\gamma, p) + M] = 0.$$

Next we calculate S_A . S_A is immediately written down by the well known rules as

$$\begin{aligned} S_A = & (1/2) \mu^{-4} \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) \\ & \times \int d^4 l (l^2 + M^2)^{-1} [(l - p_1 - p_2)^2 + M^2]^{-1} \\ & \times [(l - p_2)^2 + \mu^2]^{-1} [(l - q_2)^2 + \mu^2]^{-1} \\ & \times \langle \gamma_5(\gamma, p_2 - l) [i(\gamma, p_1 + p_2 - l) - M] \gamma_5(\gamma, q_2 - l) \rangle_1 \\ & \times \langle \gamma_5(\gamma, p_2 - l) [i(\gamma, l) - M] \gamma_5(\gamma, q_2 - l) \rangle_2, \end{aligned}$$

which can be rewritten by the use of the Dirac equations as

$$S_A = S_{A1} + S_{A2}, \quad (35)$$

$$\begin{aligned} S_{A1} = & 8(M/\mu)^4 \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) \\ & \times \int d^4 l (l^2 + M^2)^{-1} [(l - p_1 - p_2)^2 + M^2]^{-1} \\ & \times [(l - p_2)^2 + \mu^2]^{-1} [(l - q_2)^2 + \mu^2]^{-1} \\ & \times \langle i(\gamma_1 p_2) - i(\gamma, l) \rangle_1 \langle i(\gamma, l) + M \rangle_2, \end{aligned} \quad (36)$$

$$\begin{aligned}
S_{A2} = & (1/2)\mu^{-4} \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) \\
& \times \int d^4 k (k^2 + \mu^2)^{-1} [(k + p_1 - q_1)^2 + \mu^2]^{-1} \\
& \times \{8M^2 [(k + p_1)^2 + M^2]^{-1} \langle i(\gamma, k) \rangle_1 \\
& - \langle i(\gamma, k) + 2M \rangle_1 \} \langle i(\gamma, k) - 2M \rangle_2. \quad (37)
\end{aligned}$$

The factor $(l^2 + M^2)^{-1}$ appearing in (36) contains the imaginary part $\pi\delta(l^2 + M^2)$ because of the small negative imaginary part contained in M . Another factor $[(l - p_1 - p_2)^2 + M^2]^{-1}$, when multiplied by $\delta(l^2 + M^2)$, becomes $(p_1 + p_2, p_1 + p_2 - 2l)^{-1}$, which further becomes $(p_{10} + p_{20})^{-1} (p_{10} + p_{20} - 2l_0)^{-1}$ in the center of mass system. Therefore, in the expression of S_{A1} , the integrand has a pole at the energy of the intermediate state equal to the energy of the initial state. Observing (33), we see that this pole disappears when Σ_1 is subtracted from S_{A1} . In the following we calculate $S_{A1} - \Sigma_1$, S_{A2} and $-\Sigma_2$ separately.

(A₁) Contribution of $S_{A1} - \Sigma_1$

In the expression of S_{A1} (36), we first perform the integration over l_0 , the energy component of the four-vector l , remembering that M and μ have small negative imaginary part. Next we perform the integrations over k_0 and l_0 in the expression of Σ_1 (33), and subtract Σ_1 from S_{A1} . Then we have

$$S_{A1} - \Sigma_1 = i \int d^3 p_1 d^3 q_1 d^3 p_2 d^3 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) f_{A1}(p_1, q_1; p_2, q_2), \quad (38)$$

$$\begin{aligned}
f_{A1}(p, q; -p, -q) = & f'_{A1}(p, q; -p, -q) \\
& + f''_{A1}(p, q; -p, -q) + f'''_{A1}(p, q; -p, -q), \quad (39)
\end{aligned}$$

$$\begin{aligned}
f'_{A1}(p, q; -p, -q) = & -2\pi(M/\mu)^4 E_p^{-1} \\
& \times \int d^3 l E_l^{-2} [2E_l E_p + 2(l \cdot p) - 2M^2 + \mu^2]^{-1} \\
& \times [2E_l E_q + 2(l \cdot q) - 2M^2 + \mu^2]^{-1} \\
& \times \{E_l^2 \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- \rightarrow M^2 \langle 1 \rangle_+ \langle 1 \rangle_- + M \langle i(\gamma \cdot l) \rangle_+ \langle 1 \rangle_- \\
& - M \langle 1 \rangle_+ \langle i(\gamma \cdot l) \rangle_- + \langle i(\gamma \cdot l) \rangle_+ \langle i(\gamma \cdot l) \rangle_- \}, \quad (40)
\end{aligned}$$

$$\begin{aligned}
f''_{A1}(p, q; -p, -q) = & 4\pi(M/\mu)^4 \int d^3 l (l \cdot p - q)^{-1} \\
& \times \{\epsilon_{l+q}^{-1} [2E_q \epsilon_{l+q} + 2(q \cdot l + q) + \mu^2]^{-1} [2E_q \epsilon_{l+q} - 2(q \cdot l + q) - \mu^2]^{-1} \\
& \times [\epsilon_{l+q}^2 \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- + \langle i(\gamma \cdot l + q) \rangle_+ \langle i(\gamma \cdot l + q) \rangle_-] \\
& + \text{what is obtained from the preceding term by writing } -p \text{ in place of } q, \quad (41)
\end{aligned}$$

$$\begin{aligned}
f'''_{A1}(p, q; -p, -q) = & 2\pi(M/\mu)^4 E_p^{-1} \\
& \times \int d^3 l E_l^{-1} (E_l + E_p)^{-1} [2E_l E_p - 2(l \cdot p) + 2M^2 - \mu^2]^{-1} \\
& \times [2E_l E_q - 2(l \cdot q) + 2M^2 - \mu^2]^{-1} \{E_l (E_l + 2E_p) \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- \\
& - M^2 \langle 1 \rangle_+ \langle 1 \rangle_- + M E_p [\langle \gamma_4 \rangle_+ \langle 1 \rangle_- + \langle 1 \rangle_+ \langle \gamma_4 \rangle_-] \}.
\end{aligned}$$

$$\begin{aligned}
& + \langle E_p \gamma_4 - M \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_- - \langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_+ \langle E_p \gamma_4 - M \rangle_- \\
& + \langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_-,
\end{aligned} \quad (42)$$

where $E_l = \sqrt{l^2 + M^2}$, $\epsilon_k = \sqrt{k^2 + \mu^2}$. In obtaining these, we have used the Dirac equations and the relation $E_p = E_q$. f'_{A1} has been obtained by subtracting Σ_1 from the contribution of the pole due to the factor $(l^2 + M^2)^{-1}$ in (36) to S_{A1} . f''_{A1} is sum of the contributions to S_{A1} from the poles due to the factors $[(l - p_2)^2 + \mu^2]^{-1}$ and $[(l - q_2)^2 + \mu^2]^{-1}$ in (36), and f'''_{A1} is the contribution corresponding to the factor $[(l - p_1 - p_2)^2 + M^2]^{-1}$.

All of the integrals in (40)–(42) are divergent. But, if we cut off these integrals at $l \sim \mu$, the magnitudes of f'_{A1} , f''_{A1} , and f'''_{A1} are estimated as

$$\left. \begin{aligned} f'_{A1} &\sim f^4 M \mu^{-3}, \\ f''_{A1} &\sim f^4 M^2 \mu^{-4}, \\ f'''_{A1} &\sim f^4 M^{-1} \mu^{-1}. \end{aligned} \right\} \quad (43)$$

Therefore we take the first two terms from the expansion of f'_{A1} in powers of v/c , and take the first three terms from that of f''_{A1} , according to the plan of approximation explained in § 1. We neglect f'''_{A1} , because it is found to yield only the nucleon-nucleon interaction whose range is $(2M)^{-1}$.

Contribution of f'_{A1} . We neglect the last term between the curly brackets in the expression of f'_{A1} (40), since its ratio to the remaining terms is of the order of $(v/c)^2$ for all values of \boldsymbol{l} . Making use of the relation $E_p = E_q$, we can rewrite (40) as

$$\begin{aligned}
f'_{A1}(\boldsymbol{p}, \boldsymbol{q}; -\boldsymbol{p}, -\boldsymbol{q}) &= -2\pi(M/\mu)^4 E_p^{-1} \\
&\times \int_0^1 dv \int d^3 l E_l^{-2} [2E_l E_p + 2(\boldsymbol{l} \cdot \boldsymbol{J}) - 2M^2 + \mu^2]^{-2} \\
&\times \{E_l^2 \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- - M^2 \langle 1 \rangle_+ \langle 1 \rangle_- \\
&+ M[\langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_+ \langle 1 \rangle_- - \langle 1 \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \boldsymbol{l}) \rangle_-]\},
\end{aligned}$$

where

$$\boldsymbol{J} = \boldsymbol{p}v + \boldsymbol{q}(1-v).$$

Performing the integration over the directions of \boldsymbol{l} , and using the relations

$$\langle i(\boldsymbol{\gamma} \cdot \boldsymbol{J}) \rangle_{\pm} = \pm \langle E_p \gamma_4 - M \rangle_{\pm}$$

which follows from the Dirac equations, we have

$$\begin{aligned}
f'_{A1}(\boldsymbol{p}, \boldsymbol{q}; -\boldsymbol{p}, -\boldsymbol{q}) &= -2\pi^2(M/\mu)^4 \\
&\times \int_0^1 dv \{G \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- + H(\langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- - \langle 1 \rangle_+ \langle 1 \rangle_-) \\
&- K[(E_p/M)(\langle 1 \rangle_+ \langle \gamma_4 \rangle_- + \langle \gamma_4 \rangle_+ \langle 1 \rangle_-) - 2\langle 1 \rangle_+ \langle 1 \rangle_-]\}, \quad (44)
\end{aligned}$$

with

$$G = E_p^{-1} f^{-1} \int_{-\infty}^{\infty} dl l D^{-1}(1 - M^2 E_l^{-2}) \quad (45)$$

$$H = M^2 E_p^{-1} J^{-1} \int_{-\infty}^{\infty} dl \, l \, E_l^{-2} D^{-1}$$

$$K = M^2 E_p^{-1} J^{-2} \int_{-\infty}^{\infty} dl \, l \, E_l^{-2} [(2J)^{-1} \log D + l D^{-1}],$$

where

$$D = 2E_p E_l - 2lJ - 2J - 2M^2 + \mu^2. \quad (46)$$

The method of the calculations of G , H and K will be illustrated in detail in Appendix I. The outline of the method is as follows: We transform the integration variable l into u by means of the equations

$$\left. \begin{aligned} E_l + l &= (E_p + J)u, & \text{when } -\infty < l < 0, \\ E_l - l &= (E_p - J)u, & \text{when } 0 < l < \infty. \end{aligned} \right\} \quad (47)$$

After this we expand the integrands in powers of v/c and of μ/M , noting that $J/M \sim v/c$. Finally we have

$$G = \int_0^1 du \, \Delta^{-1} \{ (1-u^2)^2 (1+4u^2+u^4) u^{-2} (1+u^2)^{-2} \\ + 2g(u) [u^{-4} + 8u^2 (1-u^2) (1+u^2)^{-4} + 32(p/M)^2 u^4 (1+u^2)^{-4}] \\ + (J/M)^2 \Delta_0^{-1} [(J/M)^2 - (p/M)^2], \quad (48)$$

$$H = 8 \int_0^1 du \, \Delta^{-1} u^2 (1+u^2)^{-2} + \Delta_0^{-1} [(J/M)^2 - (p/M)^2],$$

$$K = 4 \int_0^1 du \, \Delta^{-1} u^3 (1+u^2)^{-1} \{ 1 + 2(1-u^2) (1+u^2)^{-1} \} \\ + \Delta_0^{-1} [(J/M)^2 - (p/M)^2],$$

where

$$\Delta = M^2 g(u) + k^2 u^2 v (1-v), \quad g(u) = (1-u)^2 + k^2 u, \quad (49)$$

$$\Delta_0 = \mu^2 + k^2 v (1-v), \quad (50)$$

$$k = |\mathbf{p} - \mathbf{q}|, \quad p = |\mathbf{p}| = |\mathbf{q}|.$$

Comparing (13) with (18), and observing (38) and (44), the contribution of f'_{A1} to $\langle \mathbf{r} | V | \mathbf{r}' \rangle$ is found to be

$$(2\pi)^{-3} (M/\mu)^4 \int d^3 p \, d^3 q \exp[i(\mathbf{p}\mathbf{r} - \mathbf{q}\mathbf{r}')] \\ \times \int_0^1 dv \{ G \langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} + H (\langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} - \langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}) \\ - K [(E_p/M) (\langle 1 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} + \langle \gamma_4 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}) - 2 \langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}] \}. \quad (51)$$

But, we find that, to our approximation,

$$\begin{aligned}
\langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} &= (4\pi)^2 \tau_A, \\
\langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} - \langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)} &= (4\pi)^2 \tau_A \sigma(\mathbf{p}, \mathbf{q}), \\
(E_p/M) (\langle 1 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} + \langle \gamma_4 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)} - 2 \langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}) \\
&= (4\pi)^2 \tau_A [(\mathbf{p}/M)^2 + \sigma(\mathbf{p}, \mathbf{q})],
\end{aligned}$$

where

$$\sigma(\mathbf{p}, \mathbf{q}) \equiv (2M^2)^{-1} [(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{p})(\boldsymbol{\sigma}^{(1)} \cdot \mathbf{q}) + (\boldsymbol{\sigma}^{(2)} \cdot \mathbf{p})(\boldsymbol{\sigma}^{(2)} \cdot \mathbf{q})]. \quad (52)$$

Then, by the use of the last expressions of G , H and K , (51) is changed into

$$\begin{aligned}
&4(2\pi)^{-6} \tau_A (M/\mu)^4 \int d^3\mathbf{p} d^3\mathbf{q} \exp[i(\mathbf{p}\mathbf{r} - \mathbf{q}\mathbf{r}')] \\
&\times \int_0^1 dv \left\{ \int_0^1 du \Delta^{-1} [(1-u^2)^2 (1+4u^2+u^4) u^{-2} (1+u^2)^{-2} \right. \\
&+ 2g(u) u^{-4} + 16u^2 (1-u^2) g(u) (1+u^2)^{-4}] + \mathcal{D}_0^{-1} M^{-4} (J^2 - \mathbf{p}^2)^2 \\
&+ 4(\mathbf{p}/M)^2 \int_0^1 du \Delta^{-1} u^3 (1+u^2)^{-1} [8u(1+u^2)^{-3} - 1 - 2(1-u^2)(1+u^2)^{-1}] \\
&\left. + 4\sigma(\mathbf{p}, \mathbf{q}) \int_0^1 du \Delta^{-1} u^2 (1+u^2)^{-1} [2(1+u^2)^{-1} - u - 2u(1-u^2)(1+u^2)^{-1}] \right\}.
\end{aligned} \quad (53)$$

The sum of the last two terms between the curly brackets in (53) is rewritten as

$$\begin{aligned}
&4(\mathbf{p}/M)^2 \int_0^1 du \Delta^{-1} (1-u)^2 u^3 (1+u^2)^{-4} (-3 + 2u + 2u^2 + 2u^3 + u^4) \\
&+ 4\sigma(\mathbf{p}, \mathbf{q}) \int_0^1 du \Delta^{-1} (1-u)^2 u^2 (1+u^2)^{-2} (2+u).
\end{aligned} \quad (54)$$

But this is of the relative order $(v/c)^2$ compared to the first term between the curly brackets in (53). We therefore neglect the last two terms between the curly brackets in (53). Because of the relation $J^2 = \mathbf{p}^2 + \mu^2 - \mathcal{D}_0$, the second term is rewritten as $\mathcal{D}_0^{-1} M^{-4} \times (\mu^2 - \mathcal{D}_0)^2$, which can be replaced by $(\mu/M)^4 \mathcal{D}_0^{-1}$, since we ignore the contact interaction. Thus the coefficient of $\exp[i(\mathbf{p}\mathbf{r} - \mathbf{q}\mathbf{r}')] in the integrand of (53) becomes a function of $\mathbf{k} = \mathbf{p} - \mathbf{q}$ alone. Therefore (53) represents the ordinary potential not containing the velocity dependence. To obtain the potential, we replace $\int d^3\mathbf{p} d^3\mathbf{q} \exp[i(\mathbf{p}\mathbf{r} - \mathbf{q}\mathbf{r}')] by $\int d^3\mathbf{k} \exp(i\mathbf{k}\mathbf{r})$ in (53). The integration over \mathbf{k} is carried out by means of the formula$$

$$\int_0^1 dv \int d^3\mathbf{k} \exp(i\mathbf{k}\mathbf{r}) [\alpha^2 + k^2 b^2 v(1-v)]^{-1} = (8\pi^2/b^2) r^{-1} K_0(2ab^{-1}r). \quad (55)$$

Then we finally have for the contribution of f_{A1}'

$$\begin{aligned}
&\mu\tau_A \pi^{-1} \{4\lambda^{-4} x^{-1} \int_0^1 du [(1-u^2)^2 (1+4u^2+u^4) u^{-4} (1+u^2)^{-2} \\
&+ 2g(u) u^{-6} + 16(1-u^2) g(u) (1+u^2)^{-4}] L_0(u, x) + 4x^{-1} K_0(2x)\},
\end{aligned} \quad (56)$$

where $x = \mu r$.

Contribution of f_{A1}' . We expand $[2E_q \epsilon_{l+q} \pm 2(\mathbf{q} \cdot \mathbf{l} + \mathbf{q}) \pm \mu^2]^{-1}$ appearing in (41) in powers of $[2(\mathbf{q} \cdot \mathbf{l} + \mathbf{q}) + \mu^2]/2E_q \epsilon_{l+q}$, and take the terms necessary for our approximation. Then, taking account of the relation $(\mathbf{l} \cdot \mathbf{p} - \mathbf{q}) = (\mathbf{l} + \mathbf{q} \cdot \mathbf{k}) + k^2/2 = (\mathbf{l} - \mathbf{p} \cdot \mathbf{k}) + k^2/2$ with $\mathbf{k} = \mathbf{p} - \mathbf{q}$ which follow from $|\mathbf{p}| = |\mathbf{q}|$, we have

$$\begin{aligned} f_{A1}'(\mathbf{p}, \mathbf{q}; -\mathbf{p}, -\mathbf{q}) &= \pi(M/\mu)^4 E_p^{-2} \\ &\times \left\{ \int d^3 l [(\mathbf{l} + \mathbf{q} \cdot \mathbf{k}) + k^2/2]^{-1} [\langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- \epsilon_{l+q}^{-1} \right. \\ &+ (2E_p)^{-2} \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- (2(\mathbf{q} \cdot \mathbf{l} + \mathbf{q}) + \mu^2)^2 \epsilon_{l+q}^{-3} \\ &+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}) \rangle_- \epsilon_{l+q}^{-3} \\ &\left. + \text{what is obtained by writing } -\mathbf{p} \text{ in place of } \mathbf{q} \text{ in the preceding term} \right\}. \end{aligned} \quad (57)$$

This is calculated in the following way (The detailed illustration of the method will be given in Appendix II): We write $\mathbf{l} - \mathbf{q}$ in place of \mathbf{l} in the first integral in (57), and $\mathbf{l} + \mathbf{p}$ in place of \mathbf{l} in the second integral. In doing this we must take care of the treatment of the "surface terms". After this, the integration over the directions of \mathbf{l} is carried out. Finally we remove the path of the integration over l from the real axis to the imaginary axis. We thus have

$$\begin{aligned} f_{A1}''(\mathbf{p}, \mathbf{q}; -\mathbf{p}, -\mathbf{q}) &= 4\pi^2 \int_1^\infty d\tau v (k^2 + 4\mu^2 \tau v^2)^{-1} \\ &\times \left\{ \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- [4\lambda^{-2} (\tau v^2 - 1)^{1/2} + 8(\tau v^2 - 1)^{3/2} + 8(\tau v^2 - 1)^{1/2} \right. \\ &\left. + (\tau v^2 - 1)^{-1/2}] - 4 \langle \boldsymbol{\gamma} \rangle_+ \langle \boldsymbol{\gamma} \rangle_- \lambda^{-2} (\tau v^2 - 1)^{1/2} \right\}, \end{aligned} \quad (58)$$

where $k = |\mathbf{p} - \mathbf{q}|$.

In this case, we have

$$\begin{aligned} \langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} &= (4\pi)^2 \tau_A [1 + \sigma(\mathbf{p}, \mathbf{q})/2] \\ \langle \boldsymbol{\gamma} \rangle_+^{(1)} \langle \boldsymbol{\gamma} \rangle_-^{(2)} &= (4\pi)^2 \tau_A \{ \sigma(\mathbf{p}, \mathbf{q}) \\ &+ (2M)^{-2} [k^2 - k^2(\boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + (\mathbf{k} \cdot \boldsymbol{\sigma}^{(1)})(\mathbf{k} \cdot \boldsymbol{\sigma}^{(2)})] \}, \end{aligned}$$

where $\sigma(\mathbf{p}, \mathbf{q})$ is given by (52). Then observing (13), (18) and (58), we have for the contribution of f_{A1}'' to $\langle \mathbf{r} | V | \mathbf{r}' \rangle$

$$\begin{aligned} &= \tau_A \pi^{-3} \delta(\mathbf{r} - \mathbf{r}') \int_1^\infty d\tau v \{ 4\lambda^{-2} (\tau v^2 - 1)^{1/2} + 8(\tau v^2 - 1)^{3/2} \\ &+ 8(\tau v^2 - 1)^{1/2} + (\tau v^2 - 1)^{-1/2} + (\tau v^2 - 1)^{1/2} \mu^{-2} [(1 - \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) \nabla_r^2 \\ &+ (\boldsymbol{\sigma}^{(1)} \cdot \nabla_r)(\boldsymbol{\sigma}^{(2)} \cdot \nabla_r)] \} \int d^3 k \exp(i\mathbf{k} \cdot \mathbf{r}) (k^2 + 4\mu^2 \tau v^2)^{-1} \\ &+ \tau_A \pi^{-3} \mu^{-2} [(\boldsymbol{\sigma}^{(1)} \cdot \nabla_r)(\boldsymbol{\sigma}^{(1)} \cdot \nabla_{r'}) + (\boldsymbol{\sigma}^{(2)} \cdot \nabla_r)(\boldsymbol{\sigma}^{(2)} \cdot \nabla_{r'})] \\ &\times \{ \delta(\mathbf{r} - \mathbf{r}') \int_1^\infty d\tau v (\tau v^2 - 1)^{1/2} \int d^3 k \exp(i\mathbf{k} \cdot \mathbf{r}) (k^2 + 4\mu^2 \tau v^2)^{-1} \}, \end{aligned} \quad (59)$$

where

$$\nabla_r = \partial/\partial \mathbf{r}, \quad \nabla_{r'} = \partial/\partial \mathbf{r}'.$$

Making use of the formula

$$\int_1^{\infty} d\omega (\omega^2 - 1)^{n-(1/2)} \int d^3k \exp(i\mathbf{k}\cdot\mathbf{r}) (k^2 + 4\mu^2 \omega^2)^{-1} \\ = 2\pi^2 r^{-1} [\Gamma(n + (1/2)) / \Gamma(1/2)] (\mu r)^{-n} K_n(2\mu r),$$

we have for the operator corresponding to the matrix (59)

$$\begin{aligned} & -\mu\tau_A \pi^{-1} \{ (2x^{-1} + 18x^{-3}) K_0(2x) \\ & + (4\lambda^{-2}x^{-2} + 12x^{-2} + 18x^{-4}) K_1(2x) \\ & - (\sigma^{(1)} \cdot \sigma^{(2)}) [4x^{-3} K_0(2x) + ((8/3)x^{-2} + 4x^{-4}) K_1(2x)] \\ & + S_{12} [4x^{-3} K_0(2x) + ((4/3)x^{-2} + 5x^{-4}) K_1(2x)] \\ & + (\sigma^{(1)} \cdot \mathcal{P}) x^{-2} K_1(2x) (\sigma^{(1)} \cdot \mathcal{P}) + (\sigma^{(2)} \cdot \mathcal{P}) x^{-2} K_1(2x) (\sigma^{(2)} \cdot \mathcal{P}) \}, \end{aligned} \quad (60)$$

where

$$\mathcal{P} = \partial / \partial \mathbf{x}, \quad \mathbf{x} = \mu \mathbf{r}.$$

(A₂) Contribution of S_{A2}

S_{A2} is calculated from (37) in the usual way. The divergent integrals appear in S_{A2} because of the derivative coupling. For these integrals we use the formulae

$$\begin{aligned} \int d^4k (k^2 + \Lambda)^{-2} &= \lim_{L \rightarrow \infty} i\pi^2 [2 \log(L/\mu) - 1 - \log(\Lambda/\mu^2)], \\ \int d^4k (k^2 + \Lambda)^{-1} &= \lim_{L \rightarrow \infty} i\pi^2 [L^2 - 2\Lambda \log(L/\mu) + \Lambda \log(\Lambda/\mu^2)]. \end{aligned}$$

We then obtain from (37)

$$S_{A2} = i \int d^4p_1 d^4q_1 d^4p_2 d^4q_2 \delta^1(p_1 + p_2 - q_1 - q_2) f_{A2}(p_1, q_1; p_2, q_2),$$

with

$$\begin{aligned} f_{A2}(p_1, q_1; p_2, q_2) &= 2\pi^2 \int_0^1 dv \{ (4\mu)^{-2} [(L/\mu)^2 \\ & - 4(4M^2\mu^{-2} + 1) \log(L/\mu) + 12(M/\mu)^2 + 1 \\ & - (1/18)\mu^{-2}(p_1 - q_1)^2(12 \log(L/\mu) + 1)] \langle \gamma_\mu \rangle_1 \langle \gamma_\mu \rangle_2 \\ & + \mu^{-2}(M/\mu)^2(2 \log(L/\mu) - 1) \langle 1 \rangle_1 \langle 1 \rangle_2 \\ & + 2(2\mu)^{-4} \Delta_0 \log(\Delta_0/\mu^2) \langle \gamma_\mu \rangle_1 \langle \gamma_\mu \rangle_2 - \mu^{-2}(M/\mu)^2 \log(\Delta_0/\mu^2) \langle 1 \rangle_1 \langle 1 \rangle_2 \\ & + \mu^{-2}(M/\mu)^2 \int_0^1 u du \log(\Delta/\mu^2) \langle \gamma_\mu \rangle_1 \langle \gamma_\mu \rangle_2 \\ & - 4(M/\mu)^4 \int_0^1 u(1-u) du \Delta^{-1} \langle 1 \rangle_1 \langle 1 \rangle_2 \\ & - \mu^{-1}(M/\mu)^3 \int_0^1 u(1-u)^2 du \Delta^{-1} \\ & \times [\langle 1 \rangle_1 \langle i(\gamma, p_1) \rangle_2 + \langle i(\gamma, p_2) \rangle_1 \langle 1 \rangle_2] \}, \end{aligned} \quad (61)$$

where

$$\begin{aligned} \Delta_0 &= \mu^2 + (p_1 - q_1)^2 v(1-v), \\ \Delta &= M^2 g(u) + (p_1 - q_1)^2 u^2 v(1-v). \end{aligned}$$

We can drop the first and the second terms between the curly brackets in this expression, because these terms yield only the contact interactions. If the nucleon momenta in the intermediate states are considered to be of the order of μ , the factor $1-u$ appearing in the last two terms is considered to be of the order of μ/M , as was explained in the footnote in § 1. Then we find that the fourth and the sixth terms are the leading terms, the fifth and the last terms are of the first order in v/c , and the third term is of the second order in v/c . Therefore we can replace $\langle \gamma_\mu \rangle_1 \langle \gamma_\mu \rangle_2$ in the third and the fifth terms by $\langle 1 \rangle_1 \langle 1 \rangle_2$, and $i(\gamma \cdot p_{1,2})$ in the last term by $-M$, because error introduced to these terms by doing this is of the relative order $(v/c)^2$ compared to themselves.

Then we have

$$\begin{aligned} f_{A^2}(\mathbf{p}, \mathbf{q}; -\mathbf{p}, -\mathbf{q}) &= -2\pi^2 \langle 1 \rangle_+ \langle 1 \rangle_- \int_0^1 dv \\ &\quad \{ \mu^{-2} (M/\mu)^2 \log(\Delta_0/\mu^2) + 4(M/\mu)^4 \int_0^1 u(1-u) du \Delta^{-1} \\ &\quad - 2(M/\mu)^4 \int_0^1 du u(1-u)^2 \Delta^{-1} - \mu^{-2} (M/\mu)^2 \int_0^1 u du \log(\Delta/\mu^2) \\ &\quad - 2(2\mu)^{-4} \Delta_0 \log(\Delta_0/\mu^2) \}, \end{aligned}$$

where Δ and Δ_0 are given by (49) and (50). The contribution of f_{A^2} to $\langle \mathbf{p} | T | \mathbf{p}' \rangle$ is easily obtained by observing (13), (18) and the above expression of f_{A^2} . $\langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}$ appearing in this contribution is approximated by $(4\pi)^2 \tau_A [1 - \sigma(\mathbf{p}, \mathbf{q})/2]$. The integrations over \mathbf{k} are carried out by means of the formulae (55) and

$$\begin{aligned} \int_0^1 dv \int d^3\mathbf{k} \exp(i\mathbf{k}\mathbf{r}) \log[a^2 + k^2 b^2 v(1-v)] &= -8\pi^2 ab^{-1} r^{-2} K_1(2ab^{-1}r), \\ \int_0^1 dv \int d^3\mathbf{k} \exp(i\mathbf{k}\mathbf{r}) \Delta_0 \log(\Delta_0/\mu^2) &= \mu^2 8\pi^2 [(\mu r)^{-3} K_0(2\mu r) + (\mu r)^{-1} K_1(2\mu r)]. \end{aligned}$$

Then the contribution of S_{A^2} to the nucleon-nucleon interaction is, in the operator form, found to be

$$\begin{aligned} \mu \tau_A \pi^{-1} \{ & -[4\lambda^{-2} x^{-2} + (1/2)x^{-4}] K_1(2x) - (1/2)x^{-3} K_0(2x) \\ & + 8\lambda^{-4} x^{-1} \int_0^1 du (1-u^2) u^{-1} L_0(u, x) + 4\lambda^{-3} x^{-2} \int_0^1 du [g(u)]^{1/2} L_1(u, x) \\ & + (\sigma^{(1)} \cdot \mathbf{F}) a(x) (\sigma^{(1)} \cdot \mathbf{F}) + (\sigma^{(2)} \cdot \mathbf{F}) a(x) (\sigma^{(2)} \cdot \mathbf{F}) \} \end{aligned} \quad (62)$$

with

$$a(x) = 4\lambda^{-2} x^{-1} \int_0^1 du (1-u) u^{-1} L_0(u, x) - x^{-2} K_1(2x).$$

(A₃) Contribution of $-\Sigma_2$

In the expression of Σ_2 (34), we perform the integrations over l_0 , k_0 and \mathbf{k} , and drop from the integrand the terms whose ratios to the leading terms are of the negligible order in v/c for all values of \mathbf{l} . Then we have

$$-\Sigma_2 = i \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) f_{\Sigma_2}(p_1, q_1; p_2, q_2)$$

with

$$\begin{aligned} f_{\Sigma_2}(\mathbf{p}, \mathbf{q}; -\mathbf{p}, -\mathbf{q}) = & -2\pi(M/\mu)^2 \mu^{-2} \\ & \times \int d^3 \mathbf{l} E_l^{-2} [2E_l E_p + 2(\mathbf{l} \cdot \mathbf{p}) - 2M^2 + \mu^2]^{-1} \\ & \times [2E_l E_q + 2(\mathbf{l} \cdot \mathbf{q}) - 2M^2 + \mu^2]^{-1} (E_l - M) \\ & \times \{l^2 \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- + M[\langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_+ \langle 1 \rangle_- - \langle 1 \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_-] \}. \end{aligned}$$

If we cut off the integral in this expression at $l \sim \mu$, the order of magnitude of f_{Σ_2} is estimated as follows:

$$f_{\Sigma_2} \sim f^A M^{-1} \mu^{-1}.$$

Comparing this with (43), it is expected that $-\Sigma_2$ gives negligible contribution to the nucleon-nucleon interaction at distances $\gtrsim \mu^{-1}$. But, since Σ_2 contains the integral with the strongest divergence among all terms of $S_A - \Sigma$, $-\Sigma_2$ is expected to give the interaction with the strongest singularity.

Since f_{Σ_2} is of the form similar to $f_{A'}$, the contribution of $-\Sigma_2$ is calculated in the same way as that of $f_{A'}$, and is found to be

$$\begin{aligned} \mu \tau_A \pi^{-1} \lambda^{-4} x^{-1} \int_0^1 du \{ & 2(1-u)^2(1-u^2)^2(1+4u^2+u^4)u^{-5}(1+u^2)^{-2} \\ & + 4(1-u)^2 g(u)u^{-7} + (1-u^2)^4 u^{-5}(1+u^2)^{-1} \\ & + (1-u^2)^2 g(u)u^{-7}[1+2(1+u^2)^{-2}] \} L_0(u, x). \end{aligned} \quad (63)$$

Summing up (56), (60), (62) and (63), we obtain V_A given by (21)–(25).

(B) Contribution of S_B

In this section, we use the following abbreviations:

$$\begin{aligned} \langle A \rangle_1 \langle B \rangle_2 = & \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \bar{\psi}(\mathbf{p}_1) A \tau_\rho \tau_\sigma \psi(q_1) \bar{\psi}(\mathbf{p}_2) B \tau_\sigma \tau_\rho \psi(q_2), \\ \langle A \rangle_+ \langle B \rangle_- = & \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \bar{\psi}(\mathbf{p}, E_p) A \tau_\rho \tau_\sigma \psi(\mathbf{q}, E_q) \times \bar{\psi}(-\mathbf{p}, E_p) B \tau_\sigma \tau_\rho \psi(-\mathbf{q}, E_q), \\ \langle A \rangle_+^{(1)} \langle B \rangle_-^{(2)} = & \sum_{\rho, \sigma} f_\rho^2 f_\sigma^2 \tau_\rho^{(1)} \tau_\sigma^{(1)} \tau_\sigma^{(2)} \tau_\rho^{(2)} \\ & \times A^{(1)}(\mathbf{p}) \beta^{(1)} A^{(1)}(\mathbf{q}) A^{(2)}(-\mathbf{p}) \beta^{(2)} B^{(2)} A^{(2)}(-\mathbf{q}). \end{aligned}$$

Following the Feynman-Dyson rules, the expression of S_B is immediately written down. By the use of the properties of the γ -matrices and the Dirac equations, this expression is rewritten as

$$S_B = S_{B1} + S_{B2} \quad (64)$$

$$S_{B1} = i \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) f_{B1}(p_1, q_1; p_2, q_2), \quad (65)$$

$$f_{B1}(p_1, q_1; p_2, q_2) = -i8(M/\mu)^4 \int d^4k \langle i(\gamma, k) \rangle_1 \langle i(\gamma, k) \rangle_2 \\ \times (k^2 + \mu^2)^{-1} [(k + p_1 - q_1)^2 + \mu^2]^{-1} [(k + p_1)^2 + M^2]^{-1} [(k + q_2)^2 + M^2]^{-1}, \quad (66)$$

$$f_{B2}(p_1, q_1; p_2, q_2) = -(i/2)\mu^{-4} \int d^4k (k^2 + \mu^2)^{-1} [(k + p_1 - q_1)^2 + \mu^2]^{-1} \\ \times \{ \langle i(\gamma, k) + 2M \rangle_1 \langle i(\gamma, k) + 2M \rangle_2 \\ - 4M^2 [(k + p_1)^2 + M^2]^{-1} \langle i(\gamma, k) \rangle_1 \langle i(\gamma, k) + 2M \rangle_2 \\ - 4M^2 [(k + q_2)^2 + M^2]^{-1} \langle i(\gamma, k) + 2M \rangle_1 \langle i(\gamma, k) \rangle_2 \}. \quad (67)$$

(B₁) Contribution of S_{B1}

f_{B1} is calculated in the usual way, and we have

$$f_{B1}(p_1, q_1; p_2, q_2) = -4\pi^2(M/\mu)^4 \int_0^1 du \int_0^1 dv \int_0^1 dw \\ \times \{ u(1-u) \Delta_1^{-1} \langle \gamma_\mu \rangle_1 \langle \gamma_\mu \rangle_2 - 2u(1-u)^3 \Delta_1^{-2} \\ \times \langle (i/2)(\gamma, p_2 + q_2)(1-w) - M\tau w \rangle_1 \langle (i/2)(\gamma, p_1 + q_1)\tau w - M(1-\tau w) \rangle_2 \} \quad (68)$$

where

$$\Delta_1 = \Delta - [2M^2 + (p_1, q_2) + (q_1, p_2)](1-u)^2 w(1-\tau w).$$

By the use of the Dirac equations it is found that the last line of (68) can be approximated by

$$M^2 \langle 1 \rangle_+ \langle 1 \rangle_- + M [\langle 1 \rangle_+ \langle E_p \gamma_4 - M \rangle_- + \langle E_p \gamma_4 - M \rangle_+ \langle 1 \rangle_-],$$

when $p_1 = -p_2 = p$, and $q_1 = -q_2 = q$.

In this case, we have also

$$\Delta_1 = \Delta + (4p^2 - k^2)(1-u)^2 \tau w(1-\tau w), \quad k = |p - q|.$$

We therefore expand Δ^{-1} and Δ^{-2} appearing in (68) in powers of $(4p^2 - k^2)/\Delta$. Then the integration over w is easily carried out, and we have

$$f_{B1}(p, q; -p, -q) = -4\pi^2(M/\mu)^4 \int_0^1 du \int_0^1 dv u(1-u) \Delta^{-1} \\ \times \{ \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- + \langle \gamma \rangle_+ \langle \gamma \rangle_- - 2[M^2 \langle 1 \rangle_+ \langle 1 \rangle_- \\ + M(\langle 1 \rangle_+ \langle E_p \gamma_4 - M \rangle_- + \langle E_p \gamma_4 - M \rangle_+ \langle 1 \rangle_-)](1-u)^2 \Delta^{-1} \\ + (1/6) \langle \gamma_4 \rangle_+ \langle \gamma_4 \rangle_- (1-u)^2 (k^2 - 4p^2) \Delta^{-1} \\ - (2/3) \langle 1 \rangle_+ \langle 1 \rangle_- (1-u)^4 (k^2 - 4p^2) M^2 \Delta^{-2} \}.$$

Observing (13), (18), and the above expression, and using the approximate relations

$$\langle \gamma_4 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} + \langle \gamma \rangle_+^{(1)} \langle \gamma \rangle_-^{(2)} = (4\pi)^2 \tau_B \{ 1 + (3/2) \sigma(p, q) \\ + (2M)^{-2} [k^2 (1 - \sigma^{(1)} \cdot \sigma^{(2)}) + (k \cdot \sigma^{(1)}) (k \cdot \sigma^{(2)})] \}, \\ E_p (\langle 1 \rangle_+^{(1)} \langle \gamma_4 \rangle_-^{(2)} + \langle \gamma_4 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)}) - M \langle 1 \rangle_+^{(1)} \langle 1 \rangle_-^{(2)} \\ = (4\pi)^2 \tau_B M \{ 1 + (p/M)^2 + \sigma(p, q)/2 \},$$

the contribution of S_{B1} to $\langle \mathbf{r} | V | \mathbf{r}' \rangle$ is found to be

$$\begin{aligned} & \tau_B \pi^{-3} (M/\mu)^4 \delta(\mathbf{r} - \mathbf{r}') \int_0^1 du \int_0^1 dv \int d^3 \mathbf{k} \exp(i \mathbf{k} \mathbf{r}) u(1-u) \Delta^{-1} \\ & \times \{1 + (2M)^{-2} [k^2(1 - \boldsymbol{\sigma}^{(1)} \cdot \boldsymbol{\sigma}^{(2)}) + (\mathbf{k} \cdot \boldsymbol{\sigma}^{(1)})(\mathbf{k} \cdot \boldsymbol{\sigma}^{(2)})] \\ & - 2(1-u)^2(M^2 - k^2/12) \Delta^{-1} - (2/3)(1-u)^4 M^2 k^2 \Delta^{-2}\} \\ & + \tau_B \pi^{-3} (M/\mu)^4 (2M)^{-2} [(\boldsymbol{\sigma}^{(1)} \cdot \nabla_r)(\boldsymbol{\sigma}^{(1)} \cdot \nabla_{r'}) + (\boldsymbol{\sigma}^{(2)} \cdot \nabla_r)(\boldsymbol{\sigma}^{(2)} \cdot \nabla_{r'})] \\ & \times \{\delta(\mathbf{r} - \mathbf{r}') \int_0^1 du \int_0^1 dv \int d^3 \mathbf{k} \exp(i \mathbf{k} \mathbf{r}) u(1-u) \Delta^{-1} [3 - 2(1-u)^2 M^2 \Delta^{-1}]\} \\ & + \tau_B \pi^{-6} (M/\mu)^4 (1/3) \int_0^1 du \int_0^1 dv \int d^3 \mathbf{p} d^3 \mathbf{q} \exp[i(\mathbf{p} \mathbf{r} - \mathbf{q} \mathbf{r}')] \\ & \times p^2 u(1-u)^3 \Delta^{-2} [(1-u)^2 M^2 \Delta^{-1} - 1]. \end{aligned} \quad (69)$$

An ambiguity occurs in rewriting the last term of (69) into the operator form, because (69) has been obtained under the assumption that $p = |\mathbf{p}| = |\mathbf{q}|$. In the operator form, however, this will be written in any case as a product of two differential operators and the function

$$(M/\mu)^4 \int_0^1 du \int_0^1 dv \int d^3 \mathbf{k} \exp(i \mathbf{k} \mathbf{r}) u(1-u)^3 \Delta^{-2} [(1-u)^2 M^2 \Delta^{-1} - 1].$$

This function is transformed into

$$\begin{aligned} & 4\pi^2 \mu^{-1} \lambda^{-4} \int_0^1 (1-u)^3 du \{x(1-u)^2 u^{-3} [g(u)]^{-1} L_0(u, x) \\ & + \lambda u^{-2} [g(u)]^{-1/2} [(1-u)^2/g(u) - 2] L_1(u, x)\}, \end{aligned}$$

and this is found to tend to

$$2\pi^2 \mu^{-1} \{K_3(2x) - 2x^{-1} K_2(2x) - K_1(2x)\} = 0,$$

as $\lambda \rightarrow 0$. Therefore, we can consider the ratio of the last term of (69) to the first term to be of the order of $(\mu/M)^3$, though it is apparently of the order of $(\mu/M)^2$. We therefore neglect the last term of (69).^{*} The remaining terms yield the operator

$$\begin{aligned} & \mu \tau_B \pi^{-1} \{8\lambda^{-4} x^{-1} \int_0^1 du (1-u) u^{-1} [1 + (1/3)(1-u)^2 u^{-2} \\ & + (4/3)\lambda^{-2} x^2 (1-u)^4 u^{-4} - g(u) u^{-2}] L_0(u, x) \}. \end{aligned}$$

^{*} This reasoning is farfetched one, and it is obviously one-sided treatment to drop this term, only taking the velocity-dependent term

$$\mu \sum_{i=1,2} (\boldsymbol{\sigma}^{(i)} \nabla) [\tau_A U_A^v(x) + \tau_B U_B^v(x)] (\boldsymbol{\sigma}^{(i)} \nabla)$$

which also vanishes in the adiabatic limit. However both of these terms are numerically so small that they are considered to be negligible to our approximation. Nevertheless, we took the latter under the expectation that it will give something interesting, because it contains the spin-orbit coupling

$$-2\mu \frac{1}{x} \frac{d}{dx} [\tau_A U_A^v(x) + \tau_B U_B^v(x)] (\mathbf{S} \cdot \mathbf{L}).$$

$$\begin{aligned}
& -4\lambda^{-5} \int_0^1 du (1-u) u^{-2} [g(u)]^{-1/2} [4(1-u)^2 + \lambda^2 x^{-2} g(u) \\
& + (4/3)(1-u)^4 u^{-2} + (4/3)(1-u)^2 u^{-2} g(u)] L_1(u, x) \\
& + (\sigma^{(1)} \cdot \sigma^{(2)}) [(16/3)\lambda^{-4} x^{-1} \int_0^1 du (1-u) g(u) u^{-3} L_0(u, x) \\
& + (8/3)\lambda^{-3} x^{-2} \int_0^1 du (1-u) (g(u))^{1/2} u^{-2} L_1(u, x)] \\
& - S_{12} [(8/3)\lambda^{-4} x^{-1} \int_0^1 du (1-u) g(u) u^{-3} L_0(u, x) \\
& + 2\lambda^{-2} x^{-3} \int_0^1 du (1-u) u^{-1} L_0(u, x) \\
& + (16/3)\lambda^{-3} x^{-2} \int_0^1 du (1-u) (g(u))^{1/2} u^{-2} L_1(u, x)] \\
& - (\sigma^{(1)} \cdot \nabla) b(x) (\sigma^{(1)} \cdot \nabla) - (\sigma^{(2)} \cdot \nabla) b(x) (\sigma^{(2)} \cdot \nabla) \} \quad (70)
\end{aligned}$$

with

$$\begin{aligned}
b(x) &= 6\lambda^{-2} x^{-1} \int_0^1 du (1-u) u^{-1} L_0(u, x) \\
& - 4\lambda^{-3} \int_0^1 du (1-u)^3 [g(u)]^{-1/2} u^{-2} L_1(u, x).
\end{aligned}$$

(B₂) Contribution of S_{B2}

f_{B2} given by (67) is calculated in the same way as f_{A2} , and we find that f_{B2} is obtained from the expression of f_{A2} (61) only by changing the signs of the appropriate terms of the latter. Therefore, the contribution of S_{B2} to the nucleon-nucleon interaction is obtained from that of S_{A2} (62) only by changing the signs of the appropriate terms of the latter, and is found to

$$\begin{aligned}
\mu\tau_B \pi^{-1} \{ & -4\lambda^{-2} x^{-2} K_1(2x) + (1/2)x^{-4} K_1(2x) + (1/2)x^{-3} K_0(2x) \\
& + 8\lambda^{-4} x^{-1} \int_0^1 du (1-u) (3-u) u^{-1} L_0(u, x) \\
& - 4\lambda^{-3} x^{-2} \int_0^1 du [g(u)]^{1/2} L_1(u, x) \\
& + (\sigma^{(1)} \cdot \nabla) a(x) (\sigma^{(1)} \cdot \nabla) + (\sigma^{(2)} \cdot \nabla) a(x) (\sigma^{(2)} \cdot \nabla) \}, \quad (71)
\end{aligned}$$

where $a(x)$ is given below (62).

Summing up (70) and (71), we obtain V_B given by (26)–(30).

(C) Contribution of S_C and a remark on S_D

S_C is calculated in the usual way, and we have

$$S_0 = S_{00} + S_{01},$$

$$\begin{aligned} S_{00} = & -(1/2)C \sum_{\rho, \sigma} f_{\rho}^2 f_{\sigma}^2 \mu^{-2} \int d^4 x_1 d^4 x_2 A_{F\mu\nu}(x_1 - x_2) \\ & \times \bar{\psi}(x_1) \gamma_5 \tau_{\mu} \tau_{\rho} \tau_{\sigma} \tau_{\nu} \psi(x_1) \bar{\psi}(x_2) \gamma_5 \tau_{\nu} \tau_{\sigma} \psi(x_2), \\ S_{01} = & i \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) f_{01}(p_1, q_1; p_2, q_2), \\ f_{01}(p_1, q_1; p_2, q_2) = & -8\pi^2 (M/\mu)^4 \sum_{\rho, \sigma} f_{\rho}^2 f_{\sigma}^2 \\ & \times \bar{\psi}(p_1) \gamma_5 \tau_{\rho} \tau_{\sigma} \tau_{\rho} \psi(q_1) \cdot \bar{\psi}(p_2) \gamma_5 \tau_{\sigma} \psi(q_2) \\ & \times \int_0^1 du \int_0^1 dv \{2u \log(A/A_0) [(p_1 - q_1)^2 + \mu^2]^{-1} \\ & + 2u^2(1-u)v(1-v)\mu^2 A_0^{-1} A^{-1} \\ & + u^2(1-2v)^2(p_1 - q_1)^2 [(p_1 - q_1)^2 + \mu^2]^{-1} A^{-1}\}, \end{aligned} \quad (72)$$

where

$$\begin{aligned} A &= M^2 u^2 + \mu^2(1-u) + (p_1 - q_1)^2 u^2 v(1-v), \\ A_0 &= M^2 u^2 + \mu^2(1-u) - \mu^2 u^2 v(1-v). \end{aligned}$$

S_{00} is eliminated by the renormalization of the coupling constant, though it has a divergent factor C . The magnitude of f_{01} is of the order of $f^4 \mu^{-2}$. Remembering (43), we therefore find that it is sufficient for our approximation to take only the leading term in f_{01} . Then we can neglect the second term between the curly brackets in (72). The first term is of the same order as the last term in the magnitude, but this is found to give only the interaction with the range $(2M)^{-1}$. We therefore neglect this term. In the last term, we can replace the factor $(p_1 - q_1)^2$ by $-\mu^2$, and A by $M^2 u^2$. Then we have

$$\begin{aligned} f_{01}(p_1, q_1; p_2, q_2) = & (4/3)\pi^2 (M/\mu)^2 \\ & \times \sum_{\rho, \sigma} f_{\rho}^2 f_{\sigma}^2 \bar{\psi}(p_1) \gamma_5 \tau_{\rho} \tau_{\sigma} \tau_{\rho} \psi(q_1) \cdot \bar{\psi}(p_2) \gamma_5 \tau_{\sigma} \psi(q_2). \end{aligned}$$

Thus we see that S_{01} is of the same form as the second order part of the S matrix. The contribution of S_{01} to the nucleon-nucleon interaction is therefore obtained in the same way as the second order potential, and is found to be

$$\begin{aligned} & -\mu(18\pi)^{-1} \sum_{\rho, \sigma} (f_{\rho}^2/4\pi) (f_{\sigma}^2/4\pi) \tau_{\rho}^{(1)} \tau_{\sigma}^{(1)} \tau_{\rho}^{(1)} \tau_{\sigma}^{(2)} \\ & \times \{(\sigma^{(1)} \cdot \sigma^{(2)}) + S_{12}(1 + 3x^{-1} + 3x^{-2})\} x^{-1} e^{-x}. \end{aligned}$$

This is V_0 given by (31)

S_D is calculated in the usual way, and we have, dropping the term eliminated by the renormalization,

$$S_D = i \int d^4 p_1 d^4 q_1 d^4 p_2 d^4 q_2 \delta^4(p_1 + p_2 - q_1 - q_2) f_{D1}(p_1, q_1; p_2, q_2)$$

with

$$\begin{aligned} f_{D1} = & -32\pi^2 \sum_{\sigma} f_{\sigma}^4 (M/\mu)^2 \mu^{-2} \bar{\psi}(p_1) \gamma_5 \tau_{\sigma} \psi(q_1) \bar{\psi}(p_2) \gamma_5 \tau_{\sigma} \psi(q_2) \\ & \times \{M^2 \int_0^1 v(1-v) dv \int_0^1 d\tau v [M^2 - \mu^2 v(1-v)(1-\tau v)] \\ & \times [M^2 - \mu^2 v(1-v)]^{-1} [M^2 - \mu^2 v(1-v)(1-\tau v) \\ & + (p_1 - q_1)^2 v(1-v)\tau v]^{-1} - 1/12\}. \end{aligned}$$

f_{D1} is of the same order as f_{C1} in the magnitude. But it is obvious that f_{D1} yields only the interaction with the range $(2M)^{-1}$. Therefore we neglected the contribution of S_D .

§ 4. The fourth order nucleon-nucleon interaction for the case of the pseudoscalar coupling

In this case, the interaction Hamiltonian is given by

$$H(x) = i \sum_{\sigma=1}^4 g_{\sigma} \bar{\psi}(x) \gamma_5 \tau_{\sigma} \psi(x) \varphi_{\sigma}(x).$$

As in § 3, S_4 is expressed by sum of the contributions of the graphs A, B, C, \dots in Fig. 1:

$$S_4 = S_A + S_B + S_C + \dots$$

In this case, no cancellation occurs among the leading terms even in the adiabatic limit. We therefore take the first and the second terms in each of the expansions of $S_A - \Sigma$, S_B , S_C, \dots in powers of v/c and of μ/M . Then S_C and S_D are found to give no contribution. On the other hand, we find that S_A , S_B , and Σ for the pseudoscalar coupling are obtained by multiplying S_{A1} , S_{B1} , and Σ_1 for the pseudovector coupling by $(\mu/2M)^4$ respectively, and writing g_{σ}^2 in place of f_{σ}^2 in the latters. Therefore the contributions of $S_A - \Sigma$ and of S_B to the nucleon-nucleon interaction are obtained from (56) + (60) and (70) respectively by replacing $f_{\rho}^2 f_{\sigma}^2$ by $(\mu/2M)^4 g_{\rho}^2 g_{\sigma}^2$ in the latters. To the approximation mentioned above, we thus have for the fourth order nucleon-nucleon interaction for the pseudoscalar coupling

$$V_4 = \mu(\mu/2M)^2 [\tau_A W_A(x) + \tau_B W_B(x)] \quad (73)$$

with

$$\begin{aligned} \pi W_A(x) = & \lambda^{-2} x^{-1} \int_0^1 du u^{-4} [(1-u^2)^2 (1+4u^2+u^4) (1+u^2)^{-2} \\ & + 2g(u) u^{-2}] L_0(u, x) - x^{-2} K_1(2x), \end{aligned} \quad (74)$$

$$\begin{aligned} \pi W_B(x) = & 2\lambda^{-2} x^{-1} \int_0^1 du (1-u) u^{-1} L_0(u, x) \\ & - 4\lambda^{-3} \int_0^1 du (1-u)^3 u^{-2} [g(u)]^{-1/2} L_1(u, x), \end{aligned} \quad (75)$$

where

$$\begin{aligned} \tau_A = & \sum_{\rho, \sigma} (g_{\rho}^2/4\pi) (g_{\sigma}^2/4\pi) \tau_{\rho}^{(1)} \tau_{\sigma}^{(1)} \tau_{\rho}^{(2)} \tau_{\sigma}^{(2)} \\ \tau_B = & \sum_{\rho, \sigma} (g_{\rho}^2/4\pi) (g_{\sigma}^2/4\pi) \tau_{\rho}^{(1)} \tau_{\sigma}^{(1)} \tau_{\sigma}^{(2)} \tau_{\rho}^{(2)}. \end{aligned}$$

§ 5. The adiabatic limit

In this section, we will derive the adiabatic potentials from the nucleon-nucleon interactions calculated in the preceding sections. By the adiabatic potential, we mean the

potential which is obtained under the assumption that the nucleons are at rest. For the case of the pseudoscalar coupling, however, the meaning of this assumption is that the nucleons are brought to rest after the Dyson transformation has been made. For the case of the pseudovector coupling, the adiabatic potential is obtained by making μ/M tend to zero in the nucleon-nucleon interaction calculated in § 3. For the case of the pseudoscalar coupling, the adiabatic potential consists of two terms, one of which has a factor $(\mu/M)^2$, and the other has a factor $(\mu/M)^3$. Therefore, to obtain the adiabatic potential for the pseudoscalar coupling, we must expand the potential calculated in § 4 in powers of μ/M , and take the terms in $(\mu/M)^2$ and in $(\mu/M)^3$. For both the cases of pseudovector and pseudoscalar couplings, it is necessary to expand the functions U_A^c , U_B^c , etc., W_A and W_B appearing in V_1 in powers of $\lambda = \mu/M$. For this purpose, in the integrals of the functions of u appearing in these functions, we transform the variable u into t by means of the equation

$$t = u^{-1} [\lambda^{-2}(1-u)^2 + u]^{1/2},$$

and expand the integrands in powers of λ . After this, we perform the integrations.

However, it is question whether this procedure does or does not give the convergent expansions of the functions U 's and W 's in powers of λ . By the above transformation of integration variable, the integrals of the form

$$\int_0^1 du f(u) L_n(u, x)$$

contained in the expressions of U 's and W 's are rewritten in the form

$$\int_1^\infty dt \left\{ \frac{\lambda t}{T(1-\lambda^2 t^2)} - \frac{2\lambda^2 t(1-\lambda T-\lambda^2/2)}{(1-\lambda^2 t^2)^2} \right\} f\left(\frac{1-\lambda T-\lambda^2/2}{1-\lambda^2 t^2}\right) K_n(2tx)$$

with

$$T = (t^2 - 1 + \lambda^2/4)^{1/2}.$$

In expanding the integrands in powers of λ , there occurs expansion of, say, $(1-\lambda^2 t^2)^{-1}$. But this expansion is, in principle, not allowed, because λt can take values larger than unity. One may think that, because of the factor $K_n(2tx)$, only such values of t that $t < 1/2x$ give significant contributions to the integral, so the above expansion is practically allowed, as far as $\lambda/2x \ll 1$, or $x \gg (2M)^{-1}$. However, the following example shows that the integration after this expansion will perhaps not give the convergent result: Let us consider the integral

$$\int_1^\infty \frac{e^{-tx}}{1-\lambda t} dt$$

which is of the same character as the integral in the question. Carrying out the integration after substituting $(1-\lambda t)^{-1}$ by $\sum_{n=0}^\infty \lambda^n t^n$, we obtain the series

$$(e^{-x}/x) \sum_{n=0}^\infty n! (\lambda/x)^n \sum_{m=0}^n x^m/m!,$$

which is divergent for any value of x . Therefore, when we approximate the original integral by the first few terms of this series, the error of the approximation can not easily be estimated. Therefore, accuracy of the adiabatic approximation can be estimated only by comparing the adiabatic potential with the correct one.

Case of the pseudovector coupling In this case, the expansions of U_A^c and of U_B^c begin with the terms in λ^{-2} , but it will be found that the terms in λ^{-2} as well as the terms in λ^{-1} are cancelled ultimately. We therefore take the terms in λ^{-2} , in λ^{-1} and in λ^0 , and drop positive power terms. In fact, it is meaningless to take the positive power terms, because the terms neglected in calculating V_4 will contribute the positive power terms on account of the approximation taken in § 3. The expansions of U_A^v , U_B^s , U_B^t , and U_B^v begin with the terms in λ^0 , and we take these terms only. Then we have from (22)—(25), (27)—(30)

$$\begin{aligned}\pi U_A^c(x) &= 8\lambda^{-2} \left\{ 2x^{-1} \int_1^\infty t K_0(2tx) dt - x^{-2} K_1(2x) \right\} \\ &\quad - 4\lambda^{-1} x^{-1} \int_1^\infty t dt \{ 2(t^2 - 1)^{1/2} K_0(2tx) - x^{-1} t(t^2 - 1 + \lambda^2/4)^{-1/2} K_1(2tx) \} \\ &\quad + 4x^{-1} \int_1^\infty t dt \{ 2(9t^2 - 4) K_0(2tx) - 3x^{-1} t K_1(2tx) \} \\ &\quad + (1/2) [4x^{-1} - 37x^{-3}] K_0(2x) - (1/2) [24x^{-2} + 37x^{-4}] K_1(2x), \\ \pi U_A^v(x) &= 4x^{-1} \int_1^\infty t K_0(2tx) dt - 2x^{-2} K_1(2x), \\ \pi U_B^c(x) &= 8\lambda^{-2} \int_1^\infty [3x^{-1} t K_0(2tx) - 2(t^2 - 1) K_1(2tx)] dt - 4\lambda^{-2} x^{-2} K_1(2x) \\ &\quad - \lambda^{-1} \int_1^\infty dt \{ x^{-1} t [40(t^2 - 1)^{1/2} + 12(t^2 - 1 + \lambda^2/4)^{-1/2}] K_0(2tx) \\ &\quad - [8(4t^2 - 1)(t^2 - 1)^{1/2} - 4x^{-2} t^2(t^2 - 1 + \lambda^2/4)^{-1/2}] K_1(2tx) \} \\ &\quad + (1/3) \int_1^\infty dt \{ [32xt^5 - 64(x - 2x^{-1})t^3 + 32(x - x^{-1})t] K_0(2tx) \\ &\quad - [176t^4 - 24(6 + x^{-2})t^2 + 16] K_1(2tx) \} \\ &\quad + (1/2)x^{-3} K_0(2x) + (1/2)x^{-4} K_1(2x), \\ \pi U_B^s(x) &= (8/3) \int_1^\infty [2x^{-1} t^3 K_0(2tx) + x^{-2} t^2 K_1(2tx)] dt, \\ \pi U_B^t(x) &= -(2/3)x^{-1} \int_1^\infty \{ [3x^{-2} + 4t^2] K_0(2tx) + 8x^{-1} t K_1(2tx) \} t dt, \\ \pi U_B^v(x) &= \int_1^\infty [4(t^2 - 1) K_1(2tx) - 2x^{-1} t K_0(2tx)] dt - x^{-2} K_1(2x).\end{aligned}$$

Carrying out the integrations by the use of the formulae given in Appendix III, we have

$$U_A^c(x) = -U_B^c(x) = (2\pi)^{-1} \{ (4x^{-1} + 23x^{-3}) K_0(2x) + (12x^{-2} + 23x^{-4}) K_1(2x) \}, \quad (76)$$

$$U_A^s(x) = U_B^s(x) = (3\pi)^{-3} \{ 12x^{-3} K_0(2x) + (8x^{-2} + 12x^{-4}) K_1(2x) \}, \quad (77)$$

$$U_A^t(x) = U_B^t(x) = -(3\pi)^{-1} \{ 12x^{-3} K_0(2x) + (4x^{-2} + 15x^{-4}) K_1(2x) \}, \quad (78)$$

$$U_A^v(x) = U_B^v(x) = 0. \quad (79)$$

(20), (21), (26), (31) together with (76)–(79) give the fourth order adiabatic potential. It follows from (21), (26), (76)–(79) that, for the charge symmetrical theory ($f_1=f_2=f_3=f$, $f_4=0$),

$$\begin{aligned} V_A + V_B = & \mu(f^2/4\pi)^2(2/\pi) \\ & \times \{ -\tau^{(1)} \cdot \tau^{(2)} [(4x^{-1} + 23x^{-3}) K_0(2x) + (12x^{-2} + 23x^{-4}) K_1(2x)] \\ & + \sigma^{(1)} \cdot \sigma^{(2)} [12x^{-3} K_0(2x) + (8x^{-2} + 12x^{-4}) K_1(2x)] \\ & - S_{12} [12x^{-3} K_0(2x) + (4x^{-2} + 15x^{-4}) K_1(2x)] \}. \end{aligned} \quad (80)$$

This is in agreement with the fourth order adiabatic potential calculated by Nishijima⁴⁾ and by Taketani, Machida and Onuma.⁵⁾ These authors neglected I_c , but I_c is practically negligible, because $V_c/V_2 = (6\pi)^{-1}(f^2/4\pi) \sim 0.01$.

Case of the pseudoscalar coupling. The expansions of W_A and W_B given by (74) and (75) in powers of λ begin with the terms in λ^0 . Taking the terms in λ^0 and in λ^1 only, we have

$$\begin{aligned} \pi W_A(x) = & -x^{-2} K_1(2x) \\ & + 2\lambda x^{-1} \int_1^\infty [3t(t^2-1)^{1/2} + t^3(t^2-1+\lambda^2/4)^{-1/2}] K_0(2tx) dt, \\ \pi W_B(x) = & -2 \int_1^\infty [2(t^2-1) K_1(2tx) - x^{-1} t K_0(2tx)] dt \\ & + \lambda \int_1^\infty dt \{ (8t^2-2)(t^2-1)^{1/2} K_1(2tx) \\ & - x^{-1} t [4(t^2-1)^{1/2} + (t^2-1+\lambda^2/4)^{-1/2}] K_0(2tx) \}. \end{aligned}$$

Performing the integrations by the use of the formulae given in Appendix III, we have

$$\begin{aligned} W_A(x) = W_B(x) \\ = -\pi^{-1} x^{-2} K_1(2x) + (\mu/2M) x^{-2} (1+x^{-1})^2 e^{-2x}. \end{aligned} \quad (81)$$

(73) and (81) give the fourth order adiabatic potential for the case of the pseudoscalar coupling. In the charge symmetrical theory, this potential becomes

$$V_4 = -\mu(g^2/4\pi)^2(\mu/2M)^2 \{ (6/\pi) x^{-2} K_1(2x) - (\mu/2M) 3x^{-2} (1+x^{-1})^2 e^{-2x} \}. \quad (82)$$

This is in agreement with the potential which was first calculated by Lévy,⁶⁾ and then corrected by Klein.⁷⁾

§ 6. Numerical results and qualitative discussions on the properties of the nucleon-nucleon interactions calculated in the preceding sections

Numerical values of the functions U_A^c, U_B^c, \dots appearing in V_4 for the pseudovector coupling are evaluated from (22)–(25) and (27)–(30) taking $M/\mu=6.68$, and are tabulated in Table 1. Numerical values of the functions W_A and W_B appearing in V_4 for the pseudoscalar coupling are evaluated from (74) and (75), and are tabulated in Table 2.

Remembering (47), we see that the integration variable u in the integrals appearing in these functions is related to the nucleon momentum l in the intermediate state by

$$u = M^{-1} [\sqrt{l^2 + M^2} - l].$$

However, for very large values of l , the perturbation theory adopted by us will lose its validity. Indeed, if we take all the iterations of the second order self-energy part of meson, the meson propagation function $\Delta_F(k)$ is replaced by

$$\Delta_F'(k) = \Delta_F(k) [1 - \Delta_C(k) \Delta_F(k)]^{-1},$$

where $\Delta_C(k)$ is the second order self-energy part of meson, as was shown by Hu.⁹⁾ For the pseudovector coupling, $\Delta_F'(k)$ decreases much more rapidly than $\Delta_F(k)$, as k^2 increases, for such values of k^2 that $k^2 \gtrsim M^2$, if the value of the coupling constant is taken as $f^2/4\pi \sim 0.1$. Therefore the use of Δ_F' in place of Δ_F makes the contributions of the virtual mesons with momenta $\gtrsim M$ to the nucleon-nucleon interaction much smaller than those in the previous calculations. The iterations of the second order self-energy part of nucleon will yield a similar effect. We therefore made numerical calculations of the functions

Table 1

x	U_A^c	U_A^s	U_A^t	U_A^v	U_B^c	U_B^s	U_B^t	U_B^v
0,20	9277	1963	-2374	-19,61	-1378	536,0	-789,6	-14,26
0,30	622,2	253,8	-298,8	-4,466	-243,1	93,04	-125,9	-3,466
0,35	249,4	116,0	-134,6	-2,498	-124,4	46,77	-61,18	-1,981
0,40	112,0	58,68	-67,11	-1,496	-71,00	24,22	-31,71	-1,187
0,45	56,97	32,06	-36,12	-0,9418	-44,10	14,90	-18,31	-0,7573
0,50	32,15	18,59	-20,64	-0,6174	-27,49	9,111	-10,90	-0,5024
0,60	11,73	7,173	-7,728	-0,2914	-11,03	3,792	-4,335	-0,2418
0,70	4,976	3,162	-3,310	-0,1507	-5,192	1,770	-1,943	-0,1271
0,80	2,369	1,535	-1,562	-0,0832	-2,633	0,8995	-0,9513	-0,0710
0,90	1,236	0,8006	-0,7935	-0,0483	-1,443	0,4888	-0,4983	-0,0416
1,00	0,6869	0,4418	-0,4270	-0,0293	-0,8424	0,2761	-0,2735	-0,0253
1,10	0,4047	0,2549	-0,2405	-0,0183	-0,5091	0,1629	-0,1568	-0,0159
1,20	0,2467	0,1525	-0,1407	-0,0117	-0,3167	0,0992	-0,0930	-0,0102
1,30	0,1560	0,0940	-0,0849	-0,0077	-0,2019	0,0621	-0,0568	-0,0067
1,40	0,0999	0,0594	-0,0526	-0,0051	-0,1297	0,0397	-0,0355	-0,0045
1,50	0,0665	0,0384	-0,0333	-0,0035	-0,0859	0,0259	-0,0227	-0,0031

Table 2

x	W_A	W_B	J_{cs}	J_{ct}
0,20	49,20	-3,133		-0,120
0,30	5,590	-1,142	53,74	-1,371
0,35	2,367	-0,7475	23,55	-1,552
0,40	1,083	-0,5270	11,33	-1,324
0,45	0,5224	-0,3692	5,809	-1,059
0,50	0,2611	-0,2641	3,142	-0,6452
0,60	0,0673	-0,1425	1,033	-0,3944
0,70	0,0126	-0,0814	0,3579	-0,2459
0,80	-0,0024	-0,0487	0,1248	-0,1563
0,90	-0,0059	-0,0301	0,0401	-0,1016
1,00	-0,0054	-0,0192	0,0090	-0,0671
1,10	-0,0044	-0,0125	-0,0021	-0,0450
1,20	-0,0034	-0,0083	-0,0055	-0,0305
1,30	-0,0025	-0,0056	-0,0058	-0,0209
1,40	-0,0018	-0,0038	-0,0051	-0,0145
1,50	-0,0013	-0,0026	-0,0042	

U_A^c , U_B^c , ..., W_A and W_B for the case in which the lower limit of the integrals appearing in these functions is cut off at $u=0.4$ which corresponds to $l=M$, and we found that effect of the cut off is negligible for $x > 0.4$ in the case of the pseudovector coupling, and for $x > 0.3$ in the case of the pseudoscalar coupling. It was to be expected that the effect of the cut off is larger for the pseudovector coupling than for the pseudoscalar coupling.

We compare the potentials calculated by us with the adiabatic potentials. We first consider case of the pseudovector coupling. We see from Table 1 that the ratios of U_A^r and of U_B^r to the other functions are numerically of the order of v/c . So we can neglect U_A^r and U_B^r consistently to our approximation. In the adiabatic limit, U_B^s and U_B^t are respectively equal to U_A^s and U_A^t given in Table 1. Therefore we see that the absolute values of U_B^s and U_B^t in the adiabatic limit are larger than those in Table 1 by about 50% of the latters at $x \approx 1$. We found also that this is true for U_A^c and U_B^c . Thus we find that effects of the nucleon recoil is so large that they cannot be neglected. For the pseudoscalar coupling, W_B is equal to W_A in the adiabatic limit, and we found that W_A in the adiabatic limit is numerically not so different from that in Table 2. Therefore, as is seen from Table 2 neglectation of the nucleon recoil changes general features of the nuclear potential entirely.

Pseudovector coupling, Symmetrical theory We study the properties of the nucleon-nucleon interaction for the pseudovector coupling, taking the symmetrical theory ($f_1=f_2=f_3=f_1$, $f_4=0$). Potentials for the various states of the two-nucleon system are easily obtained from (20)–(30). Including the second order potentials, the results are as follows :

Triplet even state

$$V_{3e} = -\mu(f^2/4\pi)[1 + S_{12}(1 + 3x^{-1} + 3x^{-2})]x^{-1}e^{-x} \\ + \mu(f^2/4\pi)^2[J_{3e}(x) + S_{12}J_{3e}'(x)],$$

Singlet even state

$$V_{1e} = -\mu(f^2/4\pi)x^{-1}e^{-x} + \mu(f^2/4\pi)^2J_{1e}(x),$$

Triplet odd state

$$V_{3o} = (\mu/3)(f^2/4\pi)[1 + S_{12}(1 + 3x^{-1} + 3x^{-2})]x^{-1}e^{-x} \\ + \mu(f^2/4\pi)^2[J_{3o}(x) + S_{12}J_{3o}'(x)],$$

Singlet odd state

$$V_{1o} = 3\mu(f^2/4\pi)x^{-1}e^{-x} + \mu(f^2/4\pi)^2J_{1o}(x).$$

Numerical values of the functions J 's are calculated from Table 1, and are tabulated in Table 3. V_c and the velocity dependent terms have been neglected in the reasons mentioned

Table 3

x	J_{3e}	J_{3e}'	J_{1e}	J_{3o}	J_{3o}'	J_{1o}
0,30	8334	-2312	-2750	125,7	-928,3	314
0,35	3522	-1028	-1422	-22,70	-440,5	-93,1
0,40	1677	-508,9	-782,4	-63,2	-225,7	-145,5
0,45	888,8	-270,2	-483,3	-56,9	-127,7	-86,4
0,50	511,8	-153,1	-297,8	-41,15	-75,14	-48,2
0,60	191,9	-56,55	-121,8	-17,29	-29,40	-20,84
0,70	83,51	-23,96	-57,01	-8,97	-13,03	-9,09
0,80	40,34	-11,20	-28,90	-4,767	-6,319	-4,14
0,90	21,19	-5,647	-15,71	-2,734	-3,285	-1,767
1,00	11,86	-3,023	-8,991	-1,703	-1,795	-0,735
1,10	6,975	-1,694	-5,348	-1,072	-1,025	-0,245
1,20	4,245	-0,9873	-3,283	-0,6884	-0,6057	-0,055
1,30	2,670	-0,5935	-2,067	-0,4489	-0,3689	0,0309
1,40	1,704	-0,3668	-1,322	-0,2907	-0,2300	0,0412
1,50	1,124	-0,2317	-0,8664	-0,1952	-0,1466	0,0537

before. We plot these potentials for $f^2/4\pi=0.1$ in Figs. 2-5. For the sake of comparison, we plot also the adiabatic potentials obtained from (80) plus the second order potential.

Effects of the adiabatic potentials on the neutron-proton system with low energies have been studied in detail by Taketani et. al⁽⁵⁾. They showed that the large experimental values of the triplet effective range and the quadrupole moment of deuteron are successfully explained by virtue of the strong repulsive central force in the triplet even state which is due to the fourth order contribution, and that the strong attractive force in the singlet even state mainly owing to the fourth order contribution can account for the large ex-

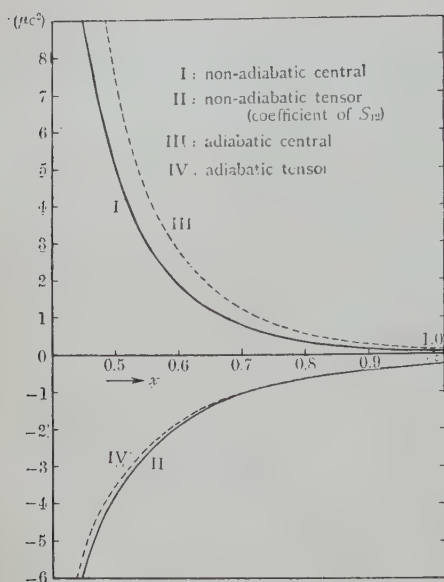


Fig. 2 Triplet even state

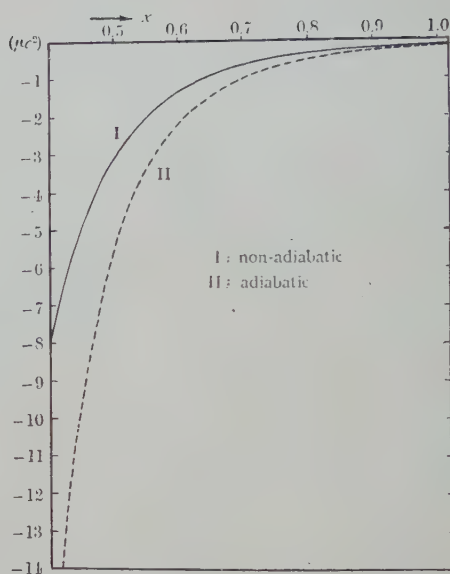


Fig. 3 Singlet even state

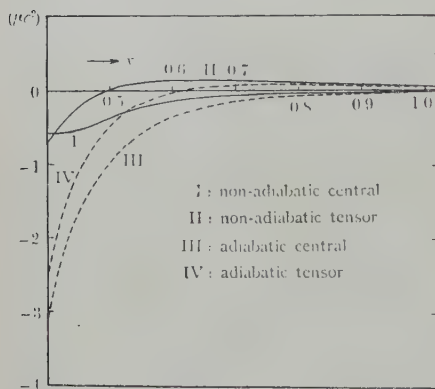


Fig. 4 Triplet odd state

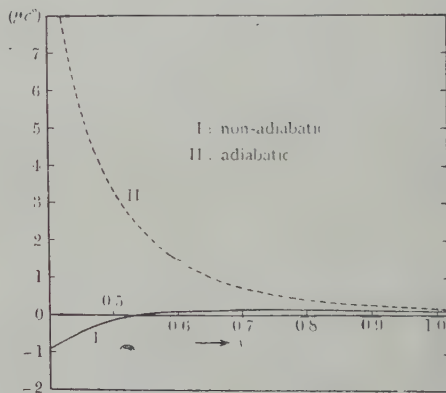


Fig. 5 Singlet odd state

perimental value of the singlet scattering length, while the second order force is too weak to explain this value. But these merits of the adiabatic potentials belong also to our potentials, as is seen from Fig. 2 and Fig. 3. Comparing with reference 5), it is obvious that the experimental data for the deuteron ground state and low energy neutron-proton scattering are well accounted for by our potentials modified by suitable cut off methods in the inside region.

On the other hand, forces in the odd states are very weak, as is seen from Fig. 4 and Fig. 5. This is due to cancellation between the second order and the fourth order contributions, and this cancellation is more complete for our potentials than for the adiabatic potentials. For the adiabatic potential, there is no cancellation in the singlet

odd state potential. As was mentioned in § 1, the weakness of the odd state forces favors to explain the angular distributions of high energy neutron-proton scattering.

Pseudoscalar coupling, symmetrical theory In this case, the fourth order potential V_4 given by (73) is written in the form

$$V_4 = \mu(\mu/2M)^2(g^2/4\pi)^2 \left[(1/2)(1 + P_x P_\sigma) J_{cs}(x) + (1/2)(1 - P_x P_\sigma) J_{ct}(x) \right].$$

Numerical values of J_{cs} and J_{ct} are given in the third and the fourth columns in Table 2. In Fig. 6-9, we plot the potentials for the various states obtained from the second order potential plus V_4 written above, for $(\mu/2M)^2(g^2/4\pi) = 0.1$. We plot also the adiabatic potentials obtained from (82) and the second order potential.

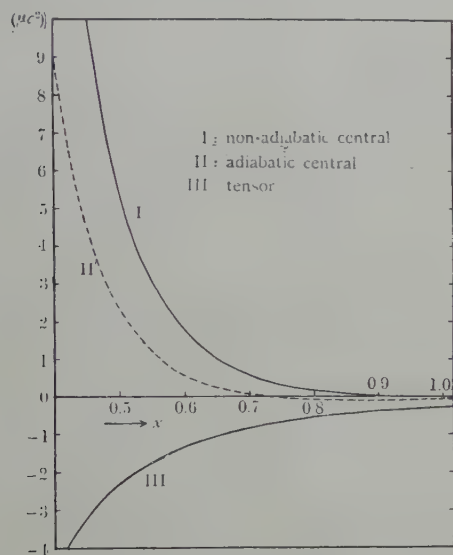


Fig. 6 Triplet even state

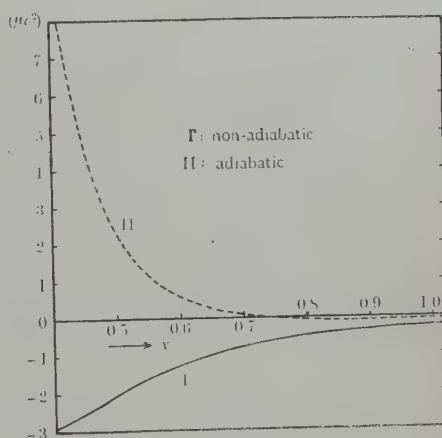


Fig. 7 Singlet even state

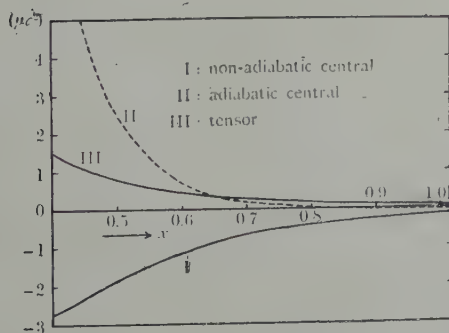


Fig. 8 Triplet odd state

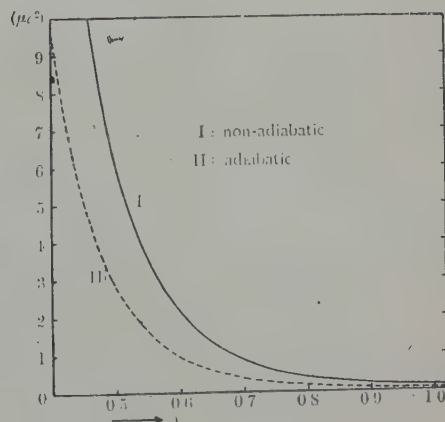


Fig. 9 Singlet odd state

In contrast with the case of pseudovector coupling, differences between our potentials and the adiabatic ones are remarkable in this case. The adiabatic potential in the singlet even state is almost repulsive, being quite incompatible with the experimental facts. We see from Fig. 7 that this is fairly improved by taking the nucleon recoil into account. As in the case of pseudovector coupling, V_1 gives the strong repulsive central force in the triplet even state and the strong attractive force in the singlet even state. Comparing Fig. 7 with Fig. 3, we see that we must choose a larger value of the coupling constant to fit the low energy singlet $n-p$ scattering data. By doing this, however, the repulsive central force in the triplet even state will be made too strong in comparison with the tensor force, because the tensor force does not contain the fourth order contribution. Therefore, in order to fit the data for deuteron ground state, it will be necessary to modify the potential in the triplet even state by means of a more artificial cut off method than in the case of pseudovector coupling.

The central potential in the triplet odd state and the potential in the singlet odd state are nearly equal to the potential in the singlet even state and the central potential in the triplet even state respectively, because of predominance of the fourth order contributions over the second order ones. These strong forces in the odd states are incompatible with the angular distributions of the high energy $n-p$ scattering, as was mentioned in § 1.

The author wishes to thank to Professor K. Nakabayasi for helpful discussions and for his kind encouragement. He is also indebted to Prof. M. Taketani, Prof. N. Fukuda, Prof. M. Sugawara, and Dr. S. Machida for interesting discussions and helpful comments.

Appendix I

To illustrate method of calculations of G , H and K appearing in (44), we calculate G as an example. G is given by (45). In (45), we transform the integration variable l into u by means of (47). Then we obtain from (46)

$$D = \Delta/u$$

with

$$\Delta = M^2(1-u)^2 + l^2 u + (\mathbf{p} - \mathbf{q})^2 u^2 v(1-v)$$

irrespective of sign of l , and we have

$$E_p J G = \frac{1}{4(E_p + J)^2} \int_0^{M/(E_p + J)} \frac{[(E_p + J)^2 u^2 - M^2]^3}{u^2 \Delta [(E_p + J)^2 u^2 + M^2]} du - (J \rightarrow -J),$$

where $(J \rightarrow -J)$ denotes the term obtained by writing $-J$ in place of J in the preceding term. This is rewritten as

$$E_p J G = I + II, \tag{A.1}$$

$$I = E_p J \int_0^{M/E_p} du u^{-2} \Delta^{-1} \{u^4 + M^4(E_p^2 - J^2)^{-2}$$

$$-8u^4 M^4 [(E_p + J)^2 u^2 + M^2]^{-1} [(E_p - J)^2 u^2 + M^2]^{-1}, \quad (\text{A} \cdot 2)$$

$$II = \frac{1}{4(E_p + J)^2} \int_{M/E_p}^{M/(E_p + J)} \frac{[(E_p + J)^2 u^2 - M^2]^3}{u^2 \Delta [(E_p + J)^2 u^2 + M^2]} du - (J \rightarrow -J). \quad (\text{A} \cdot 3)$$

Expanding the terms in the curly bracket of (A.2) in powers of p/M , and neglecting the terms of the order of $(p/M)^4$, we have

$$I = E_p J \int_0^{M/E_p} du u^{-2} \Delta^{-1} \{ (1 - u^2)^2 (1 + 4u^2 + u^4) (1 + u^2)^{-2} \\ + 32 (p/M)^2 u^6 (1 + u^2)^{-4} - 2 (k/M)^2 v (1 - v) [1 + 8u^6 (1 - u^2) (1 + u^2)^{-4}] \},$$

where $k = |p - q|$. This is rewritten as

$$I/(E_p J) = \int_0^1 du u^{-2} \Delta^{-1} \{ (1 - u^2)^2 (1 + 4u^2 + u^4) (1 + u^2)^{-2} \\ + 32 (p/M)^2 u^6 (1 + u^2)^{-4} - 2 (k/M)^2 v (1 - v) [1 + 8u^6 (1 - u^2) (1 + u^2)^{-4}] \} \\ - \int_{M/E_p}^1 du u^{-2} \Delta^{-1} \{ 32 (p/M)^2 u^6 (1 + u^2)^{-4} - 2 (k/M)^2 v (1 - v) \}.$$

In obtaining this, we have neglected terms of the order of $M^{-2}(v/c)^3$. In the first integral, we can replace $(k/M)^2 v(1-v)$ by $-g(u)u^{-2}$ since we ignore the contact interactions. In the integrand of the second integral we may put $u=1$. Then we have

$$I/(E_p J) = \int_0^1 du \Delta^{-1} \{ (1 - u^2)^2 (1 + 4u^2 + u^4) u^{-2} (1 + u^2)^{-2} \\ + 2g(u) [u^{-4} + 8u^2 (1 - u^2) (1 + u^2)^{-4}] + 32 (p/M)^2 u^4 (1 + u^2)^{-4} \} \\ - 2 (J/M)^2 [(E_p - M)/M] \Delta_0^{-1}, \quad (\text{A} \cdot 4)$$

where

$$\Delta_0 = p^2 + k^2 v(1-v).$$

Next we calculate II. We put $u = (M/E_p) - (J/M)\tau$ in the first term in the right hand side of (A.3), and put $u = (M/E_p) + (J/M)\tau$ in the second term. After this, we expand the integrands in powers of v/c . Then we have, to our approximation,

$$II/(E_p J) = (J/M)^4 \int_0^1 d\tau w (\Delta_0 + J^2 \tau^2)^{-1} (1 - w)^2 \\ \times \{ 3(1 + w^2) - 2(1 - \tau)w (\Delta_0 - J^2) (\Delta_0 + J^2 \tau^2)^{-1} \},$$

which is rewritten by the integration by parts as

$$II/(E_p J) = (J/M)^4 J^{-2} \{ [(1 - \tau)^3 (\Delta_0 - J^2) (\Delta_0 + J^2 \tau^2)^{-1}]_0^1 \\ + 3 \int_0^1 (1 - \tau)^2 d\tau \} = (J/M)^4 \Delta_0^{-1}. \quad (\text{A} \cdot 5)$$

From (A.1), (A.4) and (A.5), we obtain the expression (48) for G .

Appendix II

To illustrate the method by which (58) is obtained from (57), we calculate as an example the following integral contained in (57) :

$$R = \int d^3 \mathbf{l} \{ \epsilon_{l+q}^{-3} [(\mathbf{l} + \mathbf{q}, \mathbf{k}) + (k^2/2)]^{-1} \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}) \rangle_- + (\mathbf{q} \rightarrow -\mathbf{p}) \}.$$

We write \mathbf{l} in place of $\mathbf{l} + \mathbf{q}$ in the first term in the integrand of R , and in place of $\mathbf{l} - \mathbf{p}$ in the second term. Then we have

$$\begin{aligned} R &= R_0 + R', \\ R_0 &= 2 \int d^3 \mathbf{l} \epsilon_l^{-3} [(\mathbf{l}, \mathbf{k}) + k^2/2]^{-1} \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_-, \\ R' &= \lim_{l \rightarrow \infty} \int d\omega_l l(\mathbf{l}, \mathbf{q}) \int_0^1 d\varepsilon \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}\varepsilon) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l} + \mathbf{q}\varepsilon) \rangle_- \\ &\quad \times \epsilon_{l+q\varepsilon}^{-3} [(\mathbf{k} \cdot \mathbf{l} + \mathbf{q}\varepsilon) + k^2/2]^{-1} + (\mathbf{q} \rightarrow -\mathbf{p}). \end{aligned}$$

R' is the surface term. $\int d\omega_l$ means integration over the direction of \mathbf{l} . Dropping the terms which tend to zero as $l \rightarrow \infty$, we have

$$\begin{aligned} R' &= \lim_{l \rightarrow \infty} \int d\omega_l l(\mathbf{l}, \mathbf{q}) \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_- \\ &\quad \times \epsilon_l^{-3} \int_0^1 d\varepsilon [(\mathbf{k} \cdot \mathbf{l}) + (k^2/2)(1 - \varepsilon)]^{-1} + (\mathbf{q} \rightarrow -\mathbf{p}) \\ &= -(2/k^2) \lim_{l \rightarrow \infty} \int d\omega_l l(\mathbf{l}, \mathbf{k}) \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{l}) \rangle_- \\ &\quad \times \epsilon_l^{-3} \log |(\mathbf{l}, \mathbf{k}) + k^2/2|. \end{aligned}$$

Performing the integration, and taking account of the fact that $\langle i(\boldsymbol{\gamma} \cdot \mathbf{k}) \rangle_+ \langle i(\boldsymbol{\gamma} \cdot \mathbf{k}) \rangle_- = 0$, we have

$$\begin{aligned} R' &= \langle \boldsymbol{\gamma} \rangle_+ \langle \boldsymbol{\gamma} \rangle_- (2\pi/k) \lim_{l \rightarrow \infty} l^4 \epsilon_l^{-3} \\ &\quad \times \{ (1/4) [1 - (k/l)^2 + (k^4/4l^4)] \\ &\quad \times \log | (1 + k/2l) (1 - k/2l)^{-1} | + (5k/12l) - (k^3/16l^3) \} \\ &= (4\pi/3) \langle \boldsymbol{\gamma} \rangle_+ \langle \boldsymbol{\gamma} \rangle_-. \end{aligned}$$

We see from this that R' contributes only to the contact interactions. We therefore omit R' .

By integrating over the directions of \mathbf{l} , R_0 is transformed into

$$\begin{aligned} R_0 &= -\langle \boldsymbol{\gamma} \rangle_+ \langle \boldsymbol{\gamma} \rangle_- (2\pi/k) \int_0^\infty dl l^3 (l^2 + k^2)^{-3/2} \\ &\quad \times \{ [1 - (k/2l)^2] \log | (1 + k/2l) (1 - k/2l)^{-1} | + k/l \}. \end{aligned}$$

Integrating by parts, we obtain from this

$$R_0 = -\langle \mathbf{r} \rangle_+ \langle \mathbf{r} \rangle_- 4\pi \{k^{-1} \int_0^\infty d\ell \ell (\ell^2 + \mu^2)^{-1/2} \\ \times \log |(1 + k/2\ell)(1 - k/2\ell)^{-1} - 1|\}.$$

Integrating by parts again, we further obtain

$$R_0 = -\langle \mathbf{r} \rangle_+ \langle \mathbf{r} \rangle_- 4\pi \int_0^\infty (\ell^2 + \mu^2)^{1/2} (\ell^2 - k^2/4)^{-1} d\ell.$$

The integrand has branch points at $\ell = \pm i\mu$. We can transform path of the integration into straight line connecting $i\mu$ and $i\infty$. Putting $\ell = i\mu w$, we then have

$$R = -\langle \mathbf{r} \rangle_+ \langle \mathbf{r} \rangle_- 16\pi\mu^2 \int_1^\infty (w^2 - 1)^{1/2} (k^2 + 4\mu^2 w^2)^{-1} dw.$$

Appendix III

We collect here the formulae, by which (76)–(79) and (81) are obtained.

$$\begin{aligned} \int_1^\infty t K_0(2tx) dt &= (2x)^{-1} K_1(2x), \\ \int_1^\infty t^3 K_0(2tx) dt &= (1/2) [x^{-2} K_0(2x) + (x^{-1} + x^{-3}) K_1(2x)], \\ \int_1^\infty t^5 K_0(2tx) dt &= (1/2) [(2x^{-2} + 4x^{-4}) K_0(2x) \\ &\quad + (x^{-1} + 4x^{-3} + 4x^{-5}) K_1(2x)], \\ \int_1^\infty K_1(2tx) dt &= (2x)^{-1} K_0(2x), \\ \int_1^\infty t^2 K_1(2tx) dt &= (1/2) [x^{-1} K_0(2x) + x^{-2} K_1(2x)], \\ \int_1^\infty t^4 K_1(2tx) dt &= (1/2) [(x^{-1} + 2x^{-3}) K_0(2x) \\ &\quad + 2(x^{-2} + x^{-4}) K_1(2x)], \\ \int_1^\infty t(t^2 - 1)^{1/2} K_0(2tx) dt &= (\pi/16) (2x^{-2} + x^{-3}) e^{-2x}, \\ \int_1^\infty t^2 (t^2 - 1 + \lambda^2/4)^{-1/2} K_1(2tx) dt \\ &= (\pi/8) (2x^{-1} + x^{-2}) e^{-2x} - (\lambda/2) K_1(2x), \end{aligned}$$

$$\begin{aligned}
& \int_1^{\infty} (4t^2-1)(t^2-1)^{1/2} K_1(2tx) dt = (3\pi/8) x^{-2} (1+x^{-1})^2 e^{-2x}, \\
& \int_1^{\infty} t(t^2-1+\lambda^2/4)^{-1/2} K_0(2tx) dt \\
& \quad = (\pi/4) x^{-1} e^{-2x} - (\lambda/2) K_0(2x).
\end{aligned}$$

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Letters to the Editor

New Proposal on the Mass Spectrum of Elementary Particles

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July 8, 1953

In case one wishes to measure μ_n , the mass of each elementary particle, by the electron mass unit, Nambu's mass spectrum¹⁾ is shown as follows,

$$\mu_n = 137 n. \quad (1)$$

In this empirical formula, one feels interest that it is written with the attractive number 137 and either 'n' is full or half integral number is likely to correspond to the spin of particles being either one or the other. Moreover, since one is able to regard this formula as an arithmetical series with a small common difference 137/2, the agreement between the

calculation and observation values is somehow good. Now let this formula be called by name of Mass Spectrum-A and inserted in Table-1.

However, in spite of the consideration that 'n' may indicate the spin of particles, the equation (1) shows that the mass of an electron or a neutrino appears respectively when $n=1/137$ or $n=0$ contrary to the fact that these particles are both equally fermions. Here one inevitably meets with difficulty about interpretation of 'n' in the equation (1). In short, it may be said that Mass Spectrum-A has inconsistency in itself. The reason which caused such self-contradiction, we suppose, is that the mass spectrum has approximately been interpreted as an arithmetical series of the equation (1) whereas it may probably be a geometrical series with a small common ratio. Therefore, we first presume the following equation as the true mass spectrum at which we have aimed; that is,

$$\mu_n = 137 \omega^n, \quad (2)$$

where the electron mass unit is used as before and

Table 1. Comparison between two types of mass spectra of elementary particles

Name of Particles	Observation value		Mass spectrum-G $\mu_n = 137 \omega^n$		Mass spectrum-A $\mu_n = 137 n$	
	Mass (m_e)	Spin (\tilde{n})	Mass (m_e)	n	Mass (m_e)	n
V_1^0	~2200	$(\frac{1}{2})$	2193.1	$15\frac{1}{2}$	2260.5	$16\frac{1}{2}$
P	1836	$\frac{1}{2}$	1833.8	$14\frac{1}{2}$	1849.5	$13\frac{1}{2}$
x^\pm	1470 ± 100	(0, 1)	1402.2	13.0	1507.0	11.0
x^\pm	~1300	$(\frac{1}{2})^*$	1282.2	$12\frac{1}{2}$	1301.5	$9\frac{1}{2}$
τ^\pm	~980	(0, 1)	980.4	11.0	959.0	7.0
V_2^0	~800	(0, 1)	819.8	10.0	822.0	6.0
ζ^0	~560	(0, 1)	573.2	8.0	548.0	4.0
ζ^\pm	535 ± 35	$(\frac{1}{2})^*$	524.2	$7\frac{1}{2}$	479.5	$3\frac{1}{2}$
π^\pm	276	0, 1	280.2	4.0	274.0	2.0
μ^\pm	210	$\frac{1}{2}$	214.3	$2\frac{1}{2}$	205.5	$1\frac{1}{2}$
e^\pm	1	$\frac{1}{2}$	1	$-27\frac{1}{2}$	1	$\frac{1}{137}$
ν	~0	$\frac{1}{2}$	0	$-\infty$	0	0
γ	0	1	0	$-\infty$	0	0

Notes: $\omega = 137^{1/27.5} = 1.1959 \dots$

* As ζ^\pm is more than 10^4 times the lifetime of ζ^0 , it is assumed that ζ^\pm disintegrates into a μ^\pm -meson and two neutrinos in the same manner with x^\pm ; i.e., ζ^\pm (or x^\pm) $\rightarrow \mu^\pm + 2\nu$.

ω is a certain constant of this exponential function.

In order to know the value of ω , we ask for collation with recent observations²⁾ and provisionally settle the next assumption.

$$\omega^{3/2} = m_{\pi}/m_{\mu} \cong 276/210 \cong 1.31, \therefore \omega \cong 1.196, (3)$$

where m_{π} or m_{μ} is respectively the mass of a π - or μ -meson.

Secondly, in appli-ance of above value of ω , we investigate other mass ratio between various kinds of elementary particles under the same note as is expressed in m_V, m_P, \dots , and so forth; viz.,

$$m_{V_1}/m_P \cong 2200/1836 \cong \omega,$$

$$m_{\pi}/m_K \cong 1470/1300 \cong \omega^{1/2},$$

$$m_{\tau}/m_{V_2} \cong 980/800 \cong \omega,$$

$$m_{V_2}/m_{\nu} \cong 800/560 \cong \omega^2, \text{ etc.}$$

Here we become aware of the fact that these above mentioned relations are so suggestive that we are obliged to determine the final value of ω by the following method with reference to the equation (3); namely, if the mass of an electron should be determined when $n = -27.5$ at the equation (2), then,

$$1 = 137 \omega^{-27.5}, \therefore \omega = 137^{1/27.5} = 1.1959 \dots (4)$$

With this value of ω , we have calculated other mass values of particle in the equation (2), called by name of Mass Spectrum-G and inserted also in Table 1. The agreement between the calculation and observation values is remarkably good as well as in case of M.S.-A.

Different from other newly discovered particles, protons and electrons, because of their stability, have been so minutely studied since former days that the

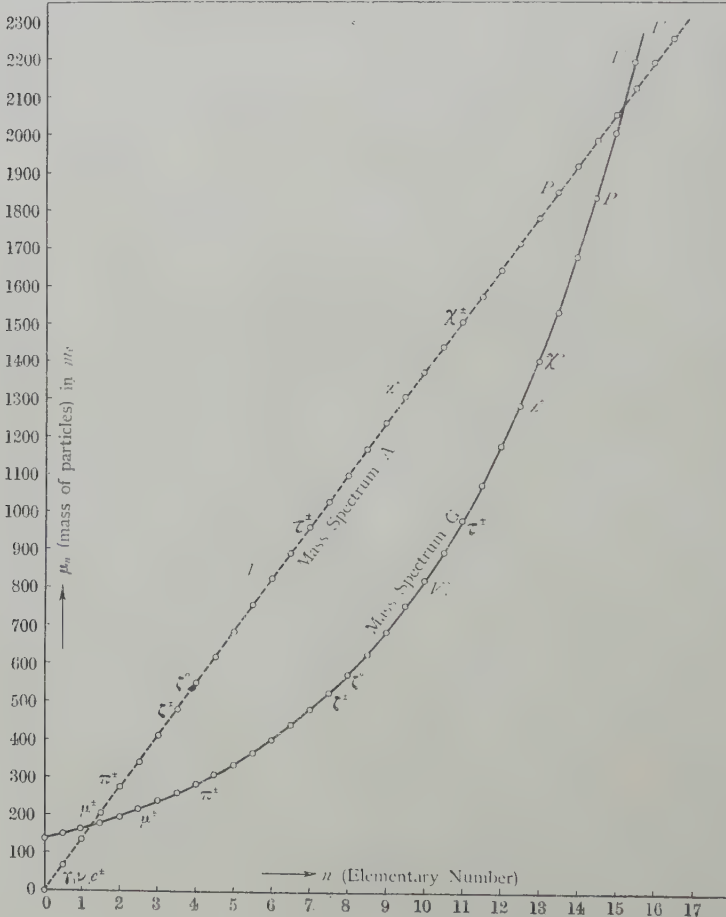


Fig. 1.

mass value of them is particularly reliable and precise. With regard to this proton mass, therefore, we will examine the deviation from the observation value in these two types of mass spectra. For the proton mass, the calculation value of M.S.-A is about $13m_e$ greater than the observation value, but in M.S.-G the calculation value is smaller only by about $2m_e$.

Further, M.S.-G has also an interesting characteristic that ' n ' may probably indicate the spin of particles, but no inconsistency is thought to be found out in itself. The electron mass is obtained when $n = -27\frac{1}{2}$ and clearly points out its spin, a half integral number. Although the neutrino mass is obtained when $n = -\infty$, it is quite favorable that infinity has the same significance for both full and half integral numbers; viz., that infinity is thought to represent, simultaneously, each case of a photon and a neutrino without any trouble.

Therefore, we dare to propose our Mass Spectrum-G because of its rationality and its good agreement with recent observations.

(Now Fig. 1 shows more clearly each characteristic of Mass Spectrum-G and-A.)

Appendix

In view of the above discussion, we infer that Mass Spectrum-G has been obtained by satisfying the following assumption;

- (1) The mass spectrum takes the shape of the exponential function as follows,

$$\mu_n = m_0 \omega^n,$$

where,

$$\left\{ \begin{array}{l} \mu_n = \text{the mass of each particle,} \\ m_0 = \text{the standard mass} = 137 m_e, \\ \omega = \text{the constant of this} \\ \quad \text{exponential function,} \\ n = \text{full or half integral number,} \end{array} \right.$$

- (2) Either ' n ' is full or half integral number which corresponds to the spin of the particles being either one or the other.
- (3) ω takes the maximum value of all that may have possibility to satisfy the above assumption (1) and (2).

The condition (3) is indispensable in order to obtain the unity of the mass spectrum.

Now the mass spectrum which may have possibility to satisfy the above assumption (1) and (2) is generally expressed as follows,

$$\mu_n = m_0 \left(\frac{m_\pi}{m_\mu} \right)^{\frac{2n}{3(2a+1)}},$$

where ' a ' is the integral number; $a \geq 0$.

Therefore, general expression of ω is,

$$\omega = \left(\frac{m_\pi}{m_\mu} \right)^{\frac{2}{3(2a+1)}}.$$

Hereupon, the maximum value of ω is obtained clearly when $a=0$, then,

$$\omega_{\max} = \left(\frac{m_\pi}{m_\mu} \right)^{\frac{2}{3}}.$$

—cf. the equation (3)—

The reason why we must choose the maximum value of ω is a future problem, but we anticipate that, in future, what it means will perhaps be understood as the state of being the most realizable in the way of distribution of the static energy $\mu_n c^2$ to each elementary particle.

The authors thank Mr. Y. Nambu, the professor of Osaka City University, for much helpful discussion and advice.

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On the Interaction Forms of the Beta-decay

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August 1, 1953

We proposed a formulation of the beta-decay on the basis of the following postulates:

- (a) The primary interactions between nucleons, leptons and bosons, by which the beta-decay interactions are derived, are renormalizable.
- (b) The Fermi and the Gamow-Teller interactions are approximately equally present in the beta-decay matrix elements.

In the previous letter,¹⁾ the primary interactions which satisfy these postulates were given by

$$H'_s = g'(\bar{\psi}_P \psi_N) \varphi^* + f'(\bar{\psi}_s \psi_N) \varphi + \text{compl. conj.}, \quad (1)$$

or

$$H'_{ps} = g'(\bar{\psi}_P \gamma_5 \psi_N) \varphi'^* + f'(\bar{\psi}_s \gamma_5 \psi_N) \varphi' + \text{compl. conj.} \quad (2)$$

where ψ_P, ψ_N, ψ_e and ψ_ν are proton, neutron, electron and neutrino wave functions, and φ or φ' is a scalar or pseudoscalar boson field.

The transformation under space inversion of fermion wave functions leads to another possible form than (1) or (2). Following Yang and Tiomno,²⁾ if it turns out that protons, neutrons and electrons are all of type A, while the neutrino is of type B, we have the following interactions instead of (1) and (2)

$$H'_S = g'(\bar{\psi}_P \gamma_5 \psi_\nu) \varphi^* + f'(\bar{\psi}_e \psi_N) \varphi + \text{compl. conj.}, \quad (3)$$

or

$$H'_T = g'_{TS}(\bar{\psi}_P \psi_\nu) \varphi'^* + f'(\bar{\psi}_e \gamma_5 \psi_N) \varphi' + \text{compl. conj.} \quad (4)$$

On the same footing mentioned in our previous letter, if we assume a mixed theory of (3) and (4), the Fierz interference factor of the beta-decay can be cancelled out. We have the beta-decay Hamiltonian which just corresponds to a definite combination of the Yang-Tiomno direct interactions ($S' - T' + P'$). The Yang-Tiomno interactions S', T' and P' are obtained by replacing the neutrino wave function ψ_ν in the Fermi direct interactions of the beta-decay S (scalar), T (tensor) and P (pseudoscalar) by $\gamma_5 \psi_\nu$.

Tokyo theoretical beta-decay group has strongly suggested ($S - T'$) or ($S - T' + P$) combinations on the basis of the $\beta - \gamma$ angular correlation of Sb^{124} and the β spectral analyses of Fe^{59} , Tc^{99} and $\text{Rb}^{87,3)}$

Our theory might give a theoretical reasoning of this combinations.

This letter resulted from discussions with Dr. H. Umezawa,⁴⁾ Messrs. S. Morita and S. Tanaka.

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On the Electronic Structure of LiH Atomic Orbital Approach with Configurational Interaction

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August 4, 1953

An atomic orbital calculation on the ground state of LiH was carried out including all the electrons. This is the case which was left untouched by Fischer.¹⁾

Supposing that the ground state of LiH is $1^1\Sigma$, we adopted the following six lowest configurations:

$$\begin{aligned} \Psi_I &= (N_I/\sqrt{48}) \{ |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta| \\ &\quad + |x_1\beta \ x_1\alpha \ x_2\beta \ x_3\alpha| \}, \\ \Psi_{II} &= (N_{II}/\sqrt{48}) \{ |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta| \\ &\quad + |x_1\beta \ x_1\alpha \ x_2\beta \ x_3\alpha| \}, \\ \Psi_{III} &= (N_{III}/\sqrt{24}) |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta|, \\ \Psi_{IV} &= (1/\sqrt{24}) |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta|, \\ \Psi_V &= (1/\sqrt{24}) |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta|, \\ \Psi_{VI} &= (1/\sqrt{48}) \{ |x_1\alpha \ x_1\beta \ x_2\alpha \ x_3\beta| \\ &\quad + |x_1\beta \ x_1\alpha \ x_2\beta \ x_3\alpha| \}, \end{aligned}$$

where x_1, x_2 and x_3 represent the $1s, 2s$ and $2p\sigma$ -orbital of Li atom and x_4 the $1s$ -orbital of H atom. α and β are the spin eigenfunctions. The same forms of χ 's were used as in the case of Fischer:

$$\begin{aligned} \chi_1 &= \sqrt{a^3/\pi} e^{-ar}, \\ \chi_2 &= [1 - (x_1, x_2)']^{-1/2} [x_2' - (x_1, x_2')x_1], \\ x_2' &= \sqrt{b^5/3\pi} r e^{-br}, \\ \chi_3 &= \sqrt{c^5/\pi} r e^{-cr} \cos \theta, \\ \chi_4 &= \sqrt{d^3/\pi} e^{-dr}, \end{aligned}$$

with $a=2.69, b=0.658, c=0.545$ and $d=1.00$.

By using the linear combination of the type

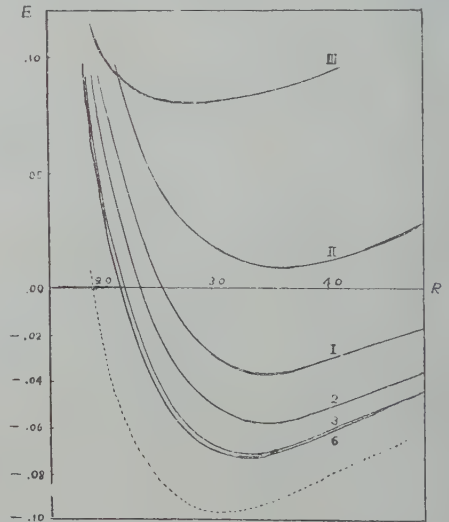


Fig. 1. Energy curves for the ground state of LiH. I, II, III: diagonal energy of single configuration Ψ_I, Ψ_{II} or Ψ_{III} . 2, 3, 6: result when 2, 3 or 6 configurations are considered. Dotted curve: observed energy. (Only in this case the origin of the scale is taken to be the true atomic state which lies 0.066 a.u. below the calculated reference state.)

$$\Psi = \sum_K^{VI} C_K \Psi_K,$$

the total energy was minimized to give the ground state for the three values of internuclear distance R (2.64, 3.02 and 3.4 a.u.). Higher configurations were taken into account successively and we obtained the results shown in Fig. 1, where the atomic reference state was taken as the calculated total energy of infinite nuclear separation.

The binding energy 0.0725 a.u., which is expected as an underestimation, amounts to 75% of the observed value. The first three configurations prove dominant in lowering the energy. This involves the presence of s , p -hybridization in the ratio $(C_{II} N_{II}) / (C_I N_I) \sim 0.5$ and also non-negligible importance of the ionic structure $\text{Li}^+ \text{H}^-$ ($C_{III} \sim 0.3$). The fundamental frequency $1.13 \times 10^3 \text{ cm}^{-1}$ is near but smaller than the experimental value $1.4 \times 10^3 \text{ cm}^{-1}$, which is considered reasonable in tendency.

Having found some errors in Fischer's table of integrals in the course of calculation, we corrected them and re-examined the valence shell treatment. (Cf. Fig. 2.) Inclusion of polar structure gives too

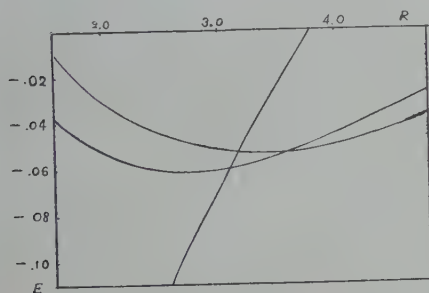


Fig. 2. Energy curves in 2-electron case. (I, II and III correspond to those in Fig. 1)

big binding energy and too small nuclear distance. Fundamental frequency is far from the observation in any case. This type of treatment, therefore, seems far less reliable than expected hitherto.²⁾

Further increase of the number of configurations amplifies the difficulty of calculation, but will give little improvement. Therefore a more practical way seems to limit ourselves to the first few configurations and to deform the component atomic orbitals in order to reach a better result.

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Interference Terms of β -ray Angular Correlations*

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August 12, 1953

A whole table of β -ray angular distribution functions $F_{LL'}^M(\theta)$ has been constructed by Yamada and the present author¹⁾ for each pure type. In this note, $F_{LL'}^M(\theta)$'s for the interference terms between different types will be given. The notation and the methods of calculations are given in our previous paper.¹⁾

The interaction Hamiltonian for β -decay is assumed to be a linear combination of five relativistic invariant terms

$$H_\beta = G_S S + G_V V + G_T T + G_A A + G_P P,$$

with the coupling constants G_i 's which are all real.

The results are:

- 1) Allowed. All isotropic.
- 2) First forbidden:

S and T

$$F_{11}^0(\theta) = G_S G_T [\{i\mathcal{M}^*(\beta\mathbf{r})\mathcal{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} \times \{(L_1 - M_0) + (K L_{12} + L_1 - N_{12}) P_2(\cos \theta)\} - \{i\mathcal{M}^*(\beta\mathbf{r})\mathcal{M}(\beta\boldsymbol{\alpha}) + \text{c.c.}\} \{((1/3) K L_0 + N_0) + 2 L_{12} P_2(\cos \theta)\}],$$

$$F_{12}^1(\theta) = G_S G_T \{\mathcal{M}^*(\beta\mathbf{r})\mathcal{M}(B_i j \beta) + \text{c.c.}\} (-K L_{12} + 3 L_1 - 3 N_{12}) (1/2\sqrt{2}) P_2(\cos \theta).$$

S and A

$$F_{11}^0(\theta) = -G_S G_A \{i\mathcal{M}^*(\beta\mathbf{r})\mathcal{M}(\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} \times [\{(2/3) K N_0 + L_1 + M_0\} + \{(1/3) K L_{12} + L_1 + N_{12}\} P_2(\cos \theta)],$$

$$F_{12}^1(\theta) = -G_S G_A \{\mathcal{M}^*(\beta\mathbf{r})\mathcal{M}(B_i j \beta) + \text{c.c.}\} \times (K L_{12} + 3 L_1 + 3 N_{12}) (1/2\sqrt{2}) P_2(\cos \theta).$$

V and T

$$F_{11}^0(\theta) = G_V G_T [\{i\mathcal{M}^*(\mathbf{r})\mathcal{M}(\beta\boldsymbol{\alpha}) + \text{c.c.}\} \times \{((1/3) K L_0 - N_0) - 2 L_{12} P_2(\cos \theta)\} + \{i\mathcal{M}^*(\mathbf{r})\mathcal{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} \{((2/3) K N_0 - L_1 - M_0) + ((1/3) K L_{12} - L_1 - N_{12}) \times P_2(\cos \theta)\} - \{\mathcal{M}^*(\boldsymbol{\alpha})\mathcal{M}(\beta\mathbf{j} \times \mathbf{r}) + \text{c.c.}\} \times \{((1/3) K L_0 - N_0) + L_{12} P_2(\cos \theta)\} + \{\mathcal{M}^*(\boldsymbol{\alpha})\mathcal{M}(\beta\boldsymbol{\alpha}) + \text{c.c.}\} L_0],$$

$$F_{12}^1(\theta) = -G_V G_T [\{i\mathcal{M}^*(\boldsymbol{\alpha})\mathcal{M}(B_i j \beta) + \text{c.c.}\} \times (3/2\sqrt{2}) L_{12} P_2(\cos \theta) + \{\mathcal{M}^*(\mathbf{r})\mathcal{M}(B_i j \beta) + \text{c.c.}\}$$

$$\times \{K L_{12} - 3 L_1 - 3 N_{12}\} (1/2\sqrt{2}) P_2(\cos \theta)].$$

V and A

$$F_{11}^0(\theta) = G_V G_A [\{i\mathcal{M}^*(\mathbf{r})\mathcal{M}(\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} \\ \times \{(L_1 - M_0) + (-L_{12} K + L_1 - N_{12}) P_2(\cos \theta)\} \\ + \{\mathcal{M}^*(\mathbf{a})\mathcal{M}(\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} \{(1/3) K L_0 + N_0 \\ - L_{12} P_2(\cos \theta)\}],$$

$$F_{12}^1(\theta) = G_V G_A [\{i\mathcal{M}^*(\mathbf{r})\mathcal{M}(B_{ij}) + \text{c.c.}\} \\ \times \{K L_{12} + 3 L_1 - 3 N_{12}\} (1/2\sqrt{2}) \\ - \{i\mathcal{M}^*(\mathbf{a})\mathcal{M}(B_{ik}) + \text{c.c.}\} (3/2\sqrt{2}) L_{12}] \\ \times P_2(\cos \theta).$$

T and P

$$F_{00}^0(\theta) = -G_T G_P \{i\mathcal{M}^*(\boldsymbol{\beta} \boldsymbol{\sigma} \cdot \mathbf{r})\mathcal{M}(\beta \gamma_5) + \text{c.c.}\} \\ \times \{(1/3) K L_0 + N_0\},$$

$$F_{02}^0(\theta) = G_T G_P \{i\mathcal{M}^*(\beta \gamma_5)\mathcal{M}(B_{ij}) + \text{c.c.}\} \\ \times \sqrt{3/2} L_{12} P_2(\cos \theta).$$

A and P

$$F_{00}^0 = G_A G_P [\{i\mathcal{M}^*(\boldsymbol{\sigma} \cdot \mathbf{r})\mathcal{M}(\beta \gamma_5) + \text{c.c.}\} \\ \times \{(1/3) K L_0 - N_0\} \\ + \{\mathcal{M}^*(\gamma_5)\mathcal{M}(\beta \gamma_5) + \text{c.c.}\} L_0],$$

$$F_{02}^0(\theta) = G_A G_P \{i\mathcal{M}^*(\beta \gamma_5)\mathcal{M}(B_{ij}) + \text{c.c.}\} \\ \times \sqrt{3/2} L_{12} P_2(\cos \theta).$$

3) Second forbidden:

S and T

$$F_{22}^0(\theta) = -G_S G_T \{i\mathcal{M}^*(R_{ijk})\mathcal{M}(A_{ijk}) + \text{c.c.}\} \\ \times [\{(1/30) K^3 L_0 + (1/6) K^2 N_0 + (1/2) K L_1 \\ + (3/2) N_1\} + \{(1/6) K^2 L_{12} + (1/2) K L_1 \\ + (9/14) L_{23} + (3/2) N_1\} P_2(\cos \theta) \\ + (27/7) L_{23} P_4(\cos \theta)] \\ + G_S G_T \{i\mathcal{M}^*(R_{ijk})\mathcal{M}(T_{ijk}) + \text{c.c.}\} \\ \times [\{(1/6) (K^2 L_1 - K^2 M_0) + (3/2) (L_2 - M_1)\} \\ + \{(1/30) K^3 L_{12} + (1/6) (K^2 L_1 - K^2 N_{12}) \\ + (5/14) K L_{23} + (12/7) L_2 - (3/2) M_1 \\ - (3/14) N_{23}\} P_2(\cos \theta) + \{(15/7) K L_{23} \\ + (9/7) (L_2 - N_{23})\} P_4(\cos \theta)],$$

$$F_{23}^1(\theta) = G_S G_T \{\mathcal{M}^*(R_{ijk})\mathcal{M}(S_{ijk}) + \text{c.c.}\} \\ \times (\sqrt{10}/\sqrt{3}) [\{(1/150) (-K^3 L_{12} + 5K^2 L_1 \\ - 5K^2 N_{12}) + (1/14) (-K L_{23} + 3L_2 - 3N_{23})\} \\ \times P_2(\cos \theta) + (5/28) (-K L_{23} + 3L_2 \\ - 3N_{23}) P_4(\cos \theta)].$$

The other terms have not been calculated. Some of them are easily deducible from the present formulae

by the rule described in our previous paper.¹⁾ However, they might not have importance. The expressions for L_i , M_i , N_i ; L_i^- , M_i^- , N_i^- and L_{ijk} , N_{ijk} have been given by Konopinski and Uhlenbeck²⁾, Smith³⁾ and Yamada and Morita¹⁾, respectively. The other notations are as follows:

$$L_{12}^- = \left(\frac{f^2}{2\pi} R \right)^{-1} \\ \times \frac{g_{-1} f_2 \cos(\delta_{-1} - \delta_2) + f_1 g_{-2} \cos(\delta_1 - \delta_{-2})}{4\pi\rho} \rightarrow 0, \\ N_{12}^- = \left(\frac{f^2}{2\pi} R \right)^{-1} \\ \times \frac{f_{-1} f_2 \cos(\delta_{-1} - \delta_2) - g_1 g_{-2} \cos(\delta_1 - \delta_{-2})}{4\pi\rho^2} \\ \rightarrow -\frac{f^2}{911}.$$

The arrow in each case indicates the approximation $aZ \ll 1$.

The author wishes to express his sincere thanks to Profs. T. Yamanouchi and S. Nakamura, and to Dr. M. Umezawa for their kind guidance throughout the work, and to Mr. M. Yamada for carrying out the check of $F_{TL,LM}(\theta)$'s in the second forbidden.

* This work has been performed during the author's stay at the Yukawa Hall, Kyoto University, Kyoto. He is also greatly indebted to Prof. M. Kobayasi for the hospitality.

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Note on Coexistence of Tensor and Axialvector Interactions in β -decay*

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August 12, 1953

The coupling constants for β -decay must be taken real (or be in phase) if the invariance with respect to the Wigner type time reversal¹⁾ or the charge conjugacy²⁾ is restricted. Then it follows easily that either the Tensor or the Axialvector interaction is absent to make the $1/W$ term in the allowed spectrum³⁾ vanish, while the Axialvector interaction is excluded by the experimental evidence concerning the $e-\nu$ P^{32} ^{4,5)} angular correlations for P^{32} ^{4,5)} and He^6 ⁶⁾.

It may, however, not yet certain whether these invariances should be persisted: We have two different kinds of time reversal⁷⁾, i.e., the Wigner and Pauli types. These two do not necessarily result in the same consequences, and we have at present no criterion to choose one of them as the correct one. It is also probable that the difference of neutron and proton does not originate entirely from their electromagnetic properties only, and this may bring further inequivalency between them.

In this stage of development, our research of β -decay is required to be phenomenological without *a priori* restrictions. The coupling constants G_i , therefore, are allowed to take complex values also. Then the $e-\nu$ angular correlation functions \mathfrak{B} will be given by the following formulae in respective cases. Fermi:

$$\begin{aligned} \mathfrak{B}_{0SV} = & |G_S \mathfrak{M}(\beta)|^2 \{L_0 - 2A_1 \cos \theta\} \\ & + |G_V \mathfrak{M}(1)|^2 \{L_0 + 2A_1 \cos \theta\} \\ & - \mathfrak{M}^*(\beta) \mathfrak{M}(1) [G_S^* G_V + G_S G_V^*] L_0 \\ & + \{i G_S^* G_V - i G_S G_V^*\} 2A_1 \cos \theta. \end{aligned} \quad (1)$$

Gamow-Teller:

$$\begin{aligned} \mathfrak{B}_{0TA} = & |G_T \mathfrak{M}(\beta\sigma)|^2 \{L_0 + (2/3)A_1 \cos \theta\} \\ & + |G_A \mathfrak{M}(\sigma)|^2 \{L_0 - (2/3)A_1 \cos \theta\} \\ & - \mathfrak{M}^*(\beta\sigma) \mathfrak{M}(\sigma) [\{G_T^* G_A + G_T G_A^*\} L_0 \\ & + \{i G_T^* G_A - i G_T G_A^*\} 2A_1 \cos \theta]. \end{aligned} \quad (2)$$

The notations are as follows:

\mathfrak{B} 's are reduced nuclear matrix elements.

$$\left. \begin{matrix} L_0 \\ L_0^- \end{matrix} \right\} = \left(\frac{p^2}{2\pi} F \right)^{-1} \frac{g_{-1}^2 \pm f_1^2}{4\pi} \rightarrow \left\{ \begin{matrix} 1 \\ 1/W \end{matrix} \right.$$

$$A_1 = \left(\frac{p^2}{2\pi} F \right)^{-1} \frac{g_{-1} f_1 \sin(\delta_{-1} - \delta_1)}{4\pi} \rightarrow \frac{p}{2W},$$

$$A_1 = \left(\frac{p^2}{2\pi} F \right)^{-1} \frac{g_{-1} f_1 \cos(\delta_{-1} - \delta_1)}{4\pi} \rightarrow \frac{aZ}{2W}.$$

The suffixes of f and g are quantum numbers κ .⁸⁾ The arrow in each formula indicates the approximation $aZ \ll 1$. Since we usually neglect in this approximation the terms of $(aZ)^n$ for $n \geq 2$, aZ is not discarded here.

By performing the integration of $\int \sin \theta d\theta$, we obtain the correction factors of β -ray spectra. The

third terms of eqs. (1) and (2), called $1/W$ terms, distort the allowed spectra. We can make these terms vanish by putting G_V/G_S or $G_A/G_T = g \exp(i\pi/2)$, where g is adjustable real number. Then, from eq. (2),

$$\mathfrak{B}_{0TA} \propto 1 + (p/3W)f(g) \cos \theta,$$

where

$$f(g) = \frac{1 + (6aZ/p)g - g^2}{1 + g^2}.$$

The data on P^{32} show $f_{\exp}(g) \geq 1$.⁴⁾ The spin of P^{32} seems to be 1. The bounds of the value of g are as follows:

$$0.18 \geq g \geq 0, \quad (W=2.2),$$

$$0.28 \geq g \geq 0, \quad (W=1.6).$$

$f(g)_{W=2.2}$ is given in Table I. We can not determine the value of $f(g)_{W=2.2}$ experimentally for $0.18 \geq g \geq 0$. The property of $f(g)_{W=1.6}$ is almost the same as that of $f(g)_{W=2.2}$.

More accurate information of g is not to be expected from the data on He.⁶ We have no choice between Scalar and Vector interactions because of the lack of experimental data.⁹⁾

The author wishes to express his sincere thanks to Profs. T. Yamanouchi and S. Nakamura, Dr. M. Umezawa and Mr. M. Yamada for their kind guidance and valuable discussions.

* This work has been performed during the author's stay at the Yukawa Hall, Kyoto University. He is greatly indebted to prof. M. Kobayasi for the hospitality.

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This is not derived from the data on He⁶, because β -ray of He⁶ has a large maximum energy W_0 and the $1/W$ term is small in high energy region. However, in the cases of some elements the β -rays have small W_0 . Then it is necessary to put $G_A G_T = 0$.

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Table I. $f(g)_{W=2.2}$.

g	$-\infty$	-11.25	-0.3	0	0.0895	0.179	0.5	∞
$f(g)$	-1	$-1.016(\text{Min.})$	0.736	1	$1.016(\text{Max.})$	1	0.743	-1

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On the Gauge Invariancy and the Structure of Elementary Particle

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August 27, 1953

In spite of the great success of the theory of renormalization, some difficulties remain dissolved even in the quantum electrodynamics, one of which is the problem of gauge invariancy, particularly the problem of the self-energy of photon.¹⁾ In this note I would like to consider the source of this problem in connection with the vacuum expectation value of the energy-momentum tensor of the system composed of the particle and the electromagnetic field.

For simplicity we confine ourselves to discuss the spinor field as vacuum particle in what follows. In the 2nd approximation of perturbation calculation, the vacuum induced current is

$$\partial j_\mu(x) = -8ie^2 \int_{-\infty}^{\infty} K_{\mu\nu}(x-x') A_\nu(x') dx', \quad (1)$$

where

$$K_{\mu\nu}(x) = 2 \frac{\partial \Delta_F(x)}{\partial x_\mu} \frac{\partial \Delta_F(x)}{\partial x_\nu} - \delta_{\mu\nu} \left(\frac{\partial \Delta_F(x)}{\partial x_\lambda} \frac{\partial \Delta_F(x)}{\partial x_\lambda} + x^2 \Delta_F(x) \Delta_F(x) \right). \quad (2)$$

Because we treat the self-energy of photon only, we can separate it by

$$\int_{-\infty}^{\infty} K_{\mu\nu}(x-x') dx' = K_{\mu\nu}, \quad (3)$$

and in this case we may write further

$$K_{\mu\nu} \equiv K \delta_{\mu\nu} = (1/4) \cdot K_{\lambda\lambda} \delta_{\mu\nu}, \quad (4)$$

so it is sufficient to know $K_{\lambda\lambda}$.

From

$$(\square - x^2) \Delta_F(x-x') = 2i \delta(x-x'), \quad (5)$$

we obtain

$$(\square - x^2) \frac{\partial \Delta_F(x-x')}{\partial x^2} = \Delta_F(x-x'). \quad (6)$$

Therefore

$$\frac{\partial}{\partial x^2} \Delta_F(x-x') = \frac{1}{2i} \int \Delta_F(x-x'-x'') \Delta_F(x'') dx'' \quad (7)$$

is obtained. From (6) we find

$$\begin{aligned} (\square - x^2) \frac{\partial}{\partial x^2} \frac{\partial}{\partial x_\mu} \Delta_F(x-x') \\ = \frac{\partial}{\partial x_\mu} \Delta_F(x-x'), \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial}{\partial x^2} \left\{ \frac{\partial}{\partial x_\mu} \Delta_F(x-x') \right\} \\ = \frac{1}{2i} \int \Delta_F(x-x'-x'') \frac{\partial}{\partial x_\mu} \Delta_F(x'') dx'' \end{aligned} \quad (9)$$

and

$$\begin{aligned} \frac{\partial}{\partial x^2} \left\{ x^2 \Delta_F(x-x') \right\} \\ = -\frac{1}{2i} \int \frac{\partial}{\partial x_\mu} \Delta_F(x-x'-x'') \frac{\partial}{\partial x_\mu} \Delta_F(x'') dx''. \end{aligned} \quad (10)$$

Therefore from (7) and (10) one obtains

$$\begin{aligned} \frac{1}{2i} K_{\lambda\lambda} &= -2 \frac{\partial}{\partial x^2} \{ x^2 \Delta_F(0) \} + 4x^2 \frac{\partial}{\partial x^2} \Delta_F(0) \\ &= 2 \left\{ x^2 \frac{\partial}{\partial x^2} - 1 \right\} \Delta_F(0). \end{aligned} \quad (11)$$

On the other hand since the vacuum expectation value of the spur of the energy-momentum tensor of whole system is

$$\langle \Theta_{\lambda\lambda} \rangle_0 = 2x^2 \Delta_F(0), \quad (12)$$

we have

$$\frac{1}{2i} K_{\lambda\lambda} = \frac{\partial}{\partial x^2} \langle \Theta_{\lambda\lambda} \rangle_0 - \frac{2}{x^2} \langle \Theta_{\lambda\lambda} \rangle_0,$$

accordingly the vacuum induced current is

$$\partial j_\mu(x) = -e^2 \left(\frac{\partial}{\partial x^2} \langle \Theta_{\lambda\lambda} \rangle_0 - \frac{2}{x^2} \langle \Theta_{\lambda\lambda} \rangle_0 \right) A_\mu(x). \quad (13)$$

Thus the problem of the self-energy of photon is reduced to $\langle \Theta_{\lambda\lambda} \rangle_0$. It is specially important that this is the equation independent of the method of the integration.

Unfortunately the present quantum field theory gives the quadratically diverging $\langle \Theta_{\lambda\lambda} \rangle_0$.³⁾ If we substitute this diverging $\langle \Theta_{\lambda\lambda} \rangle_0$ into (13), the current is

$$\partial j_\mu(x) = -\frac{e^2}{(2\pi)^2} \lim_{K \rightarrow \infty} \left(K^2 - \frac{1}{2} x^2 \right) A_\mu(x), \quad (14)$$

which agrees with the value calculated from (2) in the straight-forward manner.

It must be noted that we have only mass x as

a quantity with dimension in the starting point of field theory. Therefore from the dimensional consideration we must have

$$\langle \theta_{\lambda\lambda} \rangle_0 = a x^4 \quad (15)$$

with a dimensionless constant a . If we substitute (15) into (13), we have

$$\delta j_\mu(x) = 0$$

and the self-energy of photon vanishes.

Judging from the above fact, we may conclude as follows: *the procedure of the temporal cut-off during the calculation which contradicts with the point model of elementary particle gives the non-gauge self-energy of photon.*

This fact is analogous to the case of the self-stress, when the problem was reduced to the diverging self-energy of electron.²⁾

It is very interesting for us to examine whether (13) holds true in the higher order of perturbation calculation or not, the non-gauge term of two gamma decay of π^0 -meson and the intimate connection between the above analysis and the subtraction method of Dirac, Heisenberg, Weisskopf and Umezawa-Kawabe.⁵⁾

These points will be discussed in latter occasion.

The author gratefully acknowledges the encouragement of Prof. S. Sakata and he is also indebted to Dr. H. Umezawa, Messrs. S. Kamefuchi and S. Matsumoto for their valuable discussion.

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On the State of Solid Hydrogen

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August 28, 1953

To explain the well known anomaly in the specific heat of solid hydrogen below 10°K Schafer¹⁾ proposed a model in which each molecule is immersed in an axially symmetric field of the type $-AP_2^0(\cos \theta)$, θ being the angle between the molecular axis and the crystalline hexagonal axis.

It is interesting, therefore, to see how the newly developed method of nuclear resonance throws light

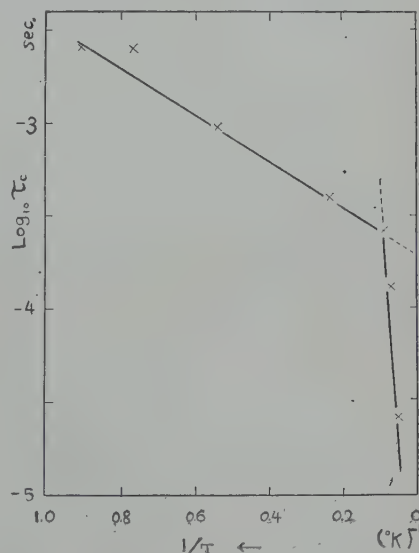


Fig. 1. Temperature dependence of characteristic time τ_c (The slope indicates the height of barrier which hinders molecular rotation.)

on this problem, and so an analysis was made of the proton resonance data observed by Hatton and Rollin.²⁾

(a) Characteristic frequency τ_0 of the molecular re-orientation was obtained from the spin-lattice relaxation by using the one parameter theory.³⁾ (cf. Fig. 1.) Its temperature dependence suggests the magnitude of the local field by which the change of orientation of each molecule is quantized or hindered. The barrier height is estimated to be approximately 3°K ($T < 11^\circ\text{K}$). (b) The separation of resonance peaks, about 50 gauss, in the lowest temperature modification was found near the value 41 gauss* which is expected when each molecule is confined to the lowest orientational state $J=1$ and $M_J=0$. This suggests the existence of an axial local field, an example of which Schafer assumed.

In order to examine the nature of the local field at the lowest temperature we calculated the crystal energy and anisotropy as a function of lattice deformation assuming the conservation of the hexagonal symmetry and also that of the volume of the unit cell. For the sake of computational convenience we studied the case of pure ortho-hydrogen, which, although absent in nature, is expected to give a good idea concerning the character of lattice deformation. Assuming that each molecule is confined to the state $J=1$ and $M_J=0$, we calculated the potential energy by using the intermolecular force according to de Boer, which includes overlap, quadrupole, dispersion and induced dipole-quadrupole forces and gives a good result for second virial coefficient.⁴⁾ Kinetic energy was estimated from a free particle model, i.e. a free particle of the molecular mass was considered to move in a spherical space (the radius of which being taken as the distance between the minimum in the two body potential and the closest distance of approach) and also to move in a deformed spheroid. The degree of deformation which gives the minimum in energy was found to be slightly affected by the kinetic energy. (cf. Fig. 2(a)).

Under similar conditions except for the adiabatic change of the orientation of the central molecule, we also obtained the anisotropy, which can be compared with the former calculation due to Nagamiya and Urano⁵⁾ using overlap force only. (cf. Fig. 2(b)).

The characteristic results are: (1) The close-packed structure gives an unstable energy and an inconsistent anisotropy. (2) Lateral contraction of 14% gives the minimum -6.5×10^{-4} a.u. in the energy, which is of the same order as the observed -2.93×10^{-4} a.u. in normal hydrogen. (3) At the position of the minimum in the energy the anisotropy

gives a consistent sign and amounts to $A=5 \times 10^{-5}$ a.u., which is approximately the same in order as that proposed by Schafer. Retaining overlap force only, we obtain essentially the same result as Nagamiya and Urano, however the situation is modified appreciably by the inclusion of the other types of intermolecular force. The magnitude of deformation seems too large possibly because it is not the natural case, and also because of the roughness of our calculation and the intermolecular force employed. An exact determination of lattice structure of a sample including ortho-hydrogen seems highly desirable.

In the higher temperature region, however, Schafer's simple theory seems inadequate in the fact

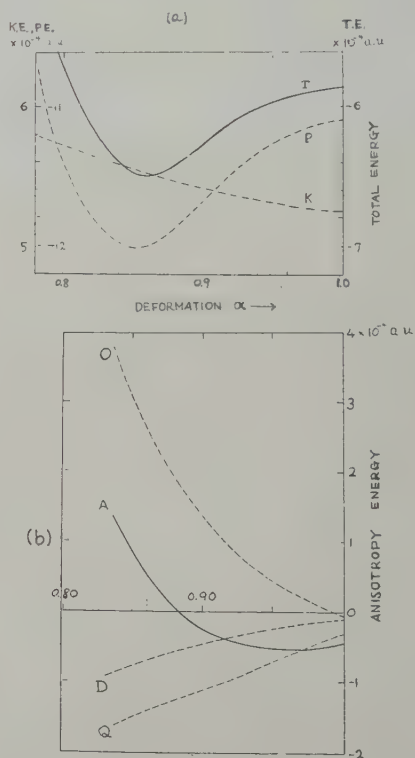


Fig. 2. (a) Calculated energy per molecule (pure ortho-hydrogen)
 T: Total energy
 P: Potential energy
 K: Kinetic energy
 (b) Calculated anisotropy (pure ortho-hydrogen)
 A: Total anisotropy energy
 O: Overlap force
 D: Van der Waals force
 Q: Quadrupole force

that it expects a curvature and peak height different from those observed in the specific heat-temperature plot. The dependence of the anomalous heat on the concentration of ortho-hydrogen and the narrowing of nuclear resonance at about 1.5°K combine to suggest a co-operative appearance of molecular rotation in the solid phase. Consideration of the statistical distribution of the ortho- and para-component in the lattice seems indispensable for the understanding of the specific heat.

* After taking polycrystalline average and considering the intermolecular interaction, we obtain a peak to peak separation about 10% smaller than this

figure, which is expected to correspond the exact observation.

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ERRATA

On the Inelastic Collision between Molecules, II

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Prog. Theor. Phys. 8 (1952), 497

On the Inelastic Collision between Molecules, III

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Prog. Theor. Phys. 9 (1953), 578

In the last part of Appendix IV, paper III, we wrote that "cross sections of the process (*) and (**) are same". However, it has been found to be wrong, because we had failed to notice that is \mathcal{J} 's are normalized

$$|(\mathcal{J}_{mm'}^{uv}(1, 2)|V|\mathcal{J}_{mm}^u(1, 2)|^2=2|(Y_{lm}(1)Y_{l'm'}(2)|V|Y_{lm}(1)Y_{lm}(2))|^2$$

for our case. The same is true for the inverse processes. This factor 2 always cancels the statistical factor $w(=1/2)$ in the equation (7) of paper III, and we must correct (11), (12), (14) in that paper as

$$Q^{(1)} : Q^{(2)} : Q^{(3)} = 1 : 0.23 : 0.11 \quad (11)$$

$$P^{(1)} = 2.00 \times 10^{-2} \quad (12)$$

$$k_1 = k_2 = k_3, \quad k_1' = k_2' = k_3', \\ x_1 = x_2 = x_3, \quad x_1' = x_2' = x_3'. \quad (14)$$

The general feature of the theoretical dispersion curve, however, does not change. For the same reason we must multiply the cross section itself of paper II with 2.

Furthermore, the following misprints should be corrected: paper II, p. 507, first line: $(l' \dots m')$ should read $(l' - m')$! paper III, p. 588, in Appendix III,

$$(4\pi/5)^2 \sum_{\mu, \mu'=-2}^2 Y_{2\mu}^* (\theta_1 \varphi_1) Y_{2\mu'}^* (\theta_2 \varphi_2) \dots$$

should read

$$(4\pi/5)^2 \sum_{\mu, \mu'=-2}^2 Y_{2\mu}^* (\theta_1 \varphi_1) Y_{2\mu'}^* (\theta_2 \varphi_2) \dots$$

Deuteron Stripping Reaction

Shiro YOSHIDA

Prog. Theor. Phys. 10 (1953) 1

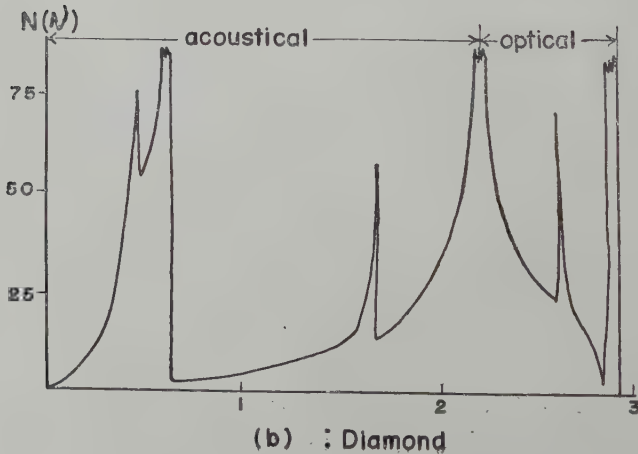
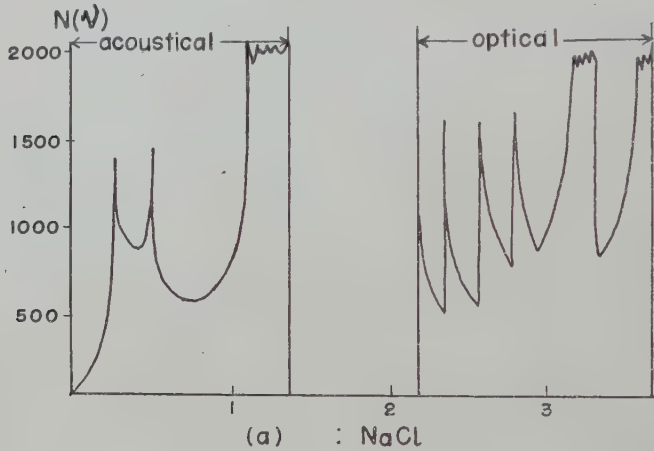
The equation (3·6) is incorrect and the right side of this equation should be multiplied by factor 2, and the right side of equation (3·22) also should be multiplied by 2. q and q' in equations (3·17b) and (3·28) should be multiplied by factor 4. The figures of the 6th and 7th columns in Table 1 should be multiplied by about 1/4. In equation (3·16) $\hbar^2/2M$ should be replaced by $2M/\hbar^2$.

Theory of the Normal Modes of Vibration in Crystal

Sadaaki YANAGAWA

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The following figures should be inserted.



On the Heisenberg Treatment of the Field Variables

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The simultaneous integral equations of the vacuum expectation values in the Heisenberg representation are proposed, by the aid of which we reduced the equations to gain the expectation values, the S -matrix and the bound state solutions. The relation of these equations with the methods of Dyson, Bethe-Salpeter and Tomonaga is obvious. The practical calculations are carried out for the system containing one Fermion and the neutral scalar mesons. The application to other general systems may be readily possible.

§ 1. Introduction

Recently the progress in the formal treatment of the field theory seems to aim for the two directions. One of them is the covariant description of the field theory. It was advanced by the treatment in the interaction representation by Tomonaga,¹⁾ Schwinger²⁾ and Dyson,³⁾ and has gained many successes for the problems of the relativistic phenomena and the relativistic subtraction. The description in the interaction representation is connected tightly with the perturbation method, because it gives the special roles to the free fields.

Another direction of the formal progress is the attempt to overcome the perturbation method. It is well known that in the nucleon- π meson system as their mutual interaction is rather strong, the perturbation method is not suitable. As to this difficulty, the treatment based on the Fock's Schroedinger equations was proposed by Tomonaga⁴⁾ for the system of one nucleon and by Tamm⁵⁾ and Dancoff⁶⁾ for the system of two nucleons. Recently Lévy has applied the extended Tamm-Dancoff method to the deuteron problem and gained very reasonable results. Thus the treatment in the Schroedinger representation is found to be effective for overcoming the limit of the perturbation method. But its non-covariant expression is inconvenient for the relativistic invariant treatment. Especially for the high energy phenomena the covariant treatment will be unavoidable.

From this consideration it should be desirable to treat the problems in the Heisenberg representation. The first study of the treatment in the Heisenberg representation was attempted by Yang-Feldman.⁸⁾ They showed that it may be possible to obtain the S -matrix in the Heisenberg representation. Recently Bethe-Salpeter⁹⁾ have proposed an integral equation for the Feynman's amplitude function and from it reduced the covariant bound state equation. The field theoretical foundation of this equation has been worked out by Gell-Mann-Low¹⁰⁾ by the use of vacuum definition and other treatments in the Heisenberg representation. But as the nucleus of the Bethe-Salpeter integral equation must be gained

from the terms of the Dyson's perturbation series, this equation is not perfectly free from the perturbation method.

It is the purpose of this paper to propose the simultaneous integral equations for the vacuum expectation values in the Heisenberg representation and to advance one step further in the direction initiated by Yang-Feldman and Gell-Mann-Low. Our equations are obtained by using the Schwinger's integral form of the equation of motion and the Gell-Mann-Low's vacuum definition, and seem to correspond to the Fock's simultaneous differential equations in the Schroedinger representation. (§ 3.) Solving the equations we shall obtain the Feynman's "scatteristic" expectation values⁽¹⁾ (which are the matrix elements of the field variables based on the free states of the infinite future and the infinite past,) and the S -matrix. If we expand it in the power series of the coupling constant, the each term of which is identified with the corresponding term of the Dyson's perturbation series the relation with the perturbation method is obvious. (§ 4.) Further with the use of the Gell-Mann-Low's method, we shall gain the equations for the bound states similar to the Bethe-Salpeter equation. It is shown that the equations correspond to the Fock-Tomonaga equations in the Schroedinger representation. (§ 5.)

These treatments are carried for the system containing one Fermion and the neutral scalar mesons. The description for other general systems may be readily possible.

§ 2. Vacuum and vacuum expectation values

(a) On the definition of the true vacuum

Gell-Mann-Low has defined the vacuum in the Heisenberg representation (the true vacuum) as follows

$$c\Psi_0 = U^{-1}(-\infty, t')\Phi_0(t') \quad (1)$$

or

$$c'\Psi_0 = U(+\infty, t')\Phi_0(t'), \quad (2)$$

where $U(t, t')$ is the transformation matrix which is taken as to identify the field variable $F(t)$ in the Heisenberg representation (the Heisenberg operator) with the field variable $F'(t)$ in the interaction representation (the free operator) at $t=t'$. The relation is written as

$$F(t) = U(t, t')F'U^{-1}(t, t'), \quad (3)$$

$$i \frac{dU(t, t')}{dt} = \int d\vec{x} H_I(\vec{x}, t)U(t, t'), \quad U(t, t) = 1, \quad (4)$$

And $\Phi_0(t')$ is the vacuum state in the interaction representation (the free vacuum) for the free operator $F'(t)$ which is transformed by $U(t, t')$ from the Heisenberg operator.

Here if we vary the time t' that the two operators in the different representations coincide, the true vacuum vector Ψ_0 will not vary except any numerical factor. Therefore in $t' \rightarrow -\infty$ the definition (1) is expressed as follows

$$c''\Psi_0 = \Phi_0(-\infty) \quad (1)'$$

namely the true vacuum is the free vacuum for $F(t)$ which will coincide with $F(t)$ in $t' \rightarrow -\infty$. Similarly in $t' \rightarrow +\infty$ from the definition (2) $\Phi_0(+\infty)$ is the true vacuum. We shall use these Gell-Mann-Low's vacuum definition hereafter.

We express the transformation matrix element from $\Phi_0(-\infty)$ to $\Phi_0(+\infty)$ as

$$(\Phi_0^*(+\infty)\Phi_0(-\infty)) = \langle \quad \rangle_0^F = \langle U(+\infty, -\infty) \rangle_0. \quad (5)$$

By the Dyson's perturbation method it may be written

$$\begin{aligned} \langle U(+\infty, -\infty) \rangle_0 &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle P(\mathbf{H}_I(x_1) \cdots \mathbf{H}_I(x_n)) \rangle_0 \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \left\{ \langle P(\mathbf{H}_I(x_1), \cdots, \mathbf{H}_I(x_n)) \rangle_0 \right. \\ &\quad + \frac{1}{2!} \sum_{p=1}^{n-1} n c_p \langle P(\mathbf{H}_I(x_1), \cdots, \mathbf{H}_I(x_p)) \rangle_0 \langle P(\mathbf{H}_I(x_{p+1}), \cdots, \mathbf{H}_I(x_n)) \rangle_0 \\ &\quad + \frac{1}{3!} \sum_p \sum_q n c_p c_q \langle \cdots \rangle_0 \langle \cdots \rangle_0 \langle \cdots \rangle_0 + \cdots \left. \right\} \\ &= 1 + i\eta + \frac{1}{2!} (i\eta)^2 + \frac{1}{3!} (i\eta)^3 + \cdots = e^{i\eta}, \end{aligned} \quad (6)$$

where

$$i\eta = \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle P(\mathbf{H}_I(x_1), \cdots, \mathbf{H}_I(x_n)) \rangle_0. \quad (7)$$

The commas between the interaction operator express that we take the connected graphs in the Feynman-Dyson diagrams. In the present field theory η has the infinite imaginary part. So as to keep the unitarity of the transformation matrix, we must make the following agreement according to Feynman.¹¹⁾ That is, we use always

$$U(+\infty, -\infty) / \langle U(+\infty, -\infty) \rangle_0,$$

instead of $U(+\infty, -\infty)$.

By some consideration it is known, this agreement is equivalent to use

$$U(t, t') / \langle U(t, t') \rangle_0$$

instead of $U(t, t')$ generally. Then for the arbitrary t' we may conclude that

$$e^{i\eta} \Psi_0 = \Phi_0(t') \quad (8)$$

namely the free vacuum $\Phi_0(t')$ is always the true vacuum.

By means of the above nature, the Feynman's vacuum expectation value of an arbitrary operator F

$$(\Phi_0^*(+\infty)F\Phi_0(-\infty)) = \langle F \rangle_0^F = \langle F U(+\infty, -\infty) \rangle_0 \quad (9)$$

we may translate to the true vacuum expectation value,

$$(\Psi_0^* F \Psi_0) = \langle F \rangle_0 \quad (10)$$

that is

$$\langle F \rangle_0 = \langle F \rangle_0^F / \langle 1 \rangle_0^F = \langle F U(+\infty, -\infty) \rangle_0 / \langle U(+\infty, -\infty) \rangle_0 \quad (11)$$

With the calculation similar to (6), we have

$$\begin{aligned} \langle F U(+\infty, -\infty) \rangle_0 &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle P(F H_1(x_1) \cdots H_n(x_n)) \rangle_0 \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle P(F, H_1(x_1), \dots, H_n(x_n)) \rangle_0 \langle U(+\infty, -\infty) \rangle_0 \end{aligned} \quad (12)$$

then

$$\langle F \rangle_0 = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dx_1 \cdots dx_n \langle P(F, H_1(x_1), \dots, H_n(x_n)) \rangle_0 \quad (13)$$

The equation (13) express to take only the connected graphs in the perturbation calculations as we are doing in practice.

(b) *Isolation of the vacuum expectation values*

A chronological ordered product of the Heisenberg operators $F_1(x_1), F_2(x_2), \dots$, accompanied ± 1 according to even or odd permutation of the Fermion operators ψ and $\bar{\psi}$, we write as follows

$$\epsilon P(F_1(x_1) F_2(x_2) \cdots) \quad (14)$$

From its true vacuum expectation value $\langle \epsilon P(F_1(x_1), F_2(x_2) \cdots) \rangle_0$ we isolate the parts composed by the products of two or more vacuum expectation values, and the remaining we express

$$\langle \epsilon P(F_1(x_1), F_2(x_2), \cdots) \rangle_0, \quad (15)$$

which means

$$\begin{aligned} \langle \epsilon P(F_1(x_1) F_2(x_2) \cdots) \rangle_0 &= \langle \epsilon P(F_1(x_1), F_2(x_2), \cdots) \rangle_0 \\ &+ \sum \epsilon \langle \epsilon P(F_1(x_1), F_2(x_2), \cdots) \rangle_0 \langle \epsilon P(\cdots) \rangle_0 \\ &+ \sum \epsilon \langle \epsilon P(\cdots) \rangle_0 \langle \epsilon P(\cdots) \rangle_0 \langle \epsilon P(\cdots) \rangle_0 + \cdots, \end{aligned} \quad (16)$$

where \sum represents the sum of all possible terms and ϵ represents ± 1 according to even or odd permutation of ψ and $\bar{\psi}$ like ϵ of (14). For instance

$$\begin{aligned} \langle \epsilon P(\psi(x_1) \psi(x_2) \bar{\psi}(x_3) \bar{\psi}(x_4)) \rangle_0 &= \langle \epsilon P(\psi(x_1), \psi(x_2), \bar{\psi}(x_3), \bar{\psi}(x_4)) \rangle_0 \\ &+ \langle \epsilon P(\psi(x_1), \bar{\psi}(x_4)) \rangle_0 \langle \epsilon P(\psi(x_2), \bar{\psi}(x_3)) \rangle_0 \\ &- \langle \epsilon P(\psi(x_1), \bar{\psi}(x_3)) \rangle_0 \langle \epsilon P(\psi(x_2), \bar{\psi}(x_4)) \rangle_0. \end{aligned} \quad (17)$$

The expectation values containing the different number of ψ and $\bar{\psi}$ vanish of course.

In the Feynman-Dyson diagrams this process corresponds to isolate the graphs decom-

posed few independent graphs. And as it is carried together with all higher order corrections of the vertices or lines in the diagrams, it has generalized meaning independent of the perturbation method.

Specially after the next section, the mixed expression

$$\langle \varepsilon P(F_1(x_1)F_2(x_2), F_3(x_3), \dots) \rangle_0 \quad (18)$$

will be necessary. It means

$$\begin{aligned} & \langle \varepsilon P(F_1(x_1)F_2(x_2), F_3(x_3), \dots) \rangle_0 \\ &= \langle \varepsilon P(F_1(x_1), F_2(x_2), F_3(x_3), \dots) \rangle_0 \\ &+ \sum \varepsilon \langle \varepsilon P(F_1(x_1), \dots) \rangle_0 \langle \varepsilon P(F_2(x_2), \dots) \rangle_0, \end{aligned} \quad (19)$$

namely it contains the terms decomposable between $F_1(x_1)$ and $F_2(x_2)$.

§ 3. Reduction of the simultaneous integral equations

For convenience we carry the process in the Fermion and neutral scalar meson system.

We express the Fermion free operator $\psi_{\sigma'}(x)$ which coincides with the Heisenberg operator $\psi(x)$ on the spacelike surface σ'

$$\begin{aligned} \psi_{\sigma'}(x) &= \int_{\sigma'} s(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} \\ &= \int_{\sigma''} s(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} + g \int_{\sigma''} dx_1 s(x-x_1) \phi(x_1) \psi(x_1), \end{aligned} \quad (20)$$

where the surface σ'' is arbitrary. These notations are all the same to Schwinger's.²⁾

If we take the plane $t' \rightarrow +\infty$ and $t' \rightarrow -\infty$ for σ' , we have

$$\psi_{+\infty}(x) = \int_{\sigma''} s(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} + g \int_{\sigma''} dx_1 s(x-x_1) \phi(x_1) \psi(x_1) \quad (21)$$

$$\psi_{-\infty}(x) = \int_{\sigma''} s(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} - g \int_{-\infty} dx_1 s(x-x_1) \phi(x_1) \psi(x_1). \quad (22)$$

From these equations we reduce

$$\begin{aligned} (\phi_{+\infty} + i\phi_{+\infty}^{(1)})(x) &= \int_{\sigma''} (s + is^{(1)})(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} \\ &+ g \int_{\sigma''} dx_1 (s + is^{(1)})(x-x_1) \phi(x_1) \psi(x_1), \end{aligned} \quad (23)$$

$$\begin{aligned} (\phi_{-\infty} - i\phi_{-\infty}^{(1)})(x) &= \int_{\sigma''} (s - is^{(1)})(x-x_1) \gamma_\mu \psi(x_1) d\sigma_{\mu'} \\ &- g \int_{-\infty} dx_1 (s - is^{(1)})(x-x_1) \phi(x_1) \psi(x_1). \end{aligned} \quad (24)$$

The definition of the free vacuum are written

$$\phi_0^*(+\infty)(\phi_{+\infty}+i\phi_{+\infty}^{(1)})(x)=0, \quad (25)$$

$$(\phi_{-\infty}-i\phi_{-\infty}^{(1)})(x)\phi_0(-\infty)=0. \quad (26)$$

By the use of the equations (23) and (24) and $\bar{\psi}(x'')$ at x'' on the surface σ'' , we make the expression as follows

$$-\frac{1}{2}\{\langle(\phi_{+\infty}+i\phi_{+\infty}^{(1)})(x')\bar{\psi}(x'')\rangle_0^F-\langle\bar{\psi}(x'')(\phi_{-\infty}-i\phi_{-\infty}^{(1)})(x')\rangle_0^F\}. \quad (27)$$

From (9), (25) and (26), we know, the value of (27) is zero.

By the equations (20), (23) and (24) this expression is reduced in the following way :

$$\begin{aligned} & \langle \int_{\sigma''} (s(x'-x_1)\gamma_\mu \frac{1}{2} [\psi(x_1), \bar{\psi}(x'')] + i s^{(1)}(x'-x_1)\gamma_\mu \frac{1}{2} \{\psi(x_1), \bar{\psi}(x'')\}) d\sigma_{\mu'} \rangle_0^F \\ & + g \int_{-\infty}^{\infty} dx_1 \frac{1}{2} (s(x'-x_1) - i\epsilon(x''-x_1)s^{(1)}(x'-x_1)) \langle P(\phi(x_1)\psi(x_1), \bar{\psi}(x'')) \rangle_0^F \\ & = \langle \int_{\sigma''} (s(x'-x_1)\gamma_\mu \epsilon(x'-x'') P(\psi(x_1), \bar{\psi}(x'')) \\ & - s(x'-x_1)\gamma_\mu \epsilon(x'-x'') \frac{1}{2} \{\psi(x_1), \bar{\psi}(x'')\} \\ & + i s^{(1)}(x'-x_1)\gamma_\mu \frac{1}{2} \{\psi(x_1), \bar{\psi}(x'')\}) d\sigma_{\mu'} \rangle_0^F \\ & + g \int_{-\infty}^{\infty} dx_1 \frac{1}{2} (\epsilon(x_1-x'')s(x'-x_1) + i s^{(1)}(x'-x_1)) \langle \epsilon P(\phi(x_1)\psi(x_1), \bar{\psi}(x'')) \rangle_0^F \\ & = \langle \epsilon(x'-x'') P(\int_{\sigma''} s(x'-x_1)\gamma_\mu \psi(x_1) d\sigma_{\mu'}, \bar{\psi}(x'')) \rangle_0^F \\ & + \left(-\frac{1}{2i} \epsilon(x'-x'')s(x'-x'') + \frac{1}{2} s^{(1)}(x'-x'') \right) \langle \quad \rangle_0^F \\ & + g \int_{-\infty}^{\infty} dx_1 \frac{1}{2} (-\epsilon(x'-x_1)s(x'-x_1) + i s^{(1)}(x'-x_1)) \langle \epsilon P(\phi(x_1)\psi(x_1), \bar{\psi}(x'')) \rangle_0^F \\ & + g \int_{\sigma''} dx_1 \epsilon(x'-x'')s(x'-x_1) \langle P(\phi(x_1)\psi(x_1), \bar{\psi}(x'')) \rangle_0^F \\ & = \langle \epsilon P(\psi(x'), \bar{\psi}(x'')) \rangle_0^F - \frac{1}{2} s_F(x'-x'') \langle \quad \rangle_0^F \\ & - g \int_{-\infty}^{\infty} dx_1 s_F(x'-x_1) \langle \epsilon P(\phi(x_1)\psi(x_1), \bar{\psi}(x'')) \rangle_0^F, \end{aligned} \quad (28)$$

where we have used the relation

$$[\psi(x'), \bar{\psi}(x'')] = \epsilon(x'-x'') (2P(\psi(x'), \bar{\psi}(x'')) - \{\psi(x'), \bar{\psi}(x'')\}) \quad (29)$$

and

$$s_F(x' - x'') = \frac{1}{2} \epsilon(x' - x'') s(x' - x'') + \frac{1}{2i} s^{(1)}(x' - x''). \quad (30)$$

Using the relation (11) and the fact that (28) is equal to zero, we get

$$\begin{aligned} & \langle \epsilon P(\phi(x'), \bar{\psi}(x'')) \rangle_0 \\ &= \frac{1}{i} s_F(x' - x'') + g \int dx_1 s_F(x' - x_1) \langle \epsilon P(\phi(x_1) \psi(x_1), \bar{\psi}(x'')) \rangle_0. \end{aligned} \quad (31)$$

By the similar calculations

$$\begin{aligned} & \langle \epsilon P(\phi(x'), \bar{\psi}(x'')) \rangle_0 \\ &= \frac{1}{i} s_F(x' - x'') + g \int dx_1 \langle \epsilon P(\phi(x_1), \bar{\psi}(x_1) \phi(x_1)) \rangle_0 s_F(x_1 - x''), \end{aligned} \quad (32)$$

$$\begin{aligned} & \langle \epsilon P(\phi(x'), \phi(x'')) \rangle_0 \\ &= i \Delta_F(x' - x'') - g \int dx_1 \Delta_F(x' - x'') \langle \epsilon P(\bar{\psi}(x_1) \psi(x_1), \phi(x'')) \rangle_0. \end{aligned} \quad (33)$$

Generally for the products of more than three variables

$$\begin{aligned} & \langle \epsilon P(\phi(x_1), \dots, \phi(x_m), \phi(y_1), \dots, \phi(y_n), \bar{\psi}(z_1), \dots, \bar{\psi}(z_m)) \rangle_0 \\ &= g \int du_1 s_F(x_1 - u_1) \langle \epsilon P(\phi(u_1) \psi(u_1), \phi(x_2), \dots, \bar{\psi}(z_m)) \rangle_0 \\ &= g \int du_1 \langle \epsilon P(\phi(x_1), \dots, \bar{\psi}(u_1) \phi(u_1), \bar{\psi}(z_2), \dots, \bar{\psi}(z_m)) \rangle_0 s_F(u_1 - z_1) \\ &= -g \int du_1 \Delta_F(y_1 - u_1) \langle \epsilon P(\phi(x_1), \dots, \bar{\psi}(u_1) \phi(u_1), \phi(y_2), \dots, \bar{\psi}(z_m)) \rangle_0. \end{aligned} \quad (34)$$

The left side functions of (34) are antisymmetric for ϕ and $\bar{\psi}$ respectively and symmetric for ϕ .

Thus the simultaneous integral equations are obtained by means of the Schwinger's integral form of the equation of motion and the Gell-Mann-Low's vacuum definition.

In order to rewrite these equations into the momentum space, we use the following expressions :

$$\left. \begin{aligned} \phi(x) &= \int \phi(k) e^{ikx} \frac{dk}{(2\pi)^2}, \\ \bar{\psi}(x) &= \int \bar{\psi}(k) e^{-ikx} \frac{dk}{(2\pi)^2}, \\ \phi(x) &= \int \phi(k) e^{ikx} \frac{dk}{(2\pi)^2}, \quad \phi^*(k) = \phi(-k), \\ \Delta_F(x) &= \frac{1}{(2\pi i)^2} \int \frac{e^{ikx}}{k^2 + \mu^2 - i\epsilon} \frac{dk}{(2\pi)^2}, \\ s_F(x) &= \frac{1}{(2\pi i)^2} \int \frac{ik\gamma - x_0}{k^2 + x_0^2 - i\epsilon} e^{ikx} \frac{dk}{(2\pi)^2}, \end{aligned} \right\} \quad (35)$$

and use the notations of the following way :

$$\langle \varepsilon P(\psi(x'), \bar{\psi}(x'')) \rangle_0 = \int \frac{dk'}{(2\pi)^2} \frac{dk''}{(2\pi)^2} e^{ik'x' - ik''x''} \langle \varepsilon P(\psi(k'), \bar{\psi}(k'')) \rangle_0. \quad (36)$$

From the equations (31), (33) and (34) we get

$$\begin{aligned} \langle \varepsilon P(\psi(k'), \bar{\psi}(k'')) \rangle_0 &= \delta(k' - k'') i \frac{ik'\gamma - x_0}{k'^2 + x_0^2} \\ &+ \frac{ig}{(2\pi)^2} i \frac{ik'\gamma - x_0}{k'^2 + x_0^2} \int dp \langle \varepsilon P(\psi(k' - p) \psi(p), \bar{\psi}(k'')) \rangle_0, \end{aligned} \quad (37)$$

$$\begin{aligned} \langle \varepsilon P(\phi(k'), \phi(k'')) \rangle_0 &= \delta(k' + k'') \frac{-i}{k'^2 + \mu^2} \\ &+ \frac{ig}{(2\pi)^2} \frac{-i}{k'^2 + \mu^2} \int dp \langle \varepsilon P(\bar{\psi}(-k' + p) \psi(p), \phi(k'')) \rangle_0, \end{aligned} \quad (38)$$

$$\begin{aligned} &\langle \varepsilon P(\psi(k_1), \dots, \phi(l_1), \dots, \bar{\psi}(l_1), \dots) \rangle_0 \\ &= \frac{ig}{(2\pi)^2} i \frac{ik_1\gamma - x_0}{k_1^2 + x_0^2} \int dp \langle \varepsilon P(\phi(k_1 - p) \psi(p), \dots) \rangle_0 \\ &= \frac{ig}{(2\pi)^2} \int dp \langle \varepsilon P(\dots, \bar{\psi}(p) \phi(-l_1 + p), \dots) \rangle_0 i \frac{il_1\gamma - x_0}{l_1^2 + x_0^2} \\ &= \frac{ig}{(2\pi)^2} \frac{-i}{l_1^2 + \mu^2} \int dp \langle \varepsilon P(\dots, \bar{\psi}(-l_1 + p) \psi(p), \dots) \rangle_0, \end{aligned} \quad (39)$$

where we omit $i\varepsilon$ for convenience.

Solving these equations successively we may obtain the same results with the Dyson's diagrams is obvious. Namely for the Fermion path, the meson path and the vertex of the emission or absorption respectively correspond $i(ik\gamma - x_0/k^2 + x_0^2)$, $-i/k^2 + \mu^2$ and $ig/(2\pi)^2$, and the energy-momentum conservation is in being for each vertex.

The total energy-momentum conservation comes into being on each expectation value and it is contained with the form $\delta(k_1 + \dots + l_1 - \dots)$. Therefore if we put

$$\begin{aligned} &\langle \varepsilon P(\psi(k_1), \dots, \phi(l_1), \dots, \bar{\psi}(l_1), \dots) \rangle_0 \\ &= i \frac{ik_1\gamma - x_0}{k_1^2 + x_0^2} \dots \frac{-i}{l_1^2 + \mu^2} \dots s(k_1, \dots; l_1, \dots) i \frac{il_1\gamma - x_0}{l_1^2 + x_0^2} \dots \\ &\times \delta(k_1 + \dots + l_1 - \dots). \end{aligned} \quad (40)$$

(37), (38) and (39) may be written as the equations of $s(k_1, \dots; l_1, \dots)$. These functions are the matrix element of S -matrix. The equations are generally non-linear, but if we neglect the pair creation of Fermions, the equations will be linear for the s -functions. The details will be discussed in the next section.

§ 4. Matrix elements for scattering

(a) Feynman expectation values and S -matrix

For the momentum expression of the free operators we express

$$\left. \begin{aligned} \phi_t(x) &= \int \phi_t(k) e^{ikx} d\vec{k}, \\ \phi_t(x) &= \int \phi_t(k) e^{-ikx} d\vec{k}, \\ \phi_t(x) &= \int \phi_t(h) e^{ihx} d\vec{h} \end{aligned} \right\} \quad (41)$$

where

$$\vec{k} = (\vec{k}, \sqrt{\vec{k}^2 + k_0^2}), \quad \vec{h} = (\vec{h}, \sqrt{\vec{h}^2 + \mu^2}). \quad (42)$$

The state where are m Fermions of momentum $\vec{k}_1, \dots, \vec{k}_m$ and n mesons of momentum $\vec{h}_1, \dots, \vec{h}_n$ in the infinite past is written as

$$\begin{aligned} &\Phi(-\infty; \vec{k}_1, \dots, \vec{k}_m; \vec{h}_1, \dots, \vec{h}_n) \\ &= \frac{1}{\sqrt{n!}} \bar{\Phi}_{-\infty}(\vec{k}_1) \dots \bar{\Phi}_{-\infty}(\vec{k}_m) \Phi_{-\infty}(-\vec{h}_1) \dots \Phi_{-\infty}(-\vec{h}_n) \Phi_0(+\infty). \end{aligned} \quad (43)$$

Similarly $\Phi(+\infty, \dots)$ is written by $\bar{\Phi}_{+\infty}(\vec{k})$ and $\Phi_{+\infty}(-\vec{h})$.

Specially if we take the Feynman expectation value of an arbitrary operator F by one Fermion states, it becomes

$$\begin{aligned} &(\Phi^*(+\infty; \vec{k}') F \Phi(-\infty; \vec{k}'')) / (\Phi_0^*(+\infty) \Phi_0(-\infty)) \\ &= \langle \epsilon P(\Phi_{+\infty}(\vec{k}'), F, \Phi_{-\infty}(\vec{k}'')) \rangle_0. \end{aligned} \quad (44)$$

From (35) and (41)

$$\phi_t(\vec{k}) = \int \phi_t(x) e^{-ikx} \frac{dx}{(2\pi)^3} = \int \frac{dk_0}{(2\pi)^2} e^{-i(k_0 - k_0)t} \phi(k). \quad (45)$$

Similarly

$$\bar{\phi}_t(\vec{k}) = \int \frac{dk_0}{(2\pi)^2} e^{i(k_0 - k_0)t} \bar{\phi}(k). \quad (46)$$

By the use of the Dirichlet integral

$$\lim_{\alpha \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \frac{\sin \alpha x}{x} dx = \pi f(0) \quad (47)$$

we get

$$\begin{aligned} &\lim_{t \rightarrow \infty} \int \frac{dk_0}{(2\pi)^2} e^{-i(k_0 - k_0)t} \left(\frac{1}{k^2 + k_0^2} + i\pi \delta(k^2 + k_0^2) \right) f(k_0) \\ &= \lim_{t \rightarrow -\infty} \int \frac{dk_0}{(2\pi)^2} e^{i(k_0 - k_0)t} \left(\frac{1}{k^2 + k_0^2} + i\pi \delta(k^2 + k_0^2) \right) f(k_0) = \frac{i}{4\pi k_0} f(k_0). \end{aligned} \quad (48)$$

On the other hand we have the relation

$$\langle \phi_t(\vec{k}') \bar{\phi}_t(\vec{k}'') \rangle_0 = \frac{-1}{2(2\pi)^3 k_0'} (i k' \gamma - k_0) \delta(\vec{k}' - \vec{k}'') \quad (49)$$

and we may put as follows

$$\langle \varepsilon P(\phi(k'), F, \bar{\phi}(k'')) \rangle_0 = i \frac{ik'\gamma - x_0}{k'^2 + x_0^2} f(k'; k'') i \frac{ik''\gamma - x_0}{k''^2 + x_0^2}. \quad (50)$$

By means of (48), (49) and (50), the expectation value (44) is written as

$$\begin{aligned} & \langle \varepsilon P(\phi_{+\infty}(\vec{k}'), F, \phi_{-\infty}(\vec{k}'')) \rangle_0 \\ &= \lim_{\substack{t' \rightarrow \infty \\ t'' \rightarrow -\infty}} \int \frac{dk'_0}{(2\pi)^2} \frac{dk''_0}{(2\pi)^2} e^{-i(k'_0 t' - k''_0 t'')} \langle \varepsilon P(\phi(k'), F, \bar{\phi}(k'')) \rangle_0 \\ &= \frac{-1}{4\pi k'_0} (ik'\gamma - k_0) f(k'; k'') \frac{-1}{4\pi k''_0} (ik''\gamma - x_0) \\ &= \int (2\pi)^2 dk' (2\pi)^2 dk'' \langle \phi_{+\infty}(\vec{k}') \bar{\phi}_{+\infty}(\vec{k}'') \rangle_0 f(k_1; k_2) \langle \phi_{-\infty}(\vec{k}_2) \bar{\phi}_{-\infty}(\vec{k}_1) \rangle_0. \end{aligned} \quad (51)$$

Therefore we may write the operator F for the Feynman expectation values by one Fermion states in the following expression:

$$F = \int (2\pi)^4 d\vec{k}_1 d\vec{k}_2 \bar{\phi}(\vec{k}_1) f(\mathbf{k}_1; \mathbf{k}_2) \phi(\vec{k}_2). \quad (52)$$

Similarly we may write the S -matrix. For instance the S -matrix for the scattering of m Fermions with the emission or absorption of n meson may be written with the s -functions defined in (40)

$$\begin{aligned} S_{mn} &= \int (2\pi)^{4m+2n} d\vec{k}_1 \cdots d\vec{k}_m d\vec{l}_1 \cdots d\vec{l}_n d\vec{l}'_1 \cdots d\vec{l}'_m \\ &\quad \times \delta(\mathbf{k}_1 + \cdots + \mathbf{k}_m + \mathbf{h}_1 + \cdots + \mathbf{h}_n - \mathbf{l}_1 - \cdots - \mathbf{l}_m) \phi(\vec{k}_1) \cdots \phi(\vec{k}_m) \\ &\quad \times \phi(-\vec{l}_1) \cdots \phi(-\vec{l}_n) s(\mathbf{k}_1, \cdots, \mathbf{k}_m; \mathbf{h}_1, \cdots, \mathbf{h}_n; \mathbf{l}_1, \cdots, \mathbf{l}_m) \phi(\vec{l}_1) \cdots \phi(\vec{l}_m). \end{aligned} \quad (53)$$

(b) Variation function

The treatment like as in previous sections should be necessary for the strong interaction as the nucleon and π -meson system. The calculations up to this time seem to show us that we may neglect the terms for the pair creation of nucleons in such condition. In this case, in the equations (37), (38) and (39) we may put

$$\left. \begin{aligned} \langle \varepsilon P(\phi(k'), \phi(k'')) \rangle_0 &= \delta(k' + k'') \frac{-i}{k'^2 + \mu^2}, \\ \langle \varepsilon P(\phi(k_1), \cdots, \phi(k_n)) \rangle_0 &= 0, \quad n > 2. \end{aligned} \right\} \quad (54)$$

For the most simple example of this approximation we write the equations in the case of one nucleon.

Applying (19), (40) and (54) in (37), (38) and (39) we have

$$s(k; k) = (-i)(-ik\gamma - x_0) + \frac{ig}{(2\pi)^3} \int dp \frac{i p \gamma - x_0}{p^2 + x_0^2} s(p; k-p; k) \frac{-i}{(k-p)^2 + \mu^2} \quad (55)$$

$$\begin{aligned}
s(k; h_1, \dots, h_n; l) &= \frac{i g^2}{(2\pi)^2} \left[\int d p i \frac{i p \gamma - x_0}{p^2 + x_0^2} s(p; k-p, h_1, \dots, h_n; l) \frac{-i}{(k-p)^2 + \mu^2} \right. \\
&\quad \left. + \sum_{s=1}^n i \frac{i(k+h_s)\gamma - x_0}{(k+h_s)^2 + x_0^2} s(k+h_s; h_1, \dots, h_{s-1}, h_{s+1}, \dots, h_n; l) \right] \\
&= \frac{i g^2}{(2\pi)^2} \left[\int d p s(k; -l+p, h_1, \dots, h_n; p) i \frac{i p \gamma - x_0}{p^2 + x_0^2} \frac{-i}{(l-p)^2 + \mu^2} \right. \\
&\quad \left. + \sum_{s=1}^n s(k; h_1, \dots, h_{s-1}, h_{s+1}, \dots, h_n; l-h_s) i \frac{i(l-h_s)\gamma - x_0}{(l-h_s)^2 + x_0^2} \right], \quad (56)
\end{aligned}$$

where

$$l = k + h_1 + \dots + h_n.$$

The above equations are linear for $s(k; h_1, \dots, h_n; l)$. Solving them we will gain the S -matrix elements (53). If we want to take account of the subtraction of nucleon self energy, we must multiple the factor $(1 + \partial x_0 i (i k \gamma - x_0 / k^2 + x_0^2))$ on the left side of (55) and (56).

The variation function, from which (55) and (56) are reduced, may be gained as

$$\begin{aligned}
K &= \int (\bar{s}(k; k) + s(k; k)) dk i \frac{i k \gamma - x_0}{k^2 + x_0^2} \left(1 + \partial x_0 i \frac{i k \gamma - x_0}{k^2 + x_0^2} \right) \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{n!} \int dk dh_1 \dots dh_n \frac{-i}{h_1^2 + \mu^2} \dots \frac{-i}{h_n^2 + \mu^2} \\
&\quad \times \left[-\bar{s}(l; -h_1, \dots, -h_n; k) i \frac{i k \gamma - x_0}{k^2 + x_0^2} \left(1 + \partial x_0 i \frac{i k \gamma - x_0}{k^2 + x_0^2} \right) s(k; h_1, \dots, h_n; l) \right. \\
&\quad \left. + \int \bar{s}(l; -h_1, \dots, -h_n; k) i \frac{i k \gamma - x_0}{k^2 + x_0^2} \frac{i g^2}{(2\pi)^2} \right. \\
&\quad \times i \frac{i(k-h)\gamma - x_0}{(k-h)^2 + x_0^2} s(k-h; h_1, \dots, h_n, h; l) \frac{-i}{h^2 + \mu^2} dh \\
&\quad \left. + \int \bar{s}(l; -h_1, \dots, -h_n, h; k-h) i \frac{i(k-h)\gamma - x_0}{(k-h)^2 + x_0^2} \frac{i g^2}{(2\pi)^2} \right. \\
&\quad \left. \times i \frac{i k \gamma - x_0}{k^2 + x_0^2} s(k; h_1, \dots, h_n; l) \frac{-i}{h^2 + \mu^2} dh \right] \times i \frac{i l \gamma - x_0}{l^2 + x_0^2}. \quad (57)
\end{aligned}$$

By the variation

$$\frac{\partial K}{\partial s} = 0, \quad \frac{\partial K}{\partial \bar{s}} = 0,$$

we gain (55) and the first and second equations of (56) with the self energy subtraction. Then $\bar{s} = s$ is concluded.

On the other hand (57) are reduced by (55) and (56) as

$$K = \int dk s(k; k) i \frac{i k \gamma - k_0}{k^2 + k_0^2} \left(1 + \partial k_0 i \frac{i k \gamma - k_0}{k^2 + k_0^2} \right). \quad (58)$$

From (58) spur K is $i\eta$ of (7) which is neglected the terms of nucleon pair creations and take into account the nucleon self energy subtraction.

Practically in the case of the strong coupling the S -matrix (53) gained by these process will not effective, because the incident particles already are not free particles but the particles bound with the meson cloud. Therefore we must use the connected treatment of the bound state and the S -matrix to gain the exact solutions.

§ 5. Bound state equations

(a) Reduction of the bound state equations

Bethe-Salpeter have used the sum of the irreducible terms in the perturbation series of the vacuum expectation value $\langle \epsilon P(\phi(x_1)\phi(x_2)\bar{\psi}(x_3)\bar{\psi}(x_4)) \rangle_0$ for the nucleus of their integral equation.⁹⁾

But it seems to be impossible that we carry the process free from the perturbation method. Therefore we should give up the good idea, the irreducible correction, for our general equations. Again we start from the equations (31), (32), (33) and (34) reduced in section 3.

Practically we treat only one Fermion system neglected the Fermion pair creations.

Using the relation (54) or

$$\left. \begin{aligned} \langle \epsilon P(\phi(x'), \phi(x'')) \rangle_0 &= i\Delta_F(x' - x''), \\ \langle \epsilon P(\phi(x_1), \dots, \phi(x_n)) \rangle_0 &= 0, \quad n > 2 \end{aligned} \right\} \quad (54')$$

from (19), (31) and (34) we get

$$\begin{aligned} \langle \epsilon P(\phi(x'), \bar{\psi}(x'')) \rangle_0 &= \frac{1}{i} s_F(x' - x'') \\ &+ g \int dx_1 s_F(x' - x_1) \langle \epsilon P(\phi(x_1)\phi(x_1), \bar{\psi}(x'')) \rangle_0, \end{aligned} \quad (59)$$

$$\begin{aligned} \langle \epsilon P(\phi(x'), \phi(y_1), \dots, \bar{\psi}(x'')) \rangle_0 \\ = g \int dx_1 s_F(x' - x_1) \left[\langle \epsilon P(\phi(x_1)\phi(x_1), \phi(y_1), \dots, \bar{\psi}(x'')) \rangle_0 \right. \\ \left. + \sum_{s=1}^n i\Delta_F(x_1 - y_s) \langle \epsilon P(\phi(x_1), \dots, \phi(y_{s-1}), \phi(y_{s+1}), \dots, \bar{\psi}(x'')) \rangle_0 \right]. \end{aligned} \quad (60)$$

Applying the Gell-Mann-Low's method¹⁰⁾ we may obtain the bound state equations. Namely when $x'_0, y_{10}, \dots, y_{n0} > x''_0$, we may put

$$\begin{aligned} \langle \epsilon P(\phi(x'), \phi(y_1), \dots, \phi(y_n), \bar{\psi}(x'')) \rangle_0 \\ = \sum_K \langle P(\phi(x'), \phi(y_1), \dots, \phi(y_n)) |_{K,K} \bar{\psi}(x'') \rangle_0, \end{aligned} \quad (61)$$

where

$$\langle F |_{K,K} = (\Psi_0^* F \Psi_K). \quad (62)$$

Ψ_K is the bound state with the total energy-momentum K .

By means of the Gell-Mann-Low's average process in $x''_0 \rightarrow \infty$, the inhomogeneous terms vanish. Therefore from (59) and (60) we get

$$\langle \psi(x) |_{\mathcal{K}} = g \int dx_1 s_F(x-x_1) \langle P(\psi(x_1), \phi(x_1)) |_{\mathcal{K}}, \quad (63)$$

$$\begin{aligned} & \langle P(\psi(x), \phi(y_1), \dots, \phi(y_n)) |_{\mathcal{K}} \\ &= g \int dx_1 s_F(x-x_1) \left[\langle P(\psi(x_1), \phi(x_1), \phi(y_1), \dots, \phi(y_n)) |_{\mathcal{K}} \right. \\ & \quad \left. + \sum i \Delta_F(x_1-y_s) \langle P(\psi(x_1), \phi(y_1), \dots, \phi(y_{s-1}), \phi(y_{s+1}), \dots, \phi(y_n)) |_{\mathcal{K}} \right]. \end{aligned} \quad (64)$$

The equations rewritten into the momentum space are

$$\langle \psi(k) |_{\mathcal{K}} = \frac{ig}{(2\pi)^2} i \frac{ik\gamma - x_0}{k^2 + x_0^2} \int dh \langle P(\psi(k-h), \phi(h)) |_{\mathcal{K}}, \quad (65)$$

$$\begin{aligned} & \langle P(\psi(k), \phi(h_1), \dots, \phi(h_n)) |_{\mathcal{K}} \\ &= \frac{ig}{(2\pi)^2} i \frac{ik\gamma - x_0}{k^2 + x_0^2} \left[\int dh \langle P(\psi(k-h), \phi(h), \phi(h_1), \dots, \phi(h_n)) |_{\mathcal{K}} \right. \\ & \quad \left. + \sum_{s=1}^n \frac{-i}{h_s^2 + \mu^2} \langle P(\psi(k+h_s), \phi(h_1), \dots, \phi(h_{s-1}), \phi(h_{s+1}), \dots) |_{\mathcal{K}} \right], \end{aligned} \quad (66)$$

where we express as

$$\langle P(\psi(k), \phi(h_1), \dots, \phi(h_n)) |_{\mathcal{K}} = \mathcal{X}(k; h_1, \dots, h_n) \delta(k+h_1+\dots+h_n-K). \quad (67)$$

We operate $i(ik\gamma + x_0)$ on (65) and (66) and integrate with $d\vec{k}$, and we get

$$\begin{aligned} & i(ik\gamma + x_0) \mathcal{X}(K) = \frac{ig}{(2\pi)^2} \int dh \mathcal{X}(K-h; h), \quad (68) \\ & i\{i(K-h_1-\dots-h_n)\gamma + x_0\} \mathcal{X}(K-h_1-\dots-h_n; h_1, \dots, h_n) \\ &= \frac{ig}{(2\pi)^2} \left[\int dh \mathcal{X}(K-h-h_1-\dots-h_n; h, h_1, \dots, h_n) \right. \\ & \quad \left. + \sum_{s=1}^n \frac{-i}{h_s^2 + \mu^2} \mathcal{X}(K-h_1-\dots-h_{s-1}-h_{s+1}-\dots-h_n; h_1, \dots, h_{s-1}, h_{s+1}, \dots) \right]. \end{aligned} \quad (69)$$

Thus we obtain the equations for the bound state solutions. Next we return to the Schroedinger representation and show the correspondence with the Fock-Tomonaga equations.

(b) *Schroedinger equations*

The operator $\psi_\nu(\vec{x})$ in the Schroedinger representation which coincide with the Heisenberg operator at t' are written

$$\psi_\nu(\vec{x}) = \psi(\vec{x}, t') \quad (70)$$

and from (35)

$$\psi_\nu(\vec{k}) = \psi(\vec{k}; t') = \int \frac{d\vec{x}}{(2\pi)^3} \psi(\vec{x}, t') e^{-i\vec{k}\cdot\vec{x}} = \int \frac{dk_0}{(2\pi)^2} \phi(\vec{k}) e^{-ik_0 t'}. \quad (71)$$

Simply we take $t'=0$.

The free state in the Schroedinger representation are written

$$\Phi_f^s(t) = \frac{1}{\sqrt{n!}} \bar{\phi}(\vec{k}; 0) \phi(-\vec{h}_1; 0) \cdots \phi(-\vec{h}_n; 0) \Psi_0 e^{-i(K_0 + \mathbf{h}_{10} + \cdots + \mathbf{h}_{n0})t} \quad (72)$$

and for the bound state we use the following expression

$$\Psi_K^s(t) = \Psi_K e^{-iK_0 t} = \sum \Psi(\vec{k}, t; \vec{h}_1, \dots, \vec{h}_n) \Phi_f^s e^{iK_0 t}. \quad (73)$$

From (62), (67), (71), (72) and (73)

$$\begin{aligned} \Psi(\vec{k}, t; \vec{h}_1, \dots, \vec{h}_n) &= (\Phi_f^s \Psi_K^s) e^{-iK_0 t} \\ &= \frac{1}{\sqrt{n!}} \int \frac{dK_0}{(2\pi)^2} \frac{dh_{10}}{(2\pi)^2} \cdots \frac{dh_{n0}}{(2\pi)^2} \langle P(\psi(\vec{k}), \phi(h_1), \dots, \phi(h_n)) |_K e^{i(\mathbf{h}_{10} + \cdots + \mathbf{h}_{n0} - K_0)t} \\ &= \frac{1}{\sqrt{n!}} \frac{1}{(2\pi)^2} \frac{dh_{10}}{(2\pi)^2} \cdots \frac{dh_{n0}}{(2\pi)^2} \chi(K - h_1 \cdots h_n; h_1, \dots, h_n) \\ &\quad \times \delta(\vec{k} + \vec{h}_1 + \cdots + \vec{h}_n - \vec{K}) e^{i(\mathbf{h}_{10} + \cdots + \mathbf{h}_{n0} - K_0)t}. \end{aligned} \quad (74)$$

On the other hand

$$\begin{aligned} &\langle P(\psi(\vec{k}; t), \phi(h_1), \dots, \phi(h_n)) |_K \\ &= \int \frac{dK_0}{(2\pi)^2} \langle P(\psi(\vec{k}, K_0), \phi(h_1), \dots, \phi(h_n)) |_K e^{-iK_0 t} \\ &= \frac{1}{(2\pi)^2} \chi(K - h_1 \cdots h_n; h_1, \dots, h_n) \delta(\vec{k} + \vec{h}_1 + \cdots + \vec{h}_n - \vec{K}) e^{i(\mathbf{h}_{10} + \cdots + \mathbf{h}_{n0} - K_0)t}. \end{aligned} \quad (75)$$

Therefore if we put

$$\begin{aligned} \Psi(\vec{k}, t; \vec{h}_1, \dots, \vec{h}_n) \\ &= \frac{1}{(2\pi)^2} \chi(\vec{K} - \vec{h}_1 \cdots \vec{h}_n; \vec{h}_1, \dots, \vec{h}_n) \delta(\vec{k} + \vec{h}_1 + \cdots + \vec{h}_n - \vec{K}) e^{i(\mathbf{h}_{10} + \cdots + \mathbf{h}_{n0} - K_0)t} \end{aligned} \quad (76)$$

and in the equations (68) and (69) instead of

$$(K_0 - h_{10} \cdots h_{n0}) \chi(K - h_1 \cdots h_n; h_1, \dots, h_n)$$

we put $(K_0 - \mathbf{h}_{10} \cdots \mathbf{h}_{n0}) \chi(K - h_1 \cdots h_n; h_1, \dots, h_n)$ and integrate with $\frac{1}{\sqrt{n!}} \frac{dh_{10}}{(2\pi)^2} \cdots \frac{dh_{n0}}{(2\pi)^2}$, we get the equations for $\chi^s(\vec{K} - \vec{h}_1 \cdots \vec{h}_n; \vec{h}_1, \dots, \vec{h}_n)$.

They are the equations as follows

$$\begin{aligned} &-i\{K_0 - \mathbf{h}_{10} \cdots \mathbf{h}_{n0} - ((\vec{K} - \vec{h}_1 \cdots \vec{h}_n)^2 + x_0^2)^{1/2}\} \chi^s(\vec{K} - \vec{h}_1 \cdots \vec{h}_n; \vec{h}_1, \dots, \vec{h}_n) \\ &= ig \left\{ \int d\vec{h} \frac{1}{\sqrt{n+1}} \chi^s(\vec{K} - \vec{h} - \vec{h}_1 \cdots \vec{h}_n; \vec{h}, \vec{h}_1, \dots, \vec{h}_n) \right. \\ &\quad \left. + \sum_{s=1}^n \int \frac{dh_{s0}}{(2\pi)^4} \frac{-i}{h_s^2 + \mu^2} \sqrt{n} \chi^s(\vec{K} - \vec{h}_1 \cdots \vec{h}_n; \vec{h}_1, \dots, \vec{h}_{s-1}, \vec{h}_{s+1}, \dots) \right\}. \end{aligned} \quad (77)$$

Neglecting the momentum recoil, namely

$$((\vec{K}-\vec{h}_1\cdots-\vec{h}_n)^2+x_0^2)^{1/2}\rightarrow x_0 \quad \text{and} \quad r_4\rightarrow 1$$

we get the following equations

$$(x_0-K_0)\chi^s(\vec{K})=g\int d\vec{h}\chi^s(\vec{K}-\vec{h};\vec{h}), \quad (78)$$

$$\begin{aligned} & (\vec{h}_{10}+\cdots\vec{h}_{n0}+x_0-K_0)\chi^s(\vec{K}-\vec{h}_1\cdots-\vec{h}_n;\vec{h}_1,\dots,\vec{h}_n) \\ & =g\left\{\int d\vec{h}\frac{1}{\sqrt{n+1}}\chi^s(\vec{K}-\vec{h}-\vec{h}_1\cdots-\vec{h}_n;\vec{h},\vec{h}_1,\dots,\vec{h}_n) \right. \\ & \quad \left. +\sum_{s=1}^n\frac{\sqrt{n}}{2(2\pi)^3h_{s0}}\chi^s(\vec{K}-\vec{h}_1\cdots-\vec{h}_n;\vec{h}_1,\dots,\vec{h}_{s-1},\vec{h}_{s+1},\dots,\vec{h}_n)\right\}. \quad (79) \end{aligned}$$

This is the equations used by Tomonaga which have a simple analytic solution.⁴⁾

In this section we have not refer to the two Fermions system, but the equations for it will be reduced from (31), (32), (33) and (34) by the similar process and will be more complicated than (68) and (69). From the first and second equations we shall obtain, as was expected, the Bethe-Salpeter equation in which the nucleus is limited only by the first term in its perturbation series.

Generally for the problems like as in the charged meson system or containing the nucleon pair creations, the equations will be much complicated. But the relation with the Dyson's perturbation or the relativistic subtraction should be obvious. This correspondence should lead to any effective strong coupling method founded on the exact field theory.

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A Remark on the Strong Coupling Approximation in Meson-nucleon Scattering

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Tomonaga's intermediate coupling theory of the meson-nucleon scattering is applied to the case of the strong meson-nucleon coupling, using the simplified model of the charged vector longitudinal mesons interacting with a heavy nucleon. The remarkable feature in the strong coupling limit is that, besides the ordinary levels of the nucleon, its isobars and their excited states, there exist another kind of the excited levels (the "higher levels" defined in Sec. 2) which play an important role for the scattering problem.

Our result agrees with the former conclusions of the Wentzel's strong coupling treatment. Comparison with the case of the intermediate coupling region is discussed.

§ 1. Introduction

Recently, Tomonaga¹⁾ proposed an approximation method which deals with the scattering of mesons by nucleon for the intermediate coupling strength of the meson-nucleon system. In this method, the meson field was expanded in the momentum space by a set of complete orthonormal functions, $\varphi_0(\mathbf{k})$, $\varphi_{\mathfrak{B}}(\mathbf{k})$, where

$$\varphi_0(\mathbf{k}) = \frac{1}{K_2} \frac{k}{K\sqrt{K}}, \quad K_2^2 = \int \frac{k^2}{K^3} d\mathbf{k},$$

and the problem was treated by the picture: how the \mathfrak{B} -mesons, which are the mesons in the configuration of $\varphi_{\mathfrak{B}}(\mathbf{k})$, is scattered by a "clothed" nucleon, where the latter is described approximately by taking into account the whole number of mesons lying in the configuration of $\varphi_0(\mathbf{k})$.

In this paper, we discuss the meson-nucleon scattering using this picture, under the assumption that the coupling is very strong.** Although our result agrees with the well known conclusions of the strong coupling theory,³⁾⁴⁾ this method will serve to clarify the physical content of the strong coupling approximation from somewhat different view point, as well as to see how the various approximations, which were assumed in I when the coupling is intermediate, break down as the coupling becomes strong.

*) We assume $\varphi_{\mathfrak{B}}(\mathbf{k})$ to be real. Although Hayakawa²⁾ proposed a method in which only the momentum representation is employed without introducing the \mathfrak{B} -representation, we shall follow the notation of Tomonaga in order to reserve closer correspondence with I.

**) This means that only the zeroth order terms in the expansion in the reciprocal coupling constant are reserved.

We use the same simplified model as in I, that is, the charged vector longitudinal mesons interacting with a heavy nucleon. The Hamiltonian is therefore

$$\begin{aligned} H = & K_{00} \{A^*A + B^*B - V[(A+B^*)Q + (A^*+B)Q^*]\} \\ & + \{K_{0\bar{3}} \{(A^*-VQ)a_{\bar{3}} + (B^*-VQ)b_{\bar{3}}\} d\bar{3} + c.c. \\ & + \{S(a_{\bar{3}}^*a_{\bar{3}} + b_{\bar{3}}^*b_{\bar{3}}) d\bar{3} \} \end{aligned} \quad (I)$$

where and hereafter the same symbols and notations are used as in I, if not specially mentioned.

§ 2. Mesonic proper field of a nucleon in the limit of strong coupling

The eigenvalue equation of the mesonic proper field of a nucleon is

$$(\mathcal{Q} - \mathcal{Q}_0)\Phi = 0, \quad (1)$$

where \mathcal{Q} is the Hamiltonian of the self-field divided by K_{00} ,

$$\mathcal{Q} = A^*A + B^*B - V[(A+B^*)Q + (A^*+B)Q^*] \quad (2)$$

and \mathcal{Q}_0 is its eigenvalue. Following Wentzel³⁾ and Tomonaga,⁴⁾ we transform (1) into differential equations. For this purpose we introduce the canonical variables x , y , p_x , and p_y by

$$\begin{cases} A = 1/2(x - iy + ip_x + p_y) \\ B = 1/2(x + iy + ip_x - p_y) \end{cases}$$

and further the polar coordinates r , θ by

$$x = r \cos \theta, \quad y = r \sin \theta,$$

where $[x, p_x] = [y, p_y] = i$ and other commutators vanish.

Then the operators A and B (and their complex conjugates) become

$$\begin{aligned} A &= \frac{1}{2} e^{-i\theta} \left(r + \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right), \\ B &= \frac{1}{2} e^{i\theta} \left(r + \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right), \\ A^* &= \frac{1}{2} e^{i\theta} \left(r - \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right), \\ B^* &= \frac{1}{2} e^{-i\theta} \left(r - \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right). \end{aligned} \quad (3)$$

The angular part is separated by putting for the wave function

$$\Phi = \begin{pmatrix} \varphi(x, y) \\ \psi(x, y) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{r}} \begin{pmatrix} f(r) e^{i(m-1/2)\theta} \\ g(r) e^{i(m+1/2)\theta} \end{pmatrix}, \quad (4)$$

$$m = \pm 1/2, \pm 3/2, \dots,$$

and (1) is transformed into a set of differential equations

$$\begin{cases} [-d^2/dr^2 + m(m-1)/r^2 + r^2 - 2(\Omega_0 + 1)] f(r) - 2Vr g(r) = 0, \\ [-d^2/dr^2 + m(m+1)/r^2 + r^2 - 2(\Omega_0 + 1)] g(r) - 2Vr f(r) = 0. \end{cases} \quad (5)$$

The solutions of (5) in the limit of strong coupling were obtained by Wentzel³⁾ using the W. K. B. approximation. We shall now summarize some relevant results for the later discussions of the scattering problem.

(a) The energy levels which follow from (5) are divided into three classes. The first class consists of the levels which lie near $\Omega_0 = -V^2/2$, and we call them hereafter the "lower levels". The levels of the second class are situated above the lower levels by the amount of $2V^2$ and are called the "higher levels". Other levels exist, forming the third class, which lie between the first and second classes. However, as will be shown later, only the first and second classes are important for the scattering problem.

(b) The solutions of the lower levels are given in the first approximation by putting in (5)

$$f(r) = g(r), \quad (6)$$

$$[-d^2/dr^2 + m^2/r^2 + r^2 - 2Vr - 2(\Omega_0 + 1)] f(r) = 0.$$

This equation gives for $f(r)$ solutions which have large values only near the point $r = V$. If we neglect the "centrifugal force" term m^2/r^2 , which gives the effect of the order of $1/V^2$, they are the solutions of the harmonic oscillator, whose origin is displaced by $-V$. Then the eigenvalue, which is exact up to the order of $1/V^2$ and belonging to the charge and oscillator quantum numbers (m, n) is

$$\Omega_{mn} = -V^2/2 + n - 1/2 + m^2/2V^2 \quad (7)$$

where $K_{00}(m^2/2V^2)$ is the well known isobar separation and $(m+1/2)e$ is the charge of the total system, bare nucleon plus its self-field. $n=0,1,2,\dots$ is the quantum number representing the excited states of each isobars (including proton or neutron).

(c) The states which belong to other levels than the lower ones have positive self-energies. For them, the potential has its minimum near the origin, and the wave functions of the less excited states have values only near the origin, while very highly excited states have wave functions which extend as far as $r \sim V$. The former states form the third class mentioned in (a) and the latter the higher level class.

Now, as mentioned in (b), the wave functions of the lower levels have their large values only in the region $r \simeq V$. Therefore, only such states which have wave functions extending as far as $r \sim V$ can give non-vanishing matrix elements for the transition from the lower levels. This is the reason that only the lower and higher levels are important in the scattering problem; and the wave functions of the higher levels, which have values near $r \simeq V$ have the approximate relation

$$f(r) = -g(r), \quad (8)$$

and the corresponding eigenvalue

$$\mathcal{Q}_h = 3/2 \quad V^2.$$

The relations (6) and (8) and the normalization of $f(r)$, both for the lower and higher levels,

$$\int f(r)^2 dr = 1/2 \quad (10)$$

will be frequently used in the subsequent discussions. That the higher levels are really relevant will be shown more explicitly in the next section.

§ 3. Transition matrix elements

Using the solutions in the last section, we calculate the matrix elements of A , B , etc., which are needed for the scattering problem. Here A , B etc. are defined as in I by

$$\begin{cases} A = A - VQ^*, & B = B - VQ, \\ A^* = A^* - VQ, & B^* = B^* - VQ^*. \end{cases} \quad (11)$$

With these operators (2) is written as

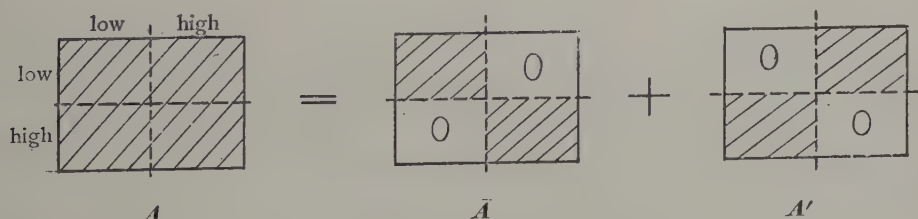
$$\mathcal{Q} = A^* A + B^* B - V^2. \quad (12)$$

In the following, we shall mainly consider about A . Similar considerations can be applied for other operators.

Corresponding to the separation of the energy levels into two classes, we divide A into two parts, in the representation in which \mathcal{Q} is diagonal:

$$A = \bar{A} + A'. \quad (13)$$

Here, \bar{A} has non-vanishing matrix elements only for the transitions among "lower levels" and among "higher levels" while A' has only those matrix elements which cause transitions between lower and higher levels. Schematically



Similarly, $Q^{(*)}$ is also divided into two parts

$$Q^{(*)} = \bar{Q}^{(*)} + Q^{(*)}'. \quad (14)$$

Then for $V \gg 1$, we have approximately

$$A' = -VQ^{*'}, \quad (15)$$

and if we introduce a new variable \bar{r} according to

$$r = \bar{r} + V,$$

\bar{A} can be expressed from (3) and (11) as

$$\begin{aligned} \bar{A} = & \frac{1}{2} e^{-i\theta} \left(\bar{r} + \frac{\partial}{\partial \bar{r}} \right) - \frac{i}{2} e^{-i\theta} \frac{1}{\bar{r} + V} \frac{\partial}{\partial \theta} \\ & + V \left(\frac{1}{2} e^{-i\theta} - \bar{Q}^* \right). \end{aligned} \quad (16)$$

Before going to the evaluation of the matrix elements, we should now investigate which matrix elements are really relevant for our problem. In the following paragraph, we shall show that only such two types of the matrix elements as

$$(l' | \bar{A} | l'') (l'' | \bar{A}^* | l) \quad (17)$$

and

$$\sum_{h'} (l' | A' | h') (h' | A^{*'} | l)$$

(and the similar ones for B , B^* and their combinations with A , A^*) are necessary to evaluate. Here, l and h stand for the "lower" and "higher" levels, respectively.

To show it, we note that what we are going to do is to describe the scattering processes with terms which do not vanish in the limit $1/V \rightarrow 0$. Now, when we write down the matrix elements of a scattering process $A \rightarrow F$, where the initial and final states A , F both belonging to the lower levels, using such quantities as

$$(l' | \bar{A} | l), \quad (h' | \bar{A} | h), \quad (l | A' | h), \quad (h | A' | l), \quad (18)$$

then we have, in the scheme of the perturbation theory, the following three kinds of the lowest order matrix elements:

$$\sum_{l''} (l_F | \bar{A}^* | l'') (l'' | \bar{A}^* | l_A) / (\mathcal{Q}_{lA} - \mathcal{Q}_{l''}), \quad (19-i)$$

$$\sum_{h'} (l_F | A' | h') (h' | A^{*'} | l_A) / (\mathcal{Q}_{lA} - \mathcal{Q}_{h'}), \quad (19-ii)$$

$$\sum_{h'l''} (l_F | A' | h'') (h'' | \bar{A} | h') (h' | A' | l_A) / [(\mathcal{Q}_{lA} - \mathcal{Q}_{h'}) (\mathcal{Q}_{lA} - \mathcal{Q}_{h''})]. \quad (19-iii)$$

Here, (19-i) stands for the processes which take places among lower levels only, (19-ii) for the processes caused by the (even number of) operators A' , and (19-iii) stands for such processes which contain some number of quantities $(h'' | \bar{A} | h')$ between the first and last transitions $(h' | A' | l_A)$ and $(l_F | A' | h'')$. Taking now into account that $(l' | \bar{A} | l)$ is $O(1)$, $(l | A' | h)$ and $(h | A' | l)$ are both $O(V)$, and $(h'' | \bar{A} | h')$ is $O(1)$ or at most $O(V)$, and remembering that

$$\mathcal{Q}_l - \mathcal{Q}_{l''} = O(1), \quad \mathcal{Q}_l - \mathcal{Q}_{h'} = O(V^2),$$

we can see that each of the matrix elements (19-i, ii, iii) has the following order of magnitude:

$$(19-i) = O(1), \quad (19-ii) = O(1), \quad (19-iii) \lesssim O(1/V).$$

This proves our statement in eqs. (17). We note that single elements such as $(l'|A'|l)$ are not necessary; the energy denominators in (19-ii) being put out of the summation symbol since $\mathcal{Q}_{k'} - \mathcal{Q}_l \cong 2|v|^2$ can be regarded as independent of the suffices l' .

Having made clear which matrix elements are really relevant, we now turn to derive some useful properties of the matrices $(l'|\bar{A}\bar{A}^*|l)$ and $(l'|A'A'^*|l)$, etc.

$$(i) \quad (l'|\bar{A}\bar{A}^*|l)$$

In the expression (16) for \bar{A} , we can immediately show that the last term has a vanishing contribution by the approximate relation $f(r) = g(r)$ (eq. (6)) for lower levels.* The second term can also be neglected since it contributes only the effect of $O(1/V)$. Since the same can be said about \bar{A}^* , \bar{B} , and B^* , we have from (3), for the transitions among lower levels

$$\begin{cases} \bar{A} \cong \frac{1}{2} e^{-i0} \left(\bar{r} + \frac{\partial}{\partial \bar{r}} \right), & \bar{B} \cong \frac{1}{2} e^{i0} \left(\bar{r} + \frac{\partial}{\partial \bar{r}} \right), \\ \bar{A}^* \cong \frac{1}{2} e^{-i0} \left(\bar{r} - \frac{\partial}{\partial \bar{r}} \right), & \bar{B}^* \cong \frac{1}{2} e^{-i0} \left(\bar{r} - \frac{\partial}{\partial \bar{r}} \right). \end{cases} \quad (20)$$

From them, we have

$$\begin{cases} [\bar{A}, \bar{A}^*] = [\bar{B}, \bar{B}^*] = 1/2, \\ \text{other commutators vanishing} \end{cases}$$

and

$$(n-1, m-1|\bar{A}|n, m) = (n-1, m+1|\bar{B}|n, m) = \sqrt{n}/\sqrt{2},$$

etc. (other elements are zero). For the later purpose, however, it is more convenient to introduce a new operator a and its conjugate by

$$\begin{cases} \sqrt{2}e^{+i0} \bar{A} = \sqrt{2}e^{-i0} \bar{B} \equiv a, \\ \sqrt{2}e^{-i0} \bar{A}^* = \sqrt{2}e^{+i0} \bar{B}^* \equiv a^*, \end{cases} \quad (21)$$

which have the commutation relation

$$[a, a^*] = 1. \quad (22)$$

Here use has been made of the fact that \bar{A} , \bar{B} etc. in (21) do not contain $\partial/\partial\theta$ and therefore commute with $\exp(\pm i\theta)$.

$$(ii) \quad (l'|A'A'^*|l)$$

From the relation (15), we have only to calculate $(l'|Q'Q'|l)$ and $(l'|Q'Q'|l)$. For $(l'|Q'Q'|l)$, we have from the definition of Q' ,

$$(l'|Q'Q'|l) = (l'|QQ|l) - \sum_{l''} (l'|Q|l'') (l''|Q|l).$$

* $f(r) - g(r)$ is at most of the order of $1/V^2$.

The first term in the right hand side is identically zero from the definition of Q ,

$$Q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and in the second term $(l'|Q|l)$ has non-vanishing values only for $(n, m+1|Q|n, m)$ and the value is $1/2$ from (4), (6) and (10). Thus we have

$$(n, m+2|Q'Q'|n, m) = -(1/4),$$

or in the operator form*,

$$Q'Q' = -(1/4) \exp(2i\theta). \quad (23)$$

Quite similarly, we can get

$$Q^{*'}Q' = Q'Q^{*'} = 1/4. \quad (24)$$

The following relation, which follows from (4), (6) and (8), and can be used for deriving (24) from (23), will also be utilized in the next section:

$$\exp(+i\theta)Q^{*'} = -\exp(-i\theta)Q'. \quad (25)$$

§ 4. Derivation of the effective Hamiltonian

We shall now proceed to the derivation of the effective Hamiltonian which is to be used in the scattering problem. Using (11), (13), (14) and (15), the Hamiltonian (I) becomes**

$$\begin{aligned} H &= K_{00}Q + H_0 + H_s + H', \\ H_0 &= \int K_{0\bar{s}} (\bar{A}^* a_{\bar{s}} + \bar{B}^* b_{\bar{s}}) d\bar{s} + c.c., \\ H_s &= \int S (a_{\bar{s}}^* a_{\bar{s}} + b_{\bar{s}}^* b_{\bar{s}}) d\bar{s}, \\ H' &= -V \int K_{0\bar{s}} [Q' (a_{\bar{s}} + b_{\bar{s}}^*) + Q (a_{\bar{s}}^* + b_{\bar{s}})] d\bar{s}. \end{aligned} \quad (26)$$

As can be seen immediately from these equations, the transitions of states of the "clothed" nucleon from lower to higher levels, and vice versa, are described by the single term H' , which contains the large coupling constant V . Then, if we can eliminate this term by a suitable canonical transformation, and if the transformed Hamiltonian, \bar{H} , is closed up to the zeroth order in $1/V$, we can use such \bar{H} to treat the scattering problem without referring to the higher levels. Such a transformation is in fact possible and given by

$$\begin{aligned} \bar{H} &= \exp(+G) H \exp(-G) \\ &= H + [G, H] + 1/2[G, [G, H]] - \dots \end{aligned} \quad (27)$$

with

*) Eqs. (23) and (24) hold also for $(h'|h')$, since to derive them we have to use only the orthonormality of the wave functions.

**) See note added in proof.

$$G = -\frac{Q_1' - Q_2'}{2V'K_{00}} \int K_{0\beta} (a_{\beta} + b_{\beta}^*) d\beta + \frac{Q_1^{*'} - Q_2^{*'}}{2V'K_{00}} \int K_{0\beta} (a_{\beta}^* + b_{\beta}) d\beta \quad (28)$$

where, in (28), the quantities Q_1' and Q_2' are defined by

$$\begin{cases} Q' = Q_1' + Q_2' \\ (l|Q_1'|h) = (l|Q'|h), \quad (h|Q_1'|l) = 0, \\ (h|Q_2'|l) = (h|Q'|l), \quad (l|Q_2'|h) = 0. \end{cases} \quad (29)$$

In order to show that this is just the required transformation, we first note that in the second term in (27):

$$[G, H] = K_{00}[G, Q] + [G, H_0] + [G, H_s] + [G, H']$$

the commutators $[G, H_0]$ and $[G, H_s]$ vanish. Then, using the relations

$$\begin{aligned} [Q, Q_1'] &= -2V^2 Q_1', & [Q, Q_1^{*'}] &= 2V^2 Q_1^{*'} \\ [Q, Q_2'] &= 2V^2 Q_2', & [Q, Q_2^{*'}] &= -2V^2 Q_2^{*'}, \end{aligned} \quad (30)$$

which follow, for instance, from

$$(l|[Q, Q_1']|h) = (Q_l - Q_h)(l|Q_1'|h) = -2V^2(l|Q_1'|h)$$

we get immediately

$$K_{00}[G, Q] = -H' \quad (31)$$

and this cancels H' in the original Hamiltonian. Next, using the relations which follow from (25), (24) and (29):

$$\begin{aligned} Q_1^{*'} &= -\exp(-2i\theta) Q_2', \\ Q_2^{*'} &= -\exp(-2i\theta) Q_1', \\ [Q_1' + Q_2', Q_1' - Q_2'] &= 1/2 \cdot \exp(2i\theta), \end{aligned} \quad (32)$$

we have

$$[G, H'] = \frac{-1}{4K_{00}} \int K_{0\beta} [e^{i\theta}(a_{\beta} + b_{\beta}^*) - e^{-i\theta}(a_{\beta}^* + b_{\beta})] d\beta. \quad (33)$$

Further, the third term in (27) is, by (31);

$$\begin{aligned} 1/2 \cdot [G, [G, H]] &= 1/2 \cdot [G, [G, K_{00}Q]] \\ &= 1/2 \cdot [G, H']. \end{aligned} \quad (34)$$

Collecting these results, we finally obtain

$$\bar{H} = K_{00}Q + H_0 + H_s + H_1,$$

$$H_1 = \frac{-1}{8K_{00}} \left(\int K_{0\beta} [e^{i\theta}(a_{\beta} + b_{\beta}^*) - e^{-i\theta}(a_{\beta}^* + b_{\beta})] d\beta \right)^2. \quad (II)$$

If we neglect the quantities of the order of $1/V$ in $K_{00} \Omega$ and H_0 , (II) can further be reduced by using the results in the preceding section. First, from (13), (14) and (15),

$$\begin{aligned} \Omega &= A^* A + B^* B - V^2 \\ &= \bar{A}^* \bar{A} + \bar{B}^* \bar{B} - V [Q'(\bar{A} + \bar{B}^*) + Q(\bar{A}^* + \bar{B})] \\ &\quad + V^2(Q'Q^{*'} + Q^{*'}Q') - V^2. \end{aligned} \quad (35)$$

Since the higher levels have been eliminated, we can express Ω by quantities of the lower levels and c -numbers. By (24), the last two terms in (35) combine together to give $-V^2/2$, which, when multiplied by K_{00} , is the main term of the nucleon self-energy, and can be dropped hereafter. Then from eqs. (20) and (25), the second term vanishes:

$$\begin{aligned} &Q'(\bar{A} + \bar{B}^*) + Q^{*'}(\bar{A}^* + \bar{B}) \\ &= (Q'e^{-i\theta} + Q^{*'}e^{+i\theta})\bar{r} = 0. \end{aligned}$$

Therefore, by introducing u and u^* of (21), we can express $K_{00} \Omega$ simply as

$$K_{00} \Omega = K_{00} u^* u. \quad (36)$$

Next, we reduce the expressions for H_0 and H_1 by introducing a new set of variables $x_{\bar{s}}$ and $y_{\bar{s}}$ defined by*

$$\begin{cases} a_{\bar{s}} = \frac{1}{\sqrt{2}} e^{-i\theta} (x_{\bar{s}} - iy_{\bar{s}}), & a_{\bar{s}}^* = \frac{1}{\sqrt{2}} e^{+i\theta} (x_{\bar{s}}^* + iy_{\bar{s}}^*), \\ b_{\bar{s}} = \frac{1}{\sqrt{2}} e^{+i\theta} (x_{\bar{s}} + iy_{\bar{s}}), & b_{\bar{s}}^* = \frac{1}{\sqrt{2}} e^{-i\theta} (x_{\bar{s}}^* - iy_{\bar{s}}^*). \end{cases} \quad (37)$$

These new variables satisfy the commutation relations

$$\begin{cases} [x_{\bar{s}}, x_{\bar{s}'}^*] = [y_{\bar{s}}, y_{\bar{s}'}^*] = \delta(\bar{s} - \bar{s}'), \\ \text{other commutators vanishing,} \end{cases} \quad (38)$$

and especially, they commute with u and u^* .

After all such procedures, the following expression is obtained for the effective Hamiltonian \bar{H} of the scattering problem:

$$\begin{aligned} \bar{H} &= H_s + H_y, \\ \begin{cases} H_s = K_{00} u^* u + \int K_{0\bar{s}} (u^* x_{\bar{s}} + u x_{\bar{s}}^*) d\bar{s} + \int S x_{\bar{s}}^* x_{\bar{s}} d\bar{s}, \\ H_y = \int S y_{\bar{s}}^* y_{\bar{s}} d\bar{s} - \frac{1}{4K_{00}} [\int K_{0\bar{s}} (y_{\bar{s}} + y_{\bar{s}}^*) d\bar{s}]^2, \end{cases} \end{aligned} \quad (III)$$

*) We could of course introduce them before we perform the transformation (27), (28).

§ 5. Scattering of mesons by a nucleon

Because H_x and H_y in the Hamiltonian (III) are commutable, we can consider them separately.

In the first place, we transform H_x into the momentum representation by introducing

$$\begin{cases} x(\mathbf{f}) = a\varphi_0(\mathbf{f}) + \int x_{\mathbf{\bar{s}}} \varphi_{\mathbf{\bar{s}}}(\mathbf{f}) d\mathbf{\bar{s}}, \\ x^*(\mathbf{f}) = a^*\varphi_0(\mathbf{f}) + \int x_{\mathbf{\bar{s}}}^* \varphi_{\mathbf{\bar{s}}}(\mathbf{f}) d\mathbf{\bar{s}}. \end{cases} \quad (39)$$

and get immediately

$$H_x = \int K x^*(\mathbf{f}) x(\mathbf{f}) d\mathbf{f}.$$

Since we have $[x(\mathbf{f}), x^*(\mathbf{f}')] = \delta(\mathbf{f} - \mathbf{f}')$ from (22) and (38), we can conclude that the x -mesons, i.e., the mesons whose total energy is represented by H_x , are not scattered by the nucleon.

If we consider for instance the case in which a positive meson is incident, this state can be regarded as a superposition of the state of the x -meson and the state of the y -meson with an equal weight (see eq. (37)). Then we can see that half of the incident positive mesons, i.e., x -mesons, will not be scattered and just the remaining half, i.e., y -mesons, of which half is the positive and another half is the negative mesons, will be scattered. This agrees exactly with the conclusion of Tomonaga's treatment⁴⁾ that the radially polarized mesons, which occupy a half of the incident positive mesons, are not scattered and only the tangential mesons, which occupy the remaining half are scattered. In fact, it is very easy to show from (37) that our x and y -mesons are just the radially and tangentially polarized mesons of Tomonaga, respectively.

To obtain the solution for y -mesons, we now transform $y_{\mathbf{\bar{s}}}$ and $y_{\mathbf{\bar{s}}}^*$ into their momentum representations by

$$\begin{cases} y(\mathbf{f}) = \int y_{\mathbf{\bar{s}}} \varphi_{\mathbf{\bar{s}}}(\mathbf{f}) d\mathbf{\bar{s}}, \\ y^*(\mathbf{f}) = \int y_{\mathbf{\bar{s}}}^* \varphi_{\mathbf{\bar{s}}}(\mathbf{f}) d\mathbf{\bar{s}} \end{cases} \quad (40)$$

and further, we introduce the real variables $p(\mathbf{f})$ and $q(\mathbf{f})$ by

$$\begin{cases} p(\mathbf{f}) = \frac{1}{i} \frac{\sqrt{K}}{\sqrt{2}} (y(\mathbf{f}) - y^*(\mathbf{f})), \\ q(\mathbf{f}) = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{K}} (y(\mathbf{f}) + y^*(\mathbf{f})). \end{cases} \quad (41)$$

Then, they have the following commutation relations:

$$\begin{cases} [q(\mathbf{f}), p(\mathbf{f}')] = i(\delta(\mathbf{f} - \mathbf{f}') - \varphi_0(\mathbf{f})\varphi_0(\mathbf{f}')), \\ \text{other commutators vanishing,} \end{cases} \quad (42)$$

and in terms of them H_y in (III) is expressed as

$$H_y = \frac{1}{2} \int \{ p(\mathbf{f})^2 + K^2 q(\mathbf{f})^2 \} d\mathbf{f}$$

$$+ \frac{1}{2K_{00}} [\int \varphi_0(\mathbf{f}) K \sqrt{K} q(\mathbf{f}) d\mathbf{f}]^2. \quad (43)$$

The scattering problem is solved by obtaining the normal modes for this Hamiltonian. The equation of motion for $q(\mathbf{f})$ is

$$q(\mathbf{f}) + K^2 q(\mathbf{f}) = \varphi_0(\mathbf{f}) \int K^2 \varphi_0(\mathbf{f}) q(\mathbf{f}) d\mathbf{f} + \frac{1}{K_{00}} (K \sqrt{K} - \mathbf{K}) \int \varphi_0(\mathbf{f}) K \sqrt{K} q(\mathbf{f}) d\mathbf{f}, \quad (44)$$

where

$$\mathbf{K} = \int K \sqrt{K} \varphi_0(\mathbf{f})^2 d\mathbf{f}.$$

This can be solved easily, and if we seek for the solution of the form

$$\phi(\mathbf{f}) = \delta(\mathbf{f} - \mathbf{p}) - \left\{ \frac{1}{K - E} + i\pi \delta(K - E) \right\} R(\mathbf{f}, \mathbf{p}), \quad (45)$$

this is the solution under the boundary condition that at infinity, besides the incident plane wave with the momentum \mathbf{p} and the energy E , there exist the outgoing spherical waves. The scattering amplitude in (45) is given by

$$R(\mathbf{p}, \mathbf{p}) = \frac{\varphi_0(\mathbf{p})^2}{\int \frac{\varphi_0(\mathbf{f})^2}{K + E} d\mathbf{f} + \int \frac{\varphi_0(\mathbf{f})^2}{K - E} d\mathbf{f} + i\pi \int \delta(K - E) \varphi_0(\mathbf{f})^2 d\mathbf{f}} \quad (46)$$

and this agrees completely with the heretofore obtained conclusion.⁴⁾

§ 6. Discussion and conclusion

From the above considerations, we can see how the strong coupling approximation is carried out in our picture, where the scattering is described by the coming in and out of the \mathfrak{s} -mesons accompanied with the transition of the nucleon state. We shall now compare this strong coupling approximation with the three approximations which were assumed in I:

- (a) The number of the \mathfrak{s} -mesons is restricted at most to one.
- (b) No negative \mathfrak{s} -mesons are considered for the process $\pi^+ + P$.
- (c) The one-level approximation.

The approximation (a) is no more adaptable when the coupling is strong, because in this case, as shown in Sec. 3, we have to take into account the transitions from the lower to higher levels, and we have the possibility that two (or more) \mathfrak{s} -mesons are (virtually) emitted when transitions from the higher to lower levels take place. If we start, however, from the Hamiltonian (II) or (III), where effects of the higher levels have been eliminated, the assumption (a) may not be so bad an approximation. In fact, if we solve the scattering problem under this assumption, with the Hamiltonian (II),

the solution is still represented in the form like (46), where the first term in the denominator

$$\int \frac{\varphi_0(\mathbf{f})^2}{K+E} d\mathbf{f} \quad (47)$$

is replaced by $(K_0+E)^{-1}$, which is not so much different from (47).

The break down of the approximation (b) is quite obvious, since when the coupling is strong, there exist stable proton isobars of higher charge numbers and the charge exchange scattering can take place.

Finally, the one-level approximation (c) does not hold in the strong coupling limit as is obvious from the fact that, besides the lower levels, we have to take account of the higher levels. If we treat the scattering problem with neglect of the higher levels, the conclusion that π -mesons are not scattered remains unchanged, but the scattering amplitude for γ -mesons, in the one γ -meson approximation, for instance, becomes

$$\frac{\varphi_0(\mathbf{f})^2}{\int \frac{\varphi_0(\mathbf{f})^2}{K-E} d\mathbf{f} + i\pi \int \delta(K-E) \varphi_0^2(\mathbf{f}) d\mathbf{f}}, \quad (48)$$

that is, the term (47) drops from the denominator of the exact solution (46).***

In this paper we considered only up to the zeroth order in $1/V$, as the limit of the very strong coupling. If the terms of $O(1/V)$ were included, we could also discuss the effect of the level ($n=0, m=3/2$), which played an essential role for the "resonance scattering" in I, and we then might have the resonance effect for the incident mesons with energies of the order of K_0/V^2 (regarding K_0/V^2 as larger than the meson mass). One of the suggested methods to include this effect is to apply the transformation as in Sec. 4 for the elimination of the term H' and then to treat the transitions between lower levels more accurately. But to carry out through such a treatment, we shall need some refinement of the theory as well as the more detailed knowledges of solutions for the self-field.

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*) In this case, the excited level for the incident particles $\pi^+ + P$ is ($n=1, m=3/2$) and a kind of the one-level approximation does hold. The original one-level in I was the ground level of the doubly charged proton isobar ($n=0, m=3/2$), and the transition matrix element to this level from the proton ground state ($n=0, m=1/2$) is vanishingly small ($O(1/V)$) in our strong coupling limit.

**) After our investigation was completed, we had the chance to read the manuscript of R. Christan and T. D. Lee, titled "The Intermediate Coupling Method for Meson Nucleon Scattering." In their treatment of the inelastic scattering in the strong coupling limit, the role of the higher levels is ignored, and this is the reason that their scattering amplitude does not agree with the correct one but is given by (48).

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Note added in proof: In the following, the high-high matrix elements, $\langle h' | \bar{A} | h'' \rangle$ etc. are disregarded the discussion of Sec. 3

Pion-nucleon Scattering and Nucleon Isobar

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The cross sections for pion-nucleon scatterings are calculated in the symmetrical f_s^s - f_s^s and f_s^s - f_v^v theory, assuming phenomenologically the effective interaction Hamiltonian which permits the virtual transitions of a nucleon to or from its excited state of spin and τ -spin $3/2$, which is described by the Rarita-Schwinger theory for spin $3/2$ particles, accompanied by one pion emission or absorption. For f_s^s - f_s^s theory, the linear f_s^s - f_s^s coupling term is transformed into the sum of the equivalent f_v^v coupling term and the meson pair term, the coupling constant for the latter being chosen effectively small relative to that for the former, taking account of the strong damping effect of nucleon pair formation. The basic equations are solved quite analogously to the well-known treatment of resonance fluorescence, in the approximation of neglecting nucleon recoils. In order to obtain the agreement of the theoretical curves with the experimental ones, the various coupling constants and the characteristic two adjustable parameters G and ΔE are determined quite sharply, G being the effective coupling constant determining the strength of virtual transitions to or from excited states and ΔE being the excitation energy of nucleon isobars. The determined values of the above coupling constants are quite reasonable and the satisfactory agreement of the theoretical curves with the experimental ones is obtained for both types of couplings. It can be said, however, that the f_s^s - f_s^s coupling is rather preferable to f_s^s - f_v^v coupling, thus concluding that the symmetrical f_s^s - f_s^s theory can be considered to be satisfactory if the appropriate accounts are taken of nucleon isobars and the present model for isobars is also satisfactory at least as far as the pion-nucleon scatterings are concerned.

§ 1. Introduction

According to the experimental results on photo-pion production and pion-nucleon scattering, it has been suggested that the nucleon isobars of spin and isotopic spin $3/2$ may be considered to exist as possible virtual states. It has not yet been successful to explain the above nucleon isobars theoretically with the present meson theory. On the other hand, various phenomenological approaches have been made, assuming the virtual transitions of nucleons to their excited states of spin and isotopic spin $3/2$ accompanied with emissions or absorptions of π -mesons. Such investigations for anomalous magnetic moments of nucleons were made by one of us¹⁾ and Hamada,²⁾ for γ - π -production and π -nucleon scattering by Minami et al.,³⁾ whose results are all considered to be satisfactory in the present status of field theory. In all the above investigations, the nucleon isobars are assumed to behave as if they were the elementary particles of spin¹⁾ and isotopic spin $3/2$ ²⁾ and an effective interaction Hamiltonian is introduced permitting the transition of a nucleon to or from its excited state of spin and isotopic spin $3/2$ accompanied with single π -meson emission or absorption. This interaction Hamiltonian contains two parameters,

one of which is the effective coupling constant G determining the strength of the above mentioned virtual transitions and the other of which is the excitation energy ΔE , i.e. the mass difference between the excited state and the ground state of a nucleon.

Starting from the same standpoint, the effects of nucleon isobars upon static nuclear potential in symmetrical pseudoscalar meson theory for both types of couplings were calculated recently by T. Matsumoto, T. Hamada and one of us,⁵⁾ who obtained the results that in general the nucleon isobars induce a very strong central attractive force and a strong tensor force of right sign. However, the quantitative results depend very much upon the above mentioned two parameters G and ΔE .⁶⁾ In order to determine these parameters, the π -nucleon scattering is the most appropriate because the theoretical curves are very sensitive to their values, as is shown below. The foregoing calculations of the π -nucleon scattering cross section are very unsatisfactory especially for p_s - p_s coupling, because the dimensionless p_s - p_s coupling constant $f^2/4\pi\hbar c$ is chosen as 1.5, which is the inevitable choice in their calculations. In p_s - p_s theory, the characteristic meson pair coupling term, which results from non-relativistic approximations to the relativistic linear coupling term and is intimately connected with nucleon pair formation,⁷⁾ gives rise to very large cross sections. Thus we must inevitably choose $f^2/4\pi\hbar c \approx 1.5$. According to Brueckner, Gell-Mann and Goldberger,⁷⁾ the reactive effects of a strongly coupled mesic self-field markedly inhibit the nucleon pair formation so that the pseudoscalar coupling terms which do not involve nucleon pairs are strongly enhanced relative to those involving pair formation. Therefore, it is not permissible to choose $f^2/4\pi\hbar c \approx 1.5$ simply, as was done by Minami et al.³⁾ In their calculations also, no detailed discussions are given about the accessible domains for the values of two adjustable parameters G and ΔE .

In order to avoid the above defects of previous calculations³⁾ and to determine the accessible domains for the values of two parameters G and ΔE and at the same time in order to see whether the good agreement between theory and experiment is obtained, based upon our phenomenological treatment of nucleon isobars, we performed the calculations of π -nucleon scattering cross sections using quite a different method from Minami et al.'s.³⁾ The starting point is quite the same as in the calculations of nuclear potential by one of us and two others,⁵⁾ to which the details are referred. Summarizing them, an effective interaction Hamiltonian mentioned above is first introduced. The p_s - p_s coupling term is transformed into the sum of the equivalent p_s - p_v coupling term and the meson pair term, the coupling constant for the latter being chosen effectively small relative to the one for the former, taking account of the strong damping effect for the meson pair term. Then the perturbation calculations are carried out up to the second order. Here we must take into account the effect of damping for the virtual transitions to excited states. We employ quite the same procedure of Weisskopf in the treatment of resonance fluorescence.⁸⁾ The essential point for the basic equations to be soluble is that there is only one virtual state which can resonate, which is the case for the present calculations if we neglect nucleon recoils. Thus we neglect nucleon recoils and take nonrelativistic approximations. In § 2, we solve basic equations in the above way. In § 3, we calculate the cross sections for both p_s and p_v couplings using the above mentioned interaction Hamiltonians. The comparison

of the theoretical curves with the experimental ones are done and the accessible domains of two adjustable parameters G and ME are determined in § 4. The satisfactory agreement with the experimental curves up to date are obtained if the adjustable parameters and coupling constants are appropriately determined, whose values are quite reasonable from theoretical standpoint. Furthermore the above parameters are specified quite sharply. The discussions and conclusions are given in the final section.

§ 2. The derivation of the scattering cross section

To avoid the difficulty of vanishing denominators of some matrix elements through nucleon isobars, we employ the Weisskopf's damping theory³⁾ for resonance fluorescence, which is generalized in this section so as to involve the intermediate states which do not resonate.

The total Hamiltonian of the system is

$$H = H_0 + H', \quad (1)$$

where H_0 is a Hamiltonian of free nucleon, nucleon isobar and meson fields and H' is an interaction Hamiltonian, which will be given in detail in § 3. A state $|\psi\rangle$, which satisfies the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle, \quad (2)$$

can be expanded as follows:

$$|\psi\rangle = \alpha_m |\phi_m\rangle e^{E_m t / i\hbar}, \quad (3)$$

where $|\phi_m\rangle$ are the eigenstates of free Hamiltonian H_0 and m specifies the number of particles, their momenta and other freedoms. From eqs. (1), (2) and (3), we obtain

$$\dot{\alpha}_n = (i\hbar)^{-1} \sum_m H'_{nm} e^{E_{mn} t / i\hbar}, \quad (4)$$

where $E_{mn} = E_m - E_n$, $H'_{nm} = \langle \phi_n | H' | \phi_m \rangle$.

To solve the basic equations (4), we assume that the initial and the final states involve one meson and one nucleon at rest, the recoils being able to be neglected entirely. Then the only one intermediate state can resonate (see § 3), which is essentially necessary in order to be able to solve eq. (4) along the following line. Neglecting, furthermore, the intermediate states which involve more than three mesons, the equations (4) are reduced as follows:

$$\begin{aligned} \dot{\alpha}_a(t) = & (i\hbar)^{-1} H'_{ac_0} \alpha_{c_0}(t) e^{E_{c_0} t / i\hbar} + (i\hbar)^{-1} \sum_{b_i} H'_{ab_i} \alpha_{b_i}(t) e^{E_{b_i} t / i\hbar} \\ & + (i\hbar)^{-1} \sum_{c_j} H'_{ac_j} \alpha_{c_j}(t) e^{E_{c_j} t / i\hbar}, \end{aligned} \quad (5a)$$

$$\dot{\alpha}_{c_0}(t) = (i\hbar)^{-1} H'_{c_0 a} \alpha_a(t) e^{E_{ac_0} t / i\hbar} + (i\hbar)^{-1} \sum_{b_i} H'_{c_0 b_i} \alpha_{b_i}(t) e^{E_{b_i c_0} t / i\hbar}, \quad (5c_0)$$

$$\dot{a}_{c_j}(t) = (i\hbar)^{-1} H'_{c_j a} a_a(t) e^{E a c_j t / i\hbar} + (i\hbar)^{-1} \sum_{b_i} \int H'_{c_j b_i} a_{b_i}(t) e^{E b_i c_j t / i\hbar}, \quad (5c_j)$$

$$\begin{aligned} \dot{a}_{b_i}(t) = & (i\hbar)^{-1} H'_{b_i c_0} a_{c_0}(t) e^{E c_0 b_i t / i\hbar} + (i\hbar)^{-1} \sum_{c_j} \int H'_{b_i c_j} a_{c_j}(t) e^{E c_j b_i t / i\hbar} \\ & + (i\hbar)^{-1} H'_{b_i a} a_a(t) e^{E a b_i t / i\hbar}, \end{aligned} \quad (5b)$$

where the symbols a , c_0 , c_j and b_i refer to initial, resonating intermediate, other intermediate and final states respectively. The symbol \int means the integrations over momentum variables and \sum_{b_i} and \sum_{c_j} the summations over other freedoms. It is to be remarked that the direct matrix elements between initial and final states appear in eqs. (5), which is necessary in our case (see § 3).

Following the procedure of Weisskopf,⁴⁾ we try to solve the differential equations (5), by putting

$$a_a(t) = e^{-\Gamma t/2}, \quad (6a)$$

$$a_{c_0}(t) = \beta_{c_0} [e^{-\Gamma t/2 + E a c_0 t / i\hbar} - e^{-\gamma t/2}], \quad (6c_0)$$

$$a_{c_j}(t) = \beta_{c_j} [e^{-\Gamma t/2 + E a c_j t / i\hbar} - 1]. \quad (6c_j)$$

Above assumptions satisfy the initial conditions $a_a(0) = 1$ and $a_{c_0}(0) = a_{c_j}(0) = 0$. As Γ is much smaller than γ , which can be confirmed in § 3, we will, in the following, confine our discussions within the time interval $1/\gamma \ll t \ll 1/\Gamma$.

Substituting (6) into (5b), (5c₀) and (5c_j) and integrating over t , we obtain

$$\begin{aligned} a_{b_i}(t) = & H'_{b_i c_0} \beta_{c_0} \left[\frac{e^{-\Gamma t/2 + E a b_i t / i\hbar} - 1}{E a b_i - i\hbar \Gamma/2} - \frac{e^{-\gamma t/2 + E c_0 b_i t / i\hbar} - 1}{E c_0 b_i - i\hbar \gamma/2} \right] \\ & + \sum_{c_j} \int H'_{b_i c_j} \beta_{c_j} \left[\frac{e^{-\Gamma t/2 + E a b_i t / i\hbar} - 1}{E a b_i - i\hbar \Gamma/2} - \frac{e^{E c_j b_i t / i\hbar} - 1}{E c_j b_i} \right] \\ & + H'_{b_i a} \frac{e^{-\Gamma t/2 + E a b_i t / i\hbar} - 1}{E a b_i - i\hbar \Gamma/2}, \end{aligned} \quad (7)$$

$$\begin{aligned} \beta_{c_0} \left[(E a c_0 - i\hbar \Gamma/2) e^{-\Gamma t/2} + \frac{i\hbar \gamma}{2} e^{-\gamma t/2 + E c_0 a t / i\hbar} \right] = & H'_{c_0 a} e^{-\Gamma t/2} \\ & + \sum_{b_i} \int H'_{c_0 b_i} H'_{b_i c_0} \beta_{c_0} \left[\frac{e^{-\Gamma t/2} - e^{E b_i a t / i\hbar}}{E a b_i - i\hbar \Gamma/2} - \frac{e^{-\gamma t/2} - e^{E b_i c_0 t / i\hbar}}{E c_0 b_i - i\hbar \gamma/2} e^{E c_0 a t / i\hbar} \right] \\ & + \sum_{c_j} \sum_{b_i} \int H'_{c_0 b_i} H'_{b_i c_j} \beta_{c_j} \left[\frac{e^{-\Gamma t/2} - e^{E b_i a t / i\hbar}}{E a b_i - i\hbar \Gamma/2} - \frac{1 - e^{E b_i c_j t / i\hbar}}{E c_j b_i} e^{E c_j a t / i\hbar} \right] \\ & + \sum_{b_i} \int H'_{c_0 b_i} H'_{b_i a} \frac{e^{-\Gamma t/2} - e^{E b_i a t / i\hbar}}{E a b_i - i\hbar \Gamma/2} = H'_{c_0 a} e^{-\Gamma t/2} \\ & + \sum_{\mathbf{k}} \overline{H'_{c_0 b_i} H'_{b_i c_0}} \rho(E b_i) |_{E b_i = E_a} (-i\pi) e^{-\Gamma t/2} \beta_{c_0} \end{aligned}$$

$$\begin{aligned}
 & - \sum_i \overline{H'_{c_0 b_i}} \overline{H'_{b_i c_0}} \rho(E_{b_i}) |_{E_{b_i}=E_{c_0}} (-i\pi) e^{-\Gamma t/2 + E_{c_0} a t / i\hbar} \beta_{c_0} \\
 & + \sum_{ij} \overline{H'_{c_0 b_i}} \overline{H'_{b_i c_j}} \beta_{c_j} \rho(E_{b_i}) |_{E_{b_i}=E_a} (-i\pi) e^{-\Gamma t/2} \\
 & + \sum_i \overline{H'_{c_0 b_i}} \overline{H'_{b_i a}} \rho(E_{b_i}) |_{E_{b_i}=E_a} (-i\pi) e^{-\Gamma t/2}, \quad (8)
 \end{aligned}$$

$$\beta_{c_j} [E_{ac_j} - i\hbar/2] e^{-\Gamma t/2} = H'_{c_j a} e^{-\Gamma t/2} + \sum_{b_i} \int H'_{c_j b_i} a_{b_i}(t) e^{E_{b_i} a t / i\hbar}, \quad (9)$$

where we used the following relations :

$$\begin{aligned}
 \sum_{b_i} \int f(b_i) \frac{e^{-\Gamma t/2} - e^{E_{b_i} a t / i\hbar}}{E_{b_i} - i\hbar/2} &= \sum_i \sum_{\text{spin}} \int f(b_i) d\Omega \rho(E_{b_i}) \frac{e^{-\Gamma t/2} - e^{E_{b_i} a t / i\hbar}}{E_{b_i} - i\hbar/2} dE_{b_i} \\
 &= \sum_i \sum_{\text{spin}} \int f(b_i) d\Omega \rho(E_{b_i}) |_{E_a=E_{b_i}} (-i\pi) e^{-\Gamma t/2} \\
 &= \sum_i f(b_i) \rho(E_{b_i}) |_{E_a=E_{b_i}} (-i\pi) e^{-\Gamma t/2},
 \end{aligned}$$

and the terms which contain the non-vanishing denominators are neglected and the symbols

\sum_i and \sum_j mean the summations over various possible final and intermediate states. Equating the coefficients of equal time factors in eq. (8), we obtain

$$\gamma = \frac{2\pi}{\hbar} \sum_i \overline{H'_{c_0 b_i}} \overline{H'_{b_i c_0}} \rho(E_{b_i}) |_{E_{b_i}=E_{c_0}}, \quad (10)$$

$$\beta_{c_0} = \frac{H'_{c_0 a} - i\pi \sum_i \rho(E_{b_i}) \{ \overline{H'_{c_0 b_i}} \overline{H'_{b_i a}} + \sum_j \overline{H'_{c_0 b_i}} \overline{H'_{b_i c_j}} \beta_{c_j} \}}{E_{ac_0} - i\hbar(\Gamma - \gamma_a)/2}, \quad (11)$$

γ being the total transition probability for the spontaneous emission of one meson by the nucleonic transition from an excited state to its ground states and γ_a being obtained from γ by putting $E_{b_i} = E_a$. Concerning the states c_j the perturbation method will be a good approximation, so that from eq. (9), neglecting the second term of the right hand side, we obtain

$$\beta_{c_j} = \frac{H'_{c_j a}}{E_{ac_j} - i\hbar\Gamma/2} = \frac{H'_{c_j a}}{E_{a c_j}}. \quad (12)$$

Finally we substitute eqs. (6) and (7) into eq. (5a) and neglect the terms involving $e^{-\Gamma t/2}$, because we are only interested in the time interval $1/\gamma \ll t \ll 1/\Gamma$, obtaining for Γ the expression

$$\begin{aligned}
 \Gamma = \frac{-2}{i\hbar} H'_{ac_0} \beta_{c_0} + \frac{2\pi}{\hbar} [\sum_i \overline{H'_{ab_i}} \overline{H'_{b_i a}} + \sum_i \overline{H'_{ab_i}} \overline{H'_{b_i c_0}} \beta_{c_0} \\
 + \sum_{ij} \overline{H'_{ab_i}} \overline{H'_{b_i c_j}} \beta_{c_j}] \rho(E_{b_i}) |_{E_{b_i}=E_a}, \quad (13)
 \end{aligned}$$

in which we again neglect the terms containing non-vanishing energy denominators. Thus the solutions are obtained completely.

The total transition probability per unit time w_i is

$$w_i = \frac{1}{t} \sum_{b_i} \int |a_{b_i}(t)|^2. \quad (14)$$

From eq. (7), the amplitude $a_{b_i}(t)$ can effectively, if we neglect the terms of the order of $1/t$ in w_i , be replaced by

$$a_{b_i}(t) = (H'_{b_i c_0} \beta_{c_0} + \sum_{c_j} H'_{b_i c_j} \beta_{c_j} + H'_{b_i a}) \frac{e^{E_{ab_i} t / \hbar} - 1}{E_{ab_i}}, \quad (15)$$

from which we obtain

$$w_i = \int w_i(\theta, \varphi) d\Omega, \quad (16)$$

$$w_i(\theta, \varphi) = \frac{2\pi}{\hbar} |H'_{b_i c_0} \beta_{c_0} + \sum_{c_j} H'_{b_i c_j} \beta_{c_j} + H'_{b_i a}|^2 \rho(E_{b_i}) |_{E_{b_i} = E_a}.$$

The differential and the total scattering cross sections are given by

$$d\sigma/d\Omega = w(\theta, \varphi) V/v, \quad (17)$$

$$\sigma = wV/v, \quad (18)$$

where V is the normalization volume and v is the velocity of the incident meson.

§ 3. Calculations of matrix elements

The interaction Hamiltonian density of our system⁵⁾ in $\not{p}S$ - $\not{p}S$ theory is given by

$$H'(x) = H^{(1)}(x) + H^{(2)}(x) + H^{(3)}(x),$$

$$H^{(1)}(x) = \frac{f^2}{2\pi\hbar c} \bar{\psi} \psi U_\alpha^2 = \frac{f^2}{2\pi\hbar c} (2U^*U + U_3^2) (\bar{\psi}_P \psi_P + \bar{\psi}_N \psi_N),$$

$$H^{(2)}(x) = \frac{f}{2\pi} \bar{\psi} i \gamma_5 \gamma_\nu \tau_\alpha \psi \frac{\partial U_\alpha}{\partial x_\nu} = \frac{\sqrt{2}f}{2\pi} \left[\bar{\psi}_P i \gamma_5 \gamma_\nu \psi_N \frac{\partial U}{\partial x_\nu} + \bar{\psi}_N i \gamma_5 \gamma_\nu \psi_P \frac{\partial U^*}{\partial x_\nu} \right. \\ \left. + \frac{1}{\sqrt{2}} (\bar{\psi}_P i \gamma_5 \gamma_\nu \psi_P - \bar{\psi}_N i \gamma_5 \gamma_\nu \psi_N) \frac{\partial U_3}{\partial x_\nu} \right] \\ \rightarrow \frac{\sqrt{2}f}{2\pi} \cdot \left[\bar{\psi}_P \sigma_i \psi_N \frac{\partial U}{\partial x_i} + \bar{\psi}_N \sigma_i \psi_P \frac{\partial U^*}{\partial x_i} + \frac{1}{\sqrt{2}} (\bar{\psi}_P \sigma_i \psi_P - \bar{\psi}_N \sigma_i \psi_N) \frac{\partial U_3}{\partial x_i} \right],$$

$$H^{(3)}(x) = \frac{G}{\mu} (\bar{\psi}_\nu T_\alpha \psi) \frac{\partial U_\alpha}{\partial x_\nu} + \frac{G}{\mu} (\bar{\psi} T_\alpha^* \psi_\nu) \frac{\partial U_\alpha}{\partial x_\nu} \\ \rightarrow \frac{\sqrt{2}G}{\mu} \left[\left(\bar{\psi}_i^+ \psi_P - \frac{1}{\sqrt{3}} \bar{\psi}_i^+ \psi_N \right) \frac{\partial U}{\partial x_i} + \left(-\frac{1}{\sqrt{3}} \bar{\psi}_i \psi_P + \bar{\psi}_i^- \psi_N \right) \frac{\partial U^*}{\partial x_i} \right. \\ \left. + \left(\frac{\sqrt{2}}{\sqrt{3}} \bar{\psi}_i^+ \psi_P - \frac{\sqrt{2}}{\sqrt{3}} \bar{\psi}_i^0 \psi_N \right) \frac{\partial U_3}{\partial x_i} \right] + \text{Comp. Conj.} \quad (19)$$

First two parts, $H^{(1)}$ and $H^{(2)}$, are the results of Dyson transformation⁶⁾ of the linear $\not{p}S$

coupling term, the former being the meson pair term and the latter the equivalent $p\bar{\nu}$ coupling term. The symbol \rightarrow means the undergoing to the approximations of nucleons at rest, i takes only 1, 2 and 3. ψ_p and ψ_n are the proton and neutron field operators respectively and Ψ_i represents nucleon isobars, the suffixes $++$, $+$, 0 and $-$ denoting the charge number of each state and λ and μ are the Compton wave length of a nucleon and a meson respectively. The meson field operators are expanded as usual;

$$\left. \begin{aligned} U(x) &= \frac{1}{\sqrt{V}} \sum_k \sqrt{\frac{\hbar}{2\omega_k c}} (a_k + b_{-k}^*) e^{ikx}, \\ U_3(x) &= \frac{1}{\sqrt{V}} \sum_k \sqrt{\frac{\hbar}{2\omega_k c}} (a_k^0 + a_{-k}^{0*}) e^{ikx}, \end{aligned} \right\} \quad (20)$$

$$\omega_k = \sqrt{\mu^2 + k^2}.$$

Four independent positive energy solutions of Rarita-Schwinger equation at rest are

$$\left. \begin{aligned} V(1); \{\Psi_\mu^{(1)}\} &= \{e_1 \psi_+, 0\}, \quad V(2); \{\Psi_\mu^{(2)}\} = \{e_2 \psi_-, 0\} \\ V(3); \{\Psi_\mu^{(3)}\} &= \left\{ \frac{1}{\sqrt{3}} e_2 \psi_+ + \sqrt{\frac{2}{3}} e_3 \psi_-, 0 \right\}, \\ V(4); \{\Psi_\mu^{(4)}\} &= \left\{ \frac{1}{\sqrt{3}} e_1 \psi_- - \sqrt{\frac{2}{3}} e_3 \psi_+, 0 \right\}, \end{aligned} \right\} \quad (21)$$

$$e_1 = \left\{ \frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}, 0 \right\}, \quad e_2 = \left\{ \frac{1}{\sqrt{2}}, -\frac{i}{\sqrt{2}}, 0 \right\}, \quad e_3 = \{0, 0, 1\},$$

where ψ_+ and ψ_- are given, in the Pauli approximation now adopted, $\psi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\psi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

a) $\pi^+ + p \rightarrow \pi^+ + p$ scattering

The possible second order diagrams are shown in Fig. 1.

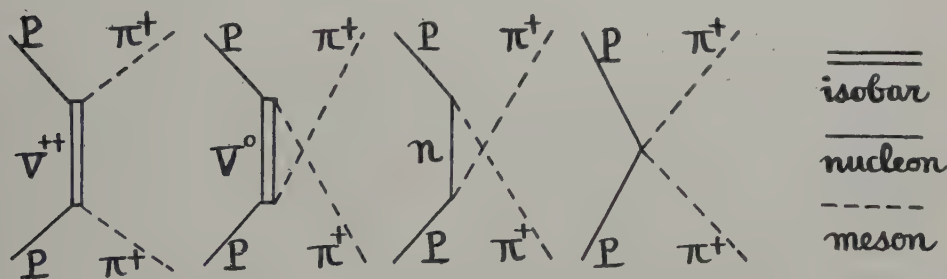


Fig. 1. The second order graphs for $\pi^+ + p \rightarrow \pi^+ + p$ scattering.

It is easy to see that the only resonating intermediate state is $V^{++}(4)$ or $V^{++}(3)$, if the spin of initial proton is parallel or antiparallel to the direction of the wave vector k_0 of the incident meson, that was necessary for the basic equations to be soluble, which we assume in the following. Then the differential cross section is given by eq. (16);

$$d\sigma(\theta, \varphi) = -\frac{2\pi}{\hbar} \sum_{\text{spin}} |H'_{b_i c_0} \beta_{c_0} + \sum_j H'_{b_i c_j} \beta_{c_j} + H'_{b_i a}|^2 \rho(E_{b_i}) \Big|_{E_{b_i} = E_a} \frac{V}{\varphi} d\Omega,$$

$$\beta_{c_0} = \frac{H'_{c_0 a} - i\pi \sum_i (\overline{H'_{c_0 b_i}} H'_{b_i a} + \sum_j \overline{H'_{c_0 b_j}} H'_{b_j c_j} \beta_{c_j}) \rho(E_{b_i})|_{E_{b_i}=E_a}}{E_{ac_0} - i\hbar(\Gamma - \gamma_a)/2},$$

$$\beta_{c_j} = \frac{H'_{c_j a}}{E_{ac_j}}, \quad \gamma_a = \sum_i \frac{2\pi}{\hbar} (\overline{H'_{c_0 b_i}} H'_{b_i c_0} \rho(E_{b_i}))|_{E_{b_i}=E_a}.$$

First of all we evaluate γ_a , which is equal for both isobar states $I'^{++}(4)$ and $I'^{++}(3)$ or for both spin directions of an initial proton and is given by

$$\gamma_a = (4/3) (G^2/4\pi\hbar c) (k/\mu)^3 \mu c, \quad (22)$$

where $k = |\mathbf{k}| = |\mathbf{k}_0| = k_0$ and \mathbf{k} is the wave vector of the scattered meson. The first term of I' is, for $E_a = E$, given by approximately from eq. (13),

$$-\left(\frac{2}{i\hbar}\right) H'_{ac_0} \beta_{c_0} \approx -\frac{H'_{ac_0} H'_{c_0 a}}{(\hbar^2/4)\gamma_a} = 8\pi \left(\frac{\mu}{k}\right) \left(\frac{\mu}{\omega}\right) \frac{\mu c}{V\mu^3},$$

which is very small compared with γ_a , as the normalization volume $V \gg 1/\mu^3$. Thus it is confirmed that $\gamma \gg \Gamma$, which was assumed in the previous section, and we can neglect Γ compared with γ_a in the denominator of β_{c_0} . The second term in the numerator of β_{c_0} can be estimated to be of the order of one-tenth of the first term, and to have a phase factor differing by just $\pi/2$ from that of the first term. Therefore we can neglect this term, because this neglect induces the error of only about one percent in the differential cross section as the term containing β_{c_0} gives the main contributions to $d\sigma(\theta, \varphi)$. The values of the typical matrix elements are

$$\left. \begin{aligned} H'_{bc_0} H'_{c_0 a} &= \left(\frac{G}{\mu}\right)^2 \frac{\hbar c}{V\omega} \left[(\mathbf{k} \cdot \mathbf{k}_0) - \frac{1}{3} (\boldsymbol{\sigma} \cdot \mathbf{k}) (\boldsymbol{\sigma} \cdot \mathbf{k}_0) \right]_{ba}, \\ H'_{bc_1} H'_{c_1 a} &= \frac{1}{3} \left(\frac{G}{\mu}\right)^2 \frac{\hbar c}{V\omega} \left[(\mathbf{k} \cdot \mathbf{k}_0) - \frac{1}{3} (\boldsymbol{\sigma} \cdot \mathbf{k}_0) (\boldsymbol{\sigma} \cdot \mathbf{k}) \right]_{ba}, \\ H'_{bc_2} H'_{c_2 a} &= \left(\frac{f}{2\pi}\right)^2 \frac{\hbar c}{V\omega} [(\boldsymbol{\sigma} \cdot \mathbf{k}_0) (\boldsymbol{\sigma} \cdot \mathbf{k})]_{ba}, \end{aligned} \right\} \quad (23)$$

where use was made of some techniques in the evaluations of matrix elements containing nucleon isobar states, which are given in detail in another paper.⁵⁾ The final state density is given by

$$\rho(E_{b_i})|_{E_{b_i}=E_a} = V\omega k / (2\pi)^3 \hbar c. \quad (24)$$

Thus the differential cross section is, summing up over the final directions of spin and averaging over the initial ones,

$$\begin{aligned} \frac{d\sigma_+}{d\Omega} &= \frac{1}{\mu^2} \left[\left(\frac{k}{\mu}\right)^4 u_G^2 \frac{4 + 12 \cos^2 \theta}{9[(\Delta\epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} + 4\alpha_f^2 \left(\frac{\mu}{2\pi}\right)^2 \right. \\ &\quad + u_G^2 \left(\frac{k}{\mu}\right)^4 \frac{(4 + 12 \cos^2 \theta)}{81(\Delta\epsilon + \epsilon)^2} + 4 \frac{\alpha_f^2}{\epsilon^2} \left(\frac{k}{\mu}\right)^4 \left(\frac{\mu}{2\pi}\right)^4 \\ &\quad \left. + \frac{\alpha_f u_G}{9\epsilon(\epsilon + \Delta\epsilon)} \left(\frac{k}{\mu}\right)^4 \left(\frac{\mu}{2\pi}\right)^2 (-8 + 24 \cos^2 \theta) \right] \end{aligned}$$

$$\begin{aligned}
 & + a_G^2 \left(\frac{k}{\mu} \right)^4 \frac{(-8 + 40 \cos^2 \theta) (\Delta \epsilon - \epsilon)}{27 (\Delta \epsilon + \epsilon) [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} \\
 & + a_f a_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2\epsilon} \right)^2 \frac{(8 + 8 \cos^2 \theta) (\Delta \epsilon - \epsilon)}{3 \epsilon [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} \\
 & - a_f a_G \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2\epsilon} \right) - \frac{16 \cos \theta}{9 (\Delta \epsilon + \epsilon)} - a_f^2 \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2\epsilon} \right)^3 \frac{8 \cos \theta}{\epsilon} \\
 & - a_f a_G \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2\epsilon} \right) \frac{16 \cos \theta (\Delta \epsilon - \epsilon)}{3 [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} \Big], \quad (25)
 \end{aligned}$$

where $a_f = f^2/4\pi\hbar c$, $a_G = G^2/4\pi\hbar c$, and $\Delta E = \Delta \epsilon mc^2$ is the isobar excitation energy, $E_\pi = \epsilon mc^2$ is the energy of the incident meson and thus $E_{c_0 a} = (\Delta \epsilon - \epsilon) mc^2$ and $E_{c_1 a} = (\Delta \epsilon + \epsilon) mc^2$, $E_{c_2 a} = \epsilon mc^2$.

The total cross section is, after the integration over the direction of final meson,

$$\begin{aligned}
 \sigma_t = & \frac{4\pi}{\mu^2} \left[a_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8}{9 [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} + 4a_f^2 \left(\frac{\mu}{2\epsilon} \right)^2 + a_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8}{81 (\Delta \epsilon + \epsilon)^2} \right. \\
 & + a_f^2 \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2\epsilon} \right)^4 \frac{4}{\epsilon^2} + a_G^2 \left(\frac{k}{\mu} \right)^4 \frac{16 (\Delta \epsilon - \epsilon)}{81 (\Delta \epsilon + \epsilon) [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} \\
 & \left. + a_f a_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2\epsilon} \right)^2 \frac{32 (\Delta \epsilon - \epsilon)}{9 \epsilon [(\Delta \epsilon - \epsilon)^2 + (\gamma_a/2\mu c)^2]} \right]. \quad (26)
 \end{aligned}$$

b) $\pi^- + p \rightarrow \pi^- + p$ and $\pi^- + p \rightarrow \pi^0 + n$ scatterings

The only possible resonating intermediate state is $V^0(4)$ or $V^0(3)$, according to whether the spin direction of an initial proton is parallel or antiparallel to the direction of an incident meson. The possible final states are (π^-, p) and (π^0, n) . The possible second order diagrams are given in Fig. 2.

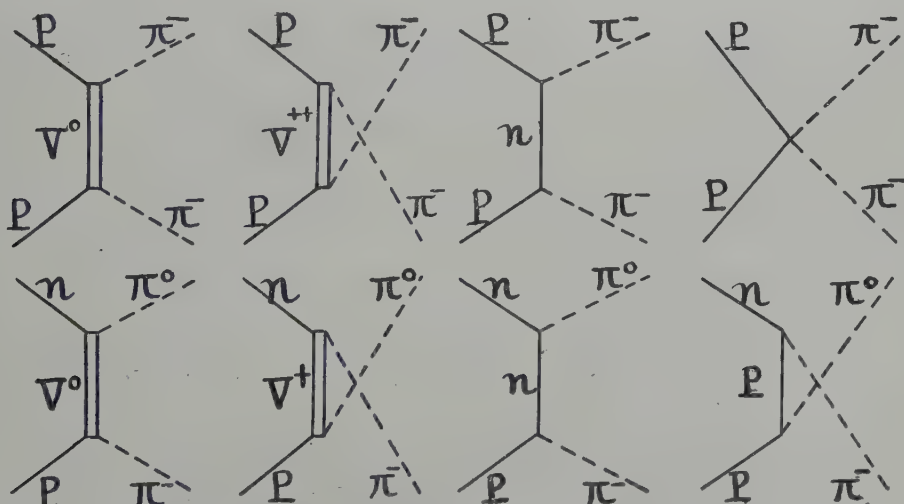


Fig. 2. The second order diagrams for the scattering $\pi^- + p \rightarrow \pi^- + p$ and $\pi^- + p \rightarrow \pi^0 + n$.

The calculations of matrix elements are almost the same as the previous ones, and only the results are given below. The decay constant γ_a in this case is the same as in the previous one. Then the differential cross sections and the total cross sections are

$$\begin{aligned} \frac{d\sigma_-}{d\Omega} = & \frac{1}{\mu^2} \left[u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{4(1+3\cos^2\theta)}{81[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} + 4u_f^2 \left(\frac{\mu}{2x} \right)^2 \right. \\ & + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{4(1+3\cos^2\theta)}{a(\Delta\epsilon+\epsilon)^2} - u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{8(1+\cos^2\theta)}{3\epsilon(\Delta\epsilon+\epsilon)} \\ & + u_f^2 \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^4 \frac{4}{\epsilon^2} + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8(-1+5\cos^2\theta)(\Delta\epsilon-\epsilon)}{27(\Delta\epsilon+\epsilon)[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \\ & - u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{8(-1+3\cos^2\theta)(\Delta\epsilon-\epsilon)}{9\epsilon[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \\ & - u_G u_f \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2x} \right) \frac{16\cos\theta}{3(\Delta\epsilon+\epsilon)} + u_f^2 \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2x} \right)^3 \frac{8\cos\theta}{\epsilon} \\ & \left. - u_G u_f \left(\frac{k}{\mu} \right)^2 \left(\frac{\mu}{2x} \right) \frac{16\cos\theta(\Delta\epsilon-\epsilon)}{9[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \right], \quad (27) \end{aligned}$$

$$\begin{aligned} \frac{d\sigma_0}{d\Omega} = & \frac{1}{\mu^2} \left[u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8(1+3\cos^2\theta)}{81[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8(1+3\cos^2\theta)}{81(\Delta\epsilon+\epsilon)^2} \right. \\ & - u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{32\cos^2\theta}{9\epsilon(\Delta\epsilon+\epsilon)} + u_f^2 \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^4 \frac{8\cos^2\theta}{\epsilon^2} \\ & - u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{16(-1+5\cos^2\theta)(\Delta\epsilon-\epsilon)}{81(\Delta\epsilon+\epsilon)[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \\ & \left. + u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{32\cos^2\theta(\Delta\epsilon-\epsilon)}{9\epsilon[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \right], \quad (28) \end{aligned}$$

$$\begin{aligned} \sigma_- = & \frac{4\pi}{\mu^2} \left[u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8}{81[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{8}{9(\Delta\epsilon+\epsilon)^2} \right. \\ & + u_f^2 \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^4 \frac{4}{\epsilon^2} - u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{32}{9\epsilon(\Delta\epsilon+\epsilon)} \\ & \left. + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{16(\Delta\epsilon-\epsilon)}{81(\Delta\epsilon+\epsilon)[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} + 4u_f^2 \left(\frac{\mu}{2x} \right)^2 \right], \quad (29) \end{aligned}$$

$$\begin{aligned} \sigma_0 = & \frac{4\pi}{\mu^2} \left[u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{16}{81[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} + u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{16}{81(\Delta\epsilon+\epsilon)^2} \right. \\ & + u_f^2 \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^4 \frac{8}{3\epsilon^2} - u_G^2 \left(\frac{k}{\mu} \right)^4 \frac{32(\Delta\epsilon-\epsilon)}{243(\Delta\epsilon+\epsilon)[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \\ & - u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{32}{27\epsilon(\Delta\epsilon+\epsilon)} \\ & \left. + u_f u_G \left(\frac{k}{\mu} \right)^4 \left(\frac{\mu}{2x} \right)^2 \frac{32(\Delta\epsilon-\epsilon)}{27\epsilon[(\Delta\epsilon-\epsilon)^2 + (\gamma_a/2\mu c)^2]} \right]. \quad (30) \end{aligned}$$

Thus far we are only concerned with $p\bar{s}$ - $p\bar{s}$ theory. In the $p\bar{s}$ - $p\bar{v}$ theory the only differences are the absence of the meson pair term $H^{(1)}$ in the interaction Hamiltonian and that the coupling constant $f/2\kappa$ is replaced by g/μ in the equivalent $p\bar{s}$ - $p\bar{v}$ coupling term $H^{(6)}$. The cross sections for this case, therefore, are readily obtained by the dropping of some terms corresponding to the meson pair interaction and the replacement of $f/2\kappa$ by g/μ in the previous expressions. They are not given here in detail.

§ 4. Comparison with experimental data

In this section the theoretical results are compared with the experimental ones, which are denoted by rectangles in the figures 3, 4, 5, 6 and 7. The data denoted by *a* refer to the measurements by Anderson, Fermi, Long and Nagle,¹⁰⁾ those denoted by *b* by Anderson, Fermi, Nagle and Yodh¹¹⁾ and the lowest energy data denoted by *c* by Angell and Perry.¹²⁾

First of all we discuss the results for $p.s.$ coupling. The essential difference between the $p.s.$ and the $p.v.$ cases is the absence of the meson pair term in the $p.v.$ case. The

contributions of the meson pair term to the cross sections σ_+ and σ_- are the terms $16\pi u_f^2(\mu/2\kappa)^2/\mu^2$ in eqs. (26) and (29), which represent an isotropic scattering cross section independent upon the incident pion energy. This term is too large ($\approx 5.98 u_f^2 mb \approx 600 mb$) to fit the experimental data if the $p.s.$ coupling constant is chosen as usual ($u_f = f^2/4\pi\hbar c \approx 10$). Thus we must take the accounts of the strong damping of the $p.s.$ coupling constant for the meson pair term.⁷⁾

If the above damping effect is neglected entirely, the permissible maximum value for u_f is 1.85. The theoretical curves for this case are represented in Figs. 3 and 4 for the σ_+ and the $\sigma_- + \sigma_0$ cases, where the abscissas represent the kinetic energy of the incident meson in *Mev*. The rough agreement with the experimental ones is obtained except the low energy data. However, the above extreme small value for u_f

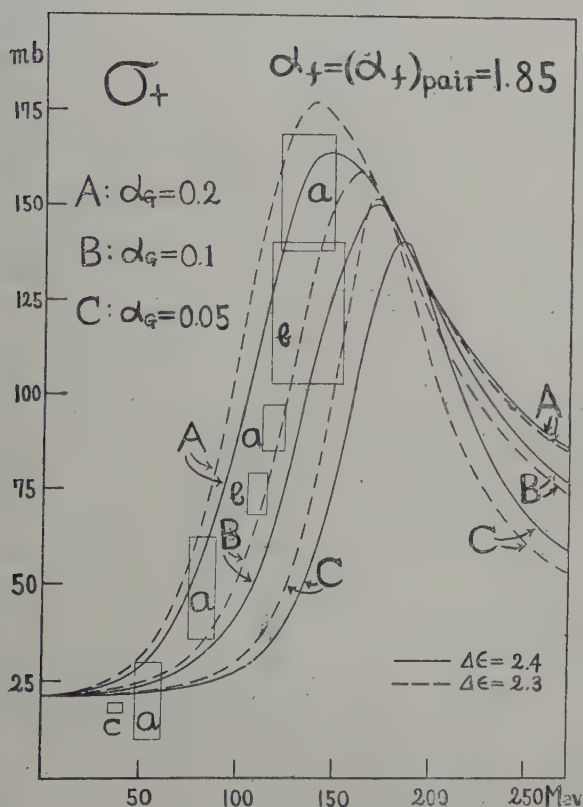


Fig. 3. The total cross section σ_+ for $u_f=1.85$ and without the damping correction for the meson pair term.

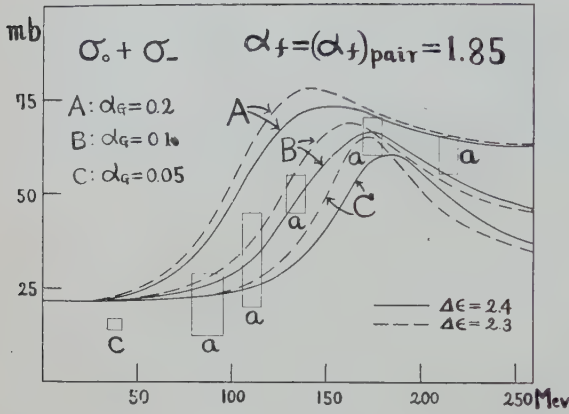


Fig. 4. The total cross section $\sigma_- + \sigma_0$ for $\alpha_f = 1.85$ and without the damping correction for the meson pair term.

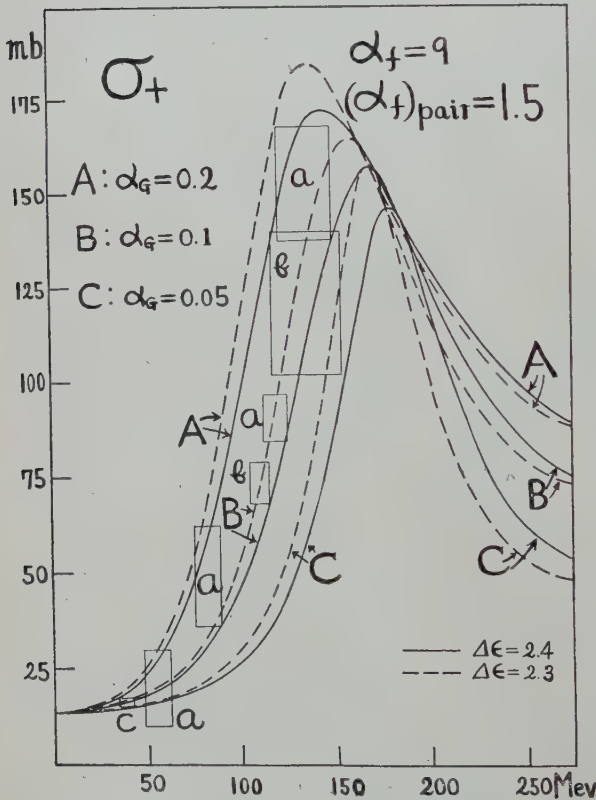


Fig. 5. The total cross section σ_+ for $\alpha_f = 9$ and $(\alpha_f)_{\text{pair}} = 1.5$.

(≈ 1.85) can hardly be appropriate. This is, as mentioned in the introduction, the most unsatisfactory point in the previous calculations.³⁾

If, on the other hand, we take into account of the effect of the damping for the meson pair term, we can assign the usual value to α_f , while $(\alpha_f)_{\text{pair}}$ can be decreased by about a factor of ten.⁷⁾ The best attainable agreement of our theoretical curves with the experimental ones can be obtained if we choose $\alpha_f = 9$ and $(\alpha_f)_{\text{pair}} = 1.5$, the damping factor for the meson pair term being 6. The curves in this case are given in Figs. 5 and 6 for the two types of scatterings. Thus we can conclude that the $p\bar{s}$ - $p\bar{s}$ meson theory can well explain the pion-nucleon scattering data, if the appropriate accounts are taken of the damping of the characteristic meson pair term. The theoretical discussions about the quantitative value for the above damping factor are given in the next section.

If we take the above damping factor ∞ , namely $(\alpha_f)_{\text{pair}} = 0$, we can make α_f as large as 12.5, which corresponds to the choice $\alpha_g = 0.07$ in the $p.v.$ coupling case, and the curves for this case are represented in Fig. 7. From the figure we can readily see that the departure from the experimental data in the lower energy region becomes very remarkable, thus obtaining another important conclusion that the effect of the small damped meson pair term can never be neglected especially in the lower energy region and therefore

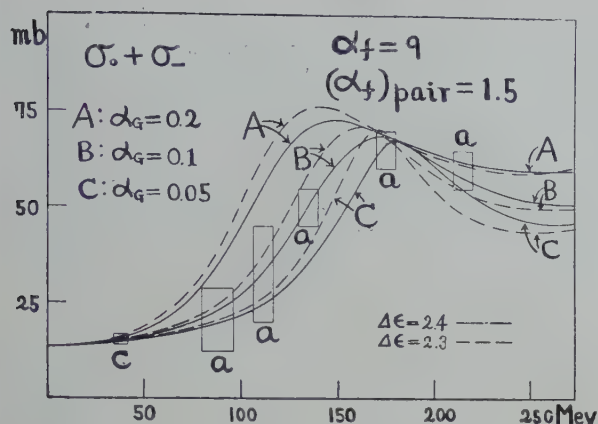


Fig. 6. The total cross section $\sigma_+ + \sigma_-$ for $\alpha_f = 9$ and $(\alpha_f)_{\text{pair}} = 1.5$.

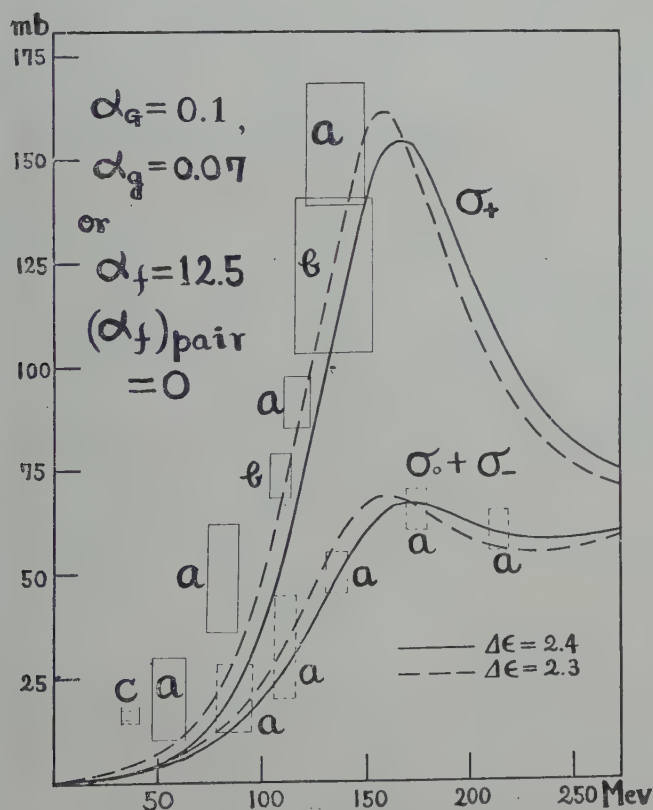


Fig. 7. The total cross sections σ_+ and $\sigma_+ + \sigma_-$ for $\alpha_G = 0.07$ (or $\alpha_f = 12.5$ and $(\alpha_f)_{\text{pair}} = 0$).

the $\bar{p}s$ - $\bar{p}s$ theory must be regarded as preferable to $\bar{p}s$ $\bar{p}v$ theory for pion-nucleon scattering data.

As to the values of the parameters $\alpha_G = G^2/4\pi\hbar c$, the effective coupling constant determining the strength of virtual transitions to or from excited states, and $\Delta E = \Delta\epsilon mc^2$, the excitation energy to excited states, they are determined sharply in spite of the very rough data of the experiments up to date, quite irrespective of the values of the other coupling constants and the types of couplings and scatterings. Namely, the most adequate values for α_G and $\Delta\epsilon$ turn out to be the same value for both types of couplings and scatterings and for the various values of other parameters. This is due to the fact that the height and the width of the resonance peak are determined only by these two parameters. From the above figures, it is concluded that we must choose as $\Delta\epsilon = 2.3 \sim 2.4$ in order to obtain the most suitable position of the resonance peak and as $\alpha_G = G^2/4\pi\hbar c = 0.1 \pm 0.05$ in order to obtain the most suitable width of the resonance peak. It must be remarked again that the above two values are determined quite sharply. Concerning the values of $\Delta\epsilon$, the further theoretical discussions

are given in the following section. The rather large value of $u_G = G^2/4\pi\hbar c$ compared with $u_G = g^2/4\pi\hbar c$ or $(\mu/2\kappa)^2 u_f = (\mu/2\kappa)^2 (f^2/4\pi\hbar c)$ means that the nucleon isobars could produce a very large effect to, for example, the static nucleon-nucleon interaction potential,⁽⁵⁾ which will be investigated in detail in another paper,⁽⁶⁾ if the above model for isobars is really a good approximation to the true situations.

As to the permissible domain of the ps - $p\bar{v}$ coupling constant $u_g = g^2/4\pi\hbar c$, we will not here give the corresponding figures and we will mention only the results. The values $0.07 \gtrsim u_g \gtrsim 0.06$ reproduce almost the same curves as those shown in Fig. 7, which corresponds to $u_g = 0.07$. The values outside the above domain produce the marked departures from the data.

Finally we will discuss the angular distribution of the scattered mesons. In the present investigations, we neglect the motions of nucleons entirely. Therefore we must recognize

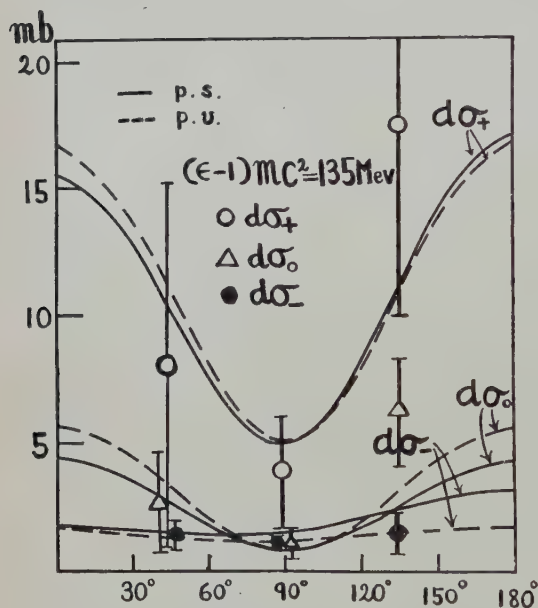


Fig. 8. The differential scattering cross sections $d\sigma_+/d\Omega$, $d\sigma_-/d\Omega$, $d\sigma_0/d\Omega$ for 135 Mev incident pion kinetic energy.

(25) and (27). The differential cross sections for $p.v.$ coupling and $d\sigma_0$ for both couplings are symmetrical with respect to the forward and the backward directions. Though the present data are very rough, they seem to show a rather predominance of the backward scatterings, thus obtaining a conclusion, that the $p.s.$ coupling seems rather preferable to the $p.v.$ coupling also with respect to the angular distributions. The quantitative differences between them, however, are very small owing to the strong damping of the meson pair term in the present calculations.

a rather large errors about the angular distributions. The curves shown in Fig. 8 are those of the differential scattering cross sections given by eqs. (25) for $d\sigma_+$ and (27) for $d\sigma_-$ and (28) for $d\sigma_0$ for the kinetic energy 135 Mev of the incident meson. The parameters being the ones which attain the best agreements concerning the cross sections σ_+ and $\sigma_- + \sigma_0$; $u_G = 0.1$, $\Delta\epsilon = 2.4$, $u_f = 9$, $(u_f)_{\text{pair}} = 1.5$ for $p.s.$ coupling and $u_g = 0.07$ for $p.v.$ coupling. The experimental data denoted by various limited vertical lines are those taken from reference 11). Nothing definite can be said at present from the above figure. The small predominance of the backward scattering of the theoretical curves for the $p.s.$ coupling compared with $p.v.$ coupling is due to the cross terms between the meson pair term and the other terms, which are proportional to $\cos\theta$ in eqs.

§ 5. Discussions and conclusions

First of all, we discuss about the quantitative value of the damping factor for the meson pair term. According to Brueckner, Gell-mann and Goldberger,⁷⁾ the *p.s.* coupling constant $f^2/4\pi\hbar c$ for the characteristic meson pair term is strongly damped owing to the presence of a strongly coupled mesonic self-field; and the considerations of a simple subset of radiative corrections to the nucleon propagation function show that the effective coupling constant for the meson pair term must be replaced by

$$(f^2/4\hbar c) / \{1 + (3/4\pi)(f^2/4\pi\hbar c)\}, \quad (31)$$

while the radiative correction to the equivalent *p.v.* coupling term is much smaller and is negligible. The most appropriate value for the $(a_f)_{\text{pair}} = (f^2/4\pi\hbar c)_{\text{pair}}$ is found to be 1.5 in our calculations, to which corresponds the value $a_f = 2.4$ according to eq. (31), which is too small to be acceptable. Our choice of $a_f = 9$ and $(a_f)_{\text{pair}} = 1.5$, which attain the best agreement with the experimental data, means a damping factor for the pair term is about 6, while this factor becomes about 3 for $a_f = 9$ according to (31). Thus the quantitative value for the damping factor is not coincide perfectly with the theoretical predictions.⁷⁾ Especially the extreme small value of $(a_f)_{\text{pair}} = 1.5$ can never be attained by any reasonable value of *p.s.* coupling constant a_f , if the damping correction is correctly given by eq. (31). Our strong damping correction factor (≈ 6) is, however, rather small compared with the theoretical predictions of Wentzel,¹³⁾ who obtained as the above damping correction factor the following expression

$$\approx 1 + (f^2/4\pi\hbar c), \quad (32)$$

whose value is 10 for our case ($a_f = 9$). It is clear, therefore, that our correction factor 6 lies just in the intermediate in the above two theoretical predictions 3 and 10. Thus we can say that our determined values $a_f = 9$ and $(a_f)_{\text{pair}} = 1.5$ may be reasonable theoretically also from quantitative considerations.

Concerning the theoretical adequacy of the two parameters $a_G = G^2/4\pi\hbar c = 0.1 \pm 0.05$ and $\Delta\epsilon = 2.1 \sim 2.4$, nothing definite can be said at present. It is, however, certain that these values are never quite unreasonable from theoretical considerations.

Finally we discuss the excitation energy $\Delta E = \Delta\epsilon mc^2 = 2.3 \sim 2.4$. As we neglect the nucleon motions entirely in our calculations, we can not distinguish between the *c.m.* system and the laboratory system. The total energy E' of the incident meson in the *c.m.* system is obtained from that in the laboratory system E by the relation

$$E' = \{EMc^2 + (mc^2)^2\} / \{(Mc^2)^2 + 2Mc^2E + (mc^2)^2\}. \quad (33)$$

Using above relation, the excitation energy in *c.m.* system is different from that in the laboratory system by about 20 percent at about the resonance peak ($\epsilon \approx 2$). As the above difference between them is rather large, we cannot, therefore, say nothing quantitative about the excitation energy ΔE from our calculations.

Finally we will summarize the main conclusions as follows:

- (i) The *ps-ps* meson theory can well explain the pion-nucleon scattering data, if the

appropriate accounts are taken of the damping of the characteristic meson pair term, the most suitable choice being 6 for the damping factor.

(ii) The effect of the small damped meson pair term can never be neglected especially in the lower energy region and therefore the ps - ps theory must be regarded as preferable to ps - pv theory for pion-nucleon scatterings.

(iii) As to the two parameters $G^2/4\pi\hbar c$ and ΔE , both of them can be determined very sharply as $G^2/4\pi\hbar c = 0.1 \pm 0.05$ and $E = 2.3 \sim 2.4 \text{ } mc^2$ quite irrespective of the types of couplings and the values of other parameters.

(iv) The rather large value of $G^2/4\pi\hbar c$ indicates that the presence of nucleon isobars may induce large effects in various other phenomena, if the model in this paper is a good approximation to the true situations.

(v) As we neglect the nucleon motion entirely, we cannot distinguish the *c.m.* system from the laboratory system and so we can not say nothing definite about the angular distribution of π -mesons and the quantitative value for the excitation energy ΔE .

(vi) The values of all the parameters determined in order to get the best agreement with experimental data are quite reasonable from the theoretical standpoint.

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Recoil Effects in the Strong Coupling Theory

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§ 1. Introduction

Recent experiments on photomeson production have been explained phenomenologically by assuming the existence of isobar states of nucleons.^{1, 2)} As is well known, isobar states can be introduced field-theoretically using the so-called strong coupling theory.^{3, 4)} However, a serious difficulty in this theory is that the recoil effect of a nucleon has been completely neglected.

On the other hand the results of experiments by MIT group⁵⁾ and also CIT group,⁶⁾ in which recoil momenta of protons were measured to determine momenta of neutral pimesons, are explained very nicely with the concept of isobar states.⁷⁾ Therefore it seems appropriate to investigate the recoil effect in the strong coupling theory.

Although several authors have already touched on this problem in different ways,^{8, 9)} they have always abandoned the recoil term in their final estimations assuming it to be very small. Recently Matthews and Salam¹⁰⁾ have extended Tomonaga's intermediate coupling theory taking into account the recoil effect. They assumed the neutral pseudoscalar field since this type of field can be solved exactly, neglecting the recoil effect. However the Hartree approximation may not be suitable in this case where angular correlations may be important, for it consists of only *S*-wave functions.

In order to avoid this difficulty we assume a charged scalar field, which is sufficient to introduce isobar states in the charge space. Fortunately Tomonaga has given a lucid explanation of the relationship between Wentzel's strong coupling theory and his own intermediate coupling theory, both of which neglect recoil effects. We shall compare our results with Tomonaga's.

§ 2. Formulation

The total hamiltonian is given in the momentum space representation as follows :

$$H = H_0 + H_1,$$

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$$\begin{aligned}
 H_0 = & \int \sqrt{k^2 + \kappa^2} \{a^*(\mathbf{k}) a(\mathbf{k}) + b^*(\mathbf{k}) b(\mathbf{k})\} d\mathbf{k} \\
 & + \int \sqrt{p^2 + \mu^2} \sum \{A_r^*(\mathbf{p}) A_r(\mathbf{p}) + B_r^*(\mathbf{p}) B_r(\mathbf{p})\} d\mathbf{p}, \\
 H_1 = & f' \int d\mathbf{p} d\mathbf{p}' d\mathbf{k} \sum \delta(\text{momentum}) \{A_r^*(\mathbf{p}') + B_r^*(\mathbf{p}')\} \\
 & \times \{A_s(\mathbf{p}) + B_s(\mathbf{p})\} u_r^*(\mathbf{p}') \beta u_s(\mathbf{p}) [\{a(\mathbf{k}) - b^*(\mathbf{k})\} \tau_+ \\
 & + \{a^*(\mathbf{k}) - b(\mathbf{k})\} \tau_-] \frac{k}{\sqrt{k^2 + \kappa^2}}, \quad (1)
 \end{aligned}$$

where we use a natural unit system $\hbar=1$, $c=1$. All notations are listed as follows:

f'	... coupling constant	$A_r^*(\mathbf{p})$... creation operator of a nucleon which has momentum \mathbf{p} and spin r
κ	... meson rest mass		
μ	... nucleon rest mass	$B_r^*(\mathbf{p})$... creation operator of an anti-nucleon which has momentum \mathbf{p} and spin r
\mathbf{k}	... meson momentum		
\mathbf{p}	... nucleon momentum		
$a^*(\mathbf{k})$... creation operator of a positive meson which has momentum \mathbf{k}	τ_+, τ_-	... charge exchange operators
$b^*(\mathbf{k})$... creation operator of a negative meson which has momentum \mathbf{k}	$u_r(\mathbf{p})$... spinor amplitude of a nucleon which has momentum \mathbf{p} and spin r

There are no nucleon pairs assumed in the following discussion even though Matthews and Salam have pointed out that they play important roles in the pseudoscalar theory. Furthermore, it is assumed that a total charge is a positive one and a nucleon is in a ground state, although it is not too difficult to solve it numerically taking into account excited states, that is, isobar states.

Let us assume that all mesons are in S -state and use the Hartree approximation for Schrödinger functions in Fock's space:

$$\left. \begin{aligned}
 \varphi_{2n} &= c_{n,n} \prod_{\lambda=1}^n f_+(k_\lambda^+) \prod_{\lambda=1}^n f_-(k_\lambda^-), \\
 \psi_{2n+1} &= c_{n+1,n} \prod_{\lambda=1}^{n+1} f_+(k_\lambda^+) \prod_{\lambda=1}^n f_-(k_\lambda^-),
 \end{aligned} \right\} \quad (2)$$

where φ_{2n} represents a proton state with n positive mesons and n negative mesons and ψ_{2n+1} represents a neutron state with $n+1$ positive mesons and n negative mesons. In this approximation we drop a nucleon part of the wave function assuming that the nucleon momentum is determined by momentum conservation for a given set of k_λ^+ and k_λ^- as in Matthews and Salam's paper. Since f_+ and f_- are normalized respectively, i.e.,

$$\int f_+^2(k) d\mathbf{k} = \int f_-^2(k) d\mathbf{k} = 1, \quad (3)$$

we have a condition for coefficients $c_{n,n}$ as follows:

$$\sum_{n=0}^{\infty} (c_{n,n}^2 + c_{n+1,n}^2) = 1, \quad (4)$$

Using these functions we calculate an expectation value of the energy of the total system

$$\bar{H} = \bar{H}_0 + \bar{H}_1. \quad (5)$$

Here we assume that recoil momentum of a nucleon is always less than μ , that is, we take only the first term of the expansion of the nucleon energy:

$$\sqrt{p^2 + \mu^2} \approx \mu + p^2/2\mu. \quad (6)$$

Considering momentum conservation and the use of S -functions only, we obtain

$$\begin{aligned} \bar{H}_0 = & \sum_{n=0}^{\infty} n \int \left(\sqrt{k^2 + x^2} + \frac{k^2}{2\mu} \right) (f_+^2(k) + f_-^2(k)) d\mathbf{k} \cdot c_{n, n}^2 \\ & + \sum_{n=0}^{\infty} \int \left(\sqrt{k^2 + x^2} + \frac{k^2}{2\mu} \right) ((n+1)f_+^2(k) + n f_-^2(k)) d\mathbf{k} \cdot c_{n+1, n}^2 + \mu. \end{aligned} \quad (7)$$

When we evaluate the other part \bar{H}_1 , we make the same assumption on nucleon momenta and find

$$u_r(\mathbf{p} \pm \mathbf{k}) \beta u_r(\mathbf{p}) = 1 - \frac{4p^2 + k^4}{8\mu^2}. \quad (8)$$

\bar{H}_1 can be calculated in the following processes, each one separately:

- 1) Production of a positive meson $\varphi_{n, n} \rightarrow \psi_{n+1, n}$,
- 2) Destruction of a negative meson $\varphi_{n, n} \rightarrow \psi_{n, n-1}$,
- 3) Production of a negative meson $\psi_{n+1, n} \rightarrow \varphi_{n+1, n+1}$,
- 4) Destruction of a positive meson $\psi_{n+1, n} \rightarrow \varphi_{n, n}$.

Then we find

$$\begin{aligned} 1) = 4) = & f' \sqrt{n+1} c_{n, n} c_{n+1, n} \left[\int \frac{k f_+(k)}{\sqrt{k^2 + x^2}} d\mathbf{k} \left(1 - \frac{n}{2\mu^2} \int k^2 (f_+^2(k) + f_-^2(k)) d\mathbf{k} \right) \right. \\ & \left. - \int \frac{k^3 f_+(k) d\mathbf{k}}{8\mu^2 \sqrt{k^2 + x^2}} \right], \end{aligned} \quad (9)$$

$$\begin{aligned} 2) = 3) = & f' \sqrt{n+1} c_{n+1, n} c_{n+1, n+1} \left[\int \frac{k f_-(k) d\mathbf{k}}{\sqrt{k^2 + x^2}} \left(1 - \frac{n+1}{2\mu^2} \int k^2 (f_+^2(k) + f_-^2(k)) d\mathbf{k} \right) \right. \\ & \left. - \int \frac{k^3 f_-(k) d\mathbf{k}}{8\mu^2 \sqrt{k^2 + x^2}} \right]. \end{aligned} \quad (10)$$

To make a comparison with Tomonaga's results, it seems convenient to introduce the following notations:

$$\left. \begin{aligned} \int \left(\sqrt{k^2 + x^2} + \frac{k^2}{2\mu} \right) f_+^2(k) d\mathbf{k} &= a, \\ \int \left(\sqrt{k^2 + x^2} + \frac{k^2}{2\mu} \right) f_-^2(k) d\mathbf{k} &= b, \\ \int \frac{k f_+(k)}{\sqrt{k^2 + x^2}} d\mathbf{k} - \int \frac{k^3 f_+(k)}{8\mu^2 \sqrt{k^2 + x^2}} d\mathbf{k} &= h, \end{aligned} \right\} \quad (11)$$

$$\left. \begin{aligned} \int \frac{k f_-(k)}{\sqrt{k^2 + x^2}} d\mathbf{k} - \int \frac{k^3 f_-(k)}{8\mu^2 \sqrt{k^2 + x^2}} d\mathbf{k} &= g', \\ \int \frac{k f_+(k)}{\sqrt{k^2 + x^2}} d\mathbf{k} \int \frac{k^2}{2\mu^2} (f_+^2(k) + f_-^2(k)) d\mathbf{k} &= h', \\ \int \frac{k f_-(k)}{\sqrt{k^2 + x^2}} d\mathbf{k} \int \frac{k^2}{2\mu^2} (f_+^2(k) + f_-^2(k)) d\mathbf{k} &= g'. \end{aligned} \right\}$$

Using these notations we rewrite the expectation value

$$\bar{H}_0 = \sum_{n=0}^{\infty} n(a+b) c_{n,n}^2 + \sum_{n=0}^{\infty} \{ (n+1)a + nb \} c_{n+1,n}^2, \quad (12)$$

$$\begin{aligned} \bar{H}_1 = & \sum_{n=0}^{\infty} 2f' \{ \sqrt{n+1} h - n \sqrt{n+1} h' \} c_{n,n} c_{n+1,n} \\ & + \sum_{n=0}^{\infty} 2f' \{ \sqrt{n+1} g - (n+1) \sqrt{n+1} g' \} c_{n+1,n} c_{n+1,n+1}. \end{aligned} \quad (13)$$

Although f_+ and f_- are unknown functions and should be chosen to give convergent integrals, we do not intend to discuss these functions in details. Here we assume these integrals themselves as numerical parameters. Substituting (12) and (13) into (5) and taking into account (4) we minimize the expectation value of the total energy (5) and obtain

$$\left. \begin{aligned} f'(\sqrt{n} - n\sqrt{n} g') c_{n,n-1} + (na + nb - E) c_{n,n} \\ + f'(\sqrt{n+1} n - n\sqrt{n+1} h') c_{n+1,n} &= 0, \\ f'(\sqrt{n+1} h - n\sqrt{n+1} h') c_{n,n} + ((n+1)a + nb - E) c_{n+1,n} \\ + f'(\sqrt{n+1} g - (n+1)\sqrt{n+1} g') c_{n+1,n+1} &= 0. \end{aligned} \right\} \quad (14)$$

§ 3. Discussion

It is seen that recoil effects appear not only through changes of magnitude of a , b , g , h but also through new terms h' and g' which are not considered in Tomonaga's calculation. If we transform (14) into simultaneous differential equations using the same procedure in Tomonaga's paper, it is found that these terms induce the third order derivatives which may be interpreted as a "Zitterbewegung". The recoil effect in the above discussion appears as an effect of fluctuation which is caused by the collision between each meson and the nucleon.

Ratios of a , b , g , and h to the same kind of quantities of the non-recoil case depend upon the function form of f_+ or f_- as follow:

$$\left. \begin{aligned} \frac{a}{a_{non}} &= 1 + \frac{1}{2\mu} \int k^2 f_+^2(k) d\mathbf{k} \left/ \int \sqrt{k^2 + x^2} f_+^2(k) d\mathbf{k} \right., \\ \frac{b}{b_{non}} &= 1 + \frac{1}{2\mu} \int k^2 f_-^2(k) d\mathbf{k} \left/ \int \sqrt{k^2 + x^2} f_-^2(k) d\mathbf{k} \right., \end{aligned} \right\} \quad (15)$$

$$\left. \begin{aligned} \frac{h}{h_{\text{non}}} &= 1 - \frac{1}{8\mu^2} \int \frac{k^3 f_+(k) dk}{\sqrt{k^2 + \kappa^2}} \bigg/ \int \frac{k f_+(k)}{\sqrt{k^2 + \kappa^2}} dk, \\ \frac{g}{g_{\text{non}}} &= 1 - \frac{1}{8\mu^2} \int \frac{k^3 f_-(k) dk}{\sqrt{k^2 + \kappa^2}} \bigg/ \int \frac{k f_-(k)}{\sqrt{k^2 + \kappa^2}} dk. \end{aligned} \right\}$$

Although these ratios are sensitive to the so-called cut-off procedure, both increases of a and b and also decreases of g and h at first sight correspond to the weakening of coupling constant f' .

However, since there are two other terms g' and h' in (14), a more careful examination of (14) should show their effects on f' . Assuming the following simplification:

$$a=b, \quad g=h, \quad \text{and} \quad g'=h',$$

we rewrite (14)

$$\left. \begin{aligned} -V(\sqrt{n} - n\sqrt{n}\lambda)c_{n,n-1} + (2n-W)c_{n,n} - V(\sqrt{n+1} - n\sqrt{n+1}\lambda)c_{n+1,n} &= 0, \\ -V(\sqrt{n+1} - n\sqrt{n+1}\lambda)c_{n,n} + (2n+1-W)c_{n+1,n} \\ -V(\sqrt{n+1} - (n+1)\sqrt{n+1}\lambda)c_{n+1,n+1} &= 0, \end{aligned} \right\} \quad (16)$$

where the same notations as in Tomonaga's paper are used, i.e.,

$$fg/a = V, \quad E/a = W, \quad g'/g = \lambda.$$

Numerical solutions of these equations are shown in Figures 1, 2, and 3 corresponding to $V^2=1, 4$, and 6 in the case of $\lambda=0.1$. The abscissas of these figures express N ;

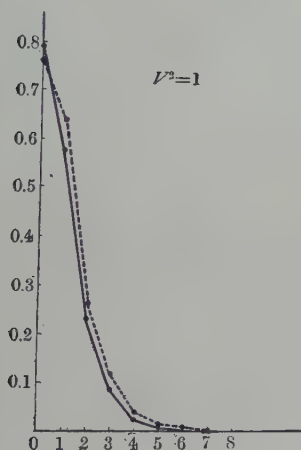


Fig. 1

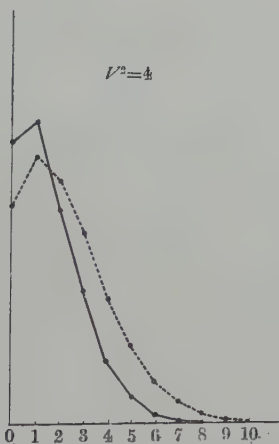


Fig. 2

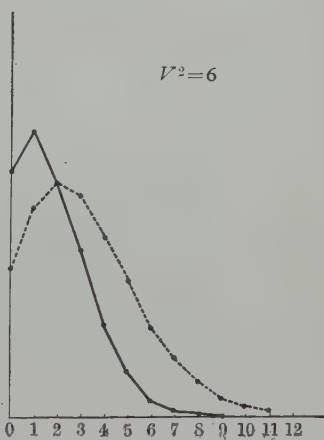


Fig. 3

the number of mesons which exist in the vicinity of the nucleon, and the coordinates express the amplitude of the probability, i.e.,

$$c_N = \begin{cases} c_{N/2, N/2} & \text{for even } N, \\ c_{N+1/2, N-1/2} & \text{for odd } N, \end{cases}$$

The dotted curves show the non-recoil cases which are given by Tomonaga.

From these results we can conclude that the recoil effect which appears through the term λ becomes stronger with increasing l^2 . With a strong effect of recoil, which causes a weakening of coupling, the bare nucleon probability always increases. According to Tomonaga his intermediate coupling theory gives almost the same results as the strong coupling theory in the case of $l^2 \lesssim 4$. Therefore it may be plausible to conclude that recoil effects reduce the coupling. In the above discussion the value of λ is taken arbitrarily for the purpose to examine the effect qualitatively. We shall discuss the magnitude of λ in the following paper.

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On the Many-body Problem in the Intermediate Coupling Theory, I

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The method of Matthews and Salam in the intermediate coupling theory is applied to the nuclear forces in the neutral $PS(\rho^0)$ meson theory. The static potentials are obtained, in one pair approximations, including the damping effects of the pair coupling term. The S -wave probabilities in the meson cloud of the two nucleon system are discussed.

§ 1. Introduction

The quantum theory of fields describes elementary particles as quanta of free fields. The states, in which definite numbers of these bare particles are present, constitute an orthonormal set. The transition probabilities between these states are given by the matrix elements of an interaction Hamiltonian. Such a perturbational treatment is very convenient to treat the annihilations or creations of elementary particles, and successful when the interaction is very small. The quanta of free fields are, however, fictitious mathematical tools and do not correspond exactly to the physical particles. Wentzel gave us one picture of a physical nucleon in his strong coupling theory.¹⁾ Tomonaga²⁾ developed the intermediate coupling theory and discussed in detail about the meson cloud surrounding a physical nucleon.³⁾⁴⁾⁵⁾ These considerations lead us to an alternative description of elementary particles.

We can imagine a group of the state functions

$$(\psi | n^A_1, n^A_2, \dots, n^B_{1'}, n^B_{2'} \dots n^C_{1''}, n^C_{2''} \dots). \quad (1.1)$$

ψ is an eigen function of the total Hamiltonian, and describes the state in which the number of physical A -particles in state 1 is n^A_1 , the number of physical A -particles in state 2 is n^A_2, \dots , the number of physical B -particles in state $1'$ is $n^{B'}_{1'}$, and so on. These state functions are not orthogonal each other in general. The angle θ between two state functions ψ_1 and ψ_2 determines the transition probability between them, and it is given by

$$\cos^2 \theta = (\psi_1, \psi_2)^2 / (\psi_1, \psi_1) (\psi_2, \psi_2). \quad (1.2)$$

Two states in which no transition can occur between them are orthogonal. Then the group of functions of (1.1) will be divided into subgroups which are orthogonal each other.

The theory must deal with the relations between the various angles θ 's given by (1.2). It is desired that only the conception of physical particles is used, but in this paper we seek the ψ in (1.1) in terms of bare particles, by the intermediate coupling

theory. Following the elegant method of Matthews and Salam⁶⁾ in which only one virtual nucleon pair is considered, we treat a few simple example.

As a simple example to construct ψ in eq. (1.1), we treat the static two nucleon problem in neutral $PS(\rho s)$ theory in I. The static potentials are given including the damping character of the pair coupling term. But our approximations are uncorrect for the small nuclear distance compared with the Compton wave length of the meson.

§ 2. Static nuclear forces

1. The method of Matthews and Salam and the Dyson-Foldy transformations.

The Hamiltonian for the interacting nucleon and neutral pseudo-scalar meson fields, with pseudoscalar coupling is, in the same notation as in Matthews and Salam,⁶⁾

$$\begin{aligned} H &= H_0 + H_1, \\ H_0 &= \int d\mathbf{p} \sum_r \{a_r^*(\mathbf{p}) a_r(\mathbf{p}) + b_r^*(\mathbf{p}) b_r(\mathbf{p})\} p_0 \\ &\quad + \int d\mathbf{K} q^*(\mathbf{K}) q(\mathbf{K}) K_0, \end{aligned} \quad (2.2)$$

$$\begin{aligned} H_1 &= f' \int d\mathbf{p} d\mathbf{p}' d\mathbf{K} \sum_s \delta(\text{momentum}) \\ &\quad \times (a_r^*(\mathbf{p}') + b_r(\mathbf{p}') (a_s(\mathbf{p}) + b_s^*(\mathbf{p})) (q^*(\mathbf{K}) + q(\mathbf{K})) \\ &\quad \times i u_r^*(\mathbf{p}') \beta \gamma_s u_s(\mathbf{p}) K_0^{-1/2}, \end{aligned} \quad (2.3)$$

$$f' = 2^{1/2} (2\pi)^{-3/2} f, \quad p_0 = (|\mathbf{p}|^2 + \kappa^2)^{1/2}, \quad K_0 = (|\mathbf{K}|^2 + \mu^2)^{1/2}. \quad (2.4)$$

The representatives of the state vector ψ in (1.1) for the two nucleons are in one pair approximations,

$$(\psi | \hat{p}_r^n, \hat{p}_s^n, \mathbf{K}^1, \dots, \mathbf{K}^n) = (\psi | 2, n) \quad (2.5)$$

and

$$(\psi | \hat{p}_r^n, \hat{p}_s^n, \hat{p}_a^n, \hat{q}_b^n, \mathbf{K}^1, \dots, \mathbf{K}^n) = (\psi | 4, n). \quad (2.6)$$

(2.5) and (2.6) represents no pair component and one pair component respectively. The conservation of momentum implies that

$$\hat{p}_r^n + \hat{p}_s^n + \sum \mathbf{K}^r = \hat{p}_r^n + \hat{p}_s^n + \hat{p}_a^n + \hat{q}_b^n + \sum \mathbf{K}^r = \text{constant},$$

and is zero in the center of mass system. The procedure parallels closely that of Matthews and Salam. The expectation value of the energy is

$$\begin{aligned} (\psi | H | \psi) &= (\psi | H | 2, n) (2, n | \psi) + (\psi | H | 4, n) (4, n | \psi) \\ &= (\psi | H | 2, n) (2, n | \psi) + E(\psi | 4, n)^2, \end{aligned} \quad (2.7)$$

where

$$(\psi | \psi) = 1. \quad (2.8)$$

We minimize X ,

$$\begin{aligned}\delta\Lambda &= \delta((\phi|H|\phi) - E(\phi|\phi)), \\ &= \delta(Y - E(\phi|2n)^2),\end{aligned}\quad (2.9)$$

where

$$\begin{aligned}Y &= (\phi|H|2, n)(2, n|\phi). \\ Y &= \sum \{ (\mathbf{d}\mathbf{K})^n \sum_i \{ (\dot{p}_0^n + \dot{p}_0'' + \sum_{s=1}^n K_s^n) | (\phi| \dot{p}_i^n, \dot{p}_u^n, \mathbf{K}^1, \dots, \mathbf{K}^n) |^2 \\ &+ f' [\sum_s n^{1/2} (U(\dot{p}_s^{n-1}, \dot{p}_i^n) (\phi| \dot{p}_s^{n-1}, \dot{p}_u^n, \mathbf{K}^1, \dots, \mathbf{K}^{n-1}) \\ &+ U(\dot{p}_s^{n-1}, \dot{p}_u^n) (\phi| \dot{p}_i^n, \dot{p}_s^{n-1}, \mathbf{K}^1, \dots, \mathbf{K}^{n-1})) \\ &\quad \times (\dot{p}_i^n, \dot{p}_u^n, \mathbf{K}^1, \dots, \mathbf{K}^n | \phi) + \text{c.c.} \\ &+ 2^{1/2} (n+1)^{1/2} \{ d\dot{p}' d\mathbf{K}^{n+1} \sum_{s,r} V(\dot{p}_s', \dot{q}_r^{n+1}, \mathbf{K}^{n+1}) \\ &\quad \times (\phi| \dot{p}_s' \dot{p}_i^n \dot{p}_u^n \dot{q}_r^{n-1}, \mathbf{K}^1, \dots, \mathbf{K}^{n+1}) (\dot{p}_i^n, \dot{p}_u^n, \mathbf{K}^1, \dots, \mathbf{K}^n | \phi) \\ &+ 2^{1/2} n^{1/2} \{ d\dot{p}' \sum_{s,r} V(\dot{p}_s' \dot{q}_r^{n-1}, \mathbf{K}^n) \\ &\quad \times (\phi| \dot{p}_s' \dot{p}_i^n, \dot{p}_u^n, \dot{q}_r^{n-1}, \mathbf{K}^1, \dots, \mathbf{K}^{n-1}) (\dot{p}_i^n, \dot{p}_u^n, \mathbf{K}^1, \dots, \mathbf{K}^n | \phi) \},\end{aligned}\quad (2.11)$$

where

$$U(\mathbf{p} + \mathbf{K}_s, \mathbf{p}_i) = i u_s^*(\mathbf{p} + \mathbf{K}) \beta \gamma_5 u_i(\mathbf{p}) (K_0)^{-1/2}, \quad (2.12)$$

$$V(\mathbf{p}_s, \mathbf{q}_r, \mathbf{K}) = i u_s^*(\mathbf{p}) \beta \gamma_5 u_r(\mathbf{q}) (K_0)^{-1/2}, \quad (2.13)$$

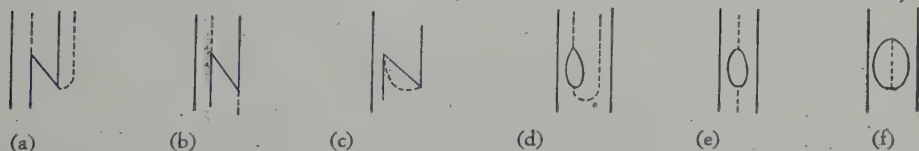
$$\begin{aligned}\dot{p}_i^{n-1} &= \dot{p}_i^n + \mathbf{K}^n, \quad \dot{p}_i^{n-1} = \dot{p}_i^n + \mathbf{K}^n, \\ \dot{q}_r^n &= -(\mathbf{p}' + \mathbf{K}^n), \quad \dot{q}_r^n = -(\mathbf{p}' - \mathbf{K}^{n+1}).\end{aligned}\quad (2.14)$$

The one pair components $(\phi|4, n)$ in eq. (2.11) are expressed by the no pair components $(\phi|2, n)$ using the equation

$$(\phi|H - E|4, n) = 0, \quad (2.15)$$

and taking only the pair creation or annihilation parts. The terms thus obtained correspond to the diagrams like Fig. 1.

Fig. 1



The terms corresponding to (a), (b) and (c) are the vacuum effects and these are omitted. The terms (d) and (e) contribute to the modifications of u_n in eq. (2.17), and (e) contribute to the short range nuclear forces, but these are neglected for simplicity.

Since we are here interested only in the static nuclear potential, we may use the following Tomonaga approximation

$$(\phi | \overset{1}{p}_i, \overset{2}{p}_i, \mathbf{K}^1, \dots, \mathbf{K}^n) = (4)^{-1/2} C_n \prod_r^n F(\mathbf{K}^r) S. \quad (2 \cdot 16)$$

where S is the wave function of the nucleons including spin functions, then we have

$$Y = \sum_n [n \alpha_n C_n^2 + 2n^{1/2} \beta_n C_n C_{n-1} + 2n^{1/2} (n-1)^{1/2} \gamma_n C_n C_{n-2}], \quad (2 \cdot 17)$$

where

$$\alpha_n = 1/n \int (d\mathbf{K})^n (\overset{1}{p}_0^n + \overset{2}{p}_0^n + \sum_r^n K_0^r) \prod_r^n |F(\mathbf{K}^r)|^2, \quad (2 \cdot 18)$$

$$\beta_n = f' \int (d\mathbf{K})^n \sum_{s,t} 1/4 \{ U(\overset{1}{p}_s^{n-1}, \overset{1}{p}_t^n) + U(\overset{1}{p}_s^{n-1}, \overset{2}{p}_t^n) \} F(\mathbf{K}^n) \prod_r^{n-1} |F(\mathbf{K}^r)|^2 + \text{c.c.}, \quad (2 \cdot 19)$$

$$\begin{aligned} \gamma_n = & - (f')^2 (4)^{-1} \int (d\mathbf{K})^n \sum_{s,t} \{ (V(\overset{1}{p}_s', \overset{1}{q}_t^{n-1}, \mathbf{K}^{n-1}) V^*(\overset{1}{p}_t^{n-2}, \overset{1}{q}_t^{n-1}, \mathbf{K}^n) \\ & + V(\overset{1}{p}_s', \overset{1}{q}_t^{n-1}, \mathbf{K}^{n-1}) V^*(\overset{2}{p}_t^{n-2}, \overset{2}{q}_t^{n-1}, \mathbf{K}^n) \} \\ & \times (\overset{1}{p}_0^{n-2} + \overset{2}{p}_0^{n-2} + \overset{1}{p}_0' + \overset{2}{p}_0' + \sum_r^{n-1} K_0^r - E)^{-1} F(\mathbf{K}^{n-1}) F(\mathbf{K}^n) \prod_r^{n-2} |F(\mathbf{K}^r)|^2 + \text{c.c.} \}. \end{aligned} \quad (2 \cdot 20)$$

We replace α_n , β_n and γ_n by the similar approximate expressions used by Matthews and Salam.⁽⁶⁾

$$\alpha_n - 2\pi/n \sim \alpha = \int K_0 |F(\mathbf{K})|^2 d\mathbf{K}. \quad (2 \cdot 21)$$

In the case of one physical nucleon placed at the origin,

$$\sum_{s,t} 1/2 U(\overset{1}{p}_s^{n-1}, \overset{1}{p}_t^n) = iK_x (x + \overset{1}{p}_0^n)^{-1} (K_0^n)^{-1/2} \mathfrak{G}(\overset{1}{p}^n),$$

where

$$\mathfrak{G}(\overset{1}{p}) = [1 + \overset{1}{p}^2 / (x^2 + \overset{1}{p}_0^2)]^{-1/2}.$$

If we make the nucleon momentum zero, then

$$1/2 \cdot \sum_{s,t} U(\overset{1}{p}_s^{n-1}, \overset{1}{p}_t^n) \sim iK_x (K_0)^{-1/2} (2x)^{-1}.$$

In our problem, we fix one nucleon at \mathbf{R}_1 , and the other at \mathbf{R}_2 . Then, we may have

$$1/2 \sum_{s,t} U(\overset{1}{p}_s^{n-1}, \overset{1}{p}_t^n) \sim iK_{x1} (K_0)^{-1/2} (2x)^{-1} e^{i\mathbf{K} \cdot \mathbf{R}_1},$$

$$1/2 \sum_{s,t} U(\overset{1}{p}_s^{n-1}, \overset{2}{p}_t^n) \sim iK_{x2} (K_0)^{-1/2} (2x)^{-1} e^{i\mathbf{K} \cdot \mathbf{R}_2},$$

where x_1 and x_2 is spin directions of nucleons 1 and 2 respectively and

$$\beta_n \sim f' \int G(\mathbf{K}) F(\mathbf{K}) d\mathbf{K} + \text{c.c.} \equiv \beta, \quad (2 \cdot 22)$$

$$G(\mathbf{K}) = i(K_0)^{-1/2} (4x)^{-1} (K_{x1} e^{i\mathbf{K} \cdot \mathbf{R}_1} + K_{x2} e^{i\mathbf{K} \cdot \mathbf{R}_2}). \quad (2 \cdot 23)$$

We evaluate γ_n in a similar approximation.

$$\gamma_n \sim (f')^2 (4x)^{-1} \left[\int K_0^{-1/2} \{ e^{-i\mathbf{K} \cdot \mathbf{R}_1} + e^{-i\mathbf{K} \cdot \mathbf{R}_2} \} F(\mathbf{K}) \right]^2 + \text{c.c.}, \quad (2 \cdot 24)$$

where we assumed that the virtual nucleon momenta are very small compared with x .

Matthews and Salam showed that the approximations used in evaluating α_n and β_n are not bad in the case of one nucleon system. The substitution of β and γ given by (2.22) and (2.24) for β_n and γ_n in eq. (2.17) corresponds to the Dyson and Foldy approximations. The interaction Hamiltonian is transformed by the Dyson-Foldy transformations⁽⁷⁾⁽⁸⁾ into two parts, taking the leading terms in f ,

$$H' = H_1 + H_2, \quad (2.25)$$

where H_1 is the gradient coupling term,

$$H_1 = \frac{f}{(2\pi)} \int \{(\sigma_1 \Delta) \delta(\mathbf{R}_1) + (\sigma_2 \Delta) \delta(\mathbf{R}_2)\} \phi(\mathbf{r}) d\mathbf{r}, \quad (2.26)$$

and H_2 is the pair coupling term,

$$H_2 = \frac{(f')^2}{2\pi} \int \{\delta(\mathbf{R}_1) + \delta(\mathbf{R}_2)\} \phi^2(\mathbf{r}) d\mathbf{r}. \quad (2.27)$$

The state vector ψ' has in this case, only no pair components

$$(\psi' | \hat{p}_1^n, \hat{p}_2^n, \mathbf{K}^1 \dots \mathbf{K}^n) = (\psi' | 2, n). \quad (2.28)$$

Here again, $(\psi' | 2, n)$ are assumed to have the form

$$(\psi' | 2, n) = (4)^{-1/2} C_n \prod^n F(\mathbf{K}^r) S. \quad (2.29)$$

We can obtain the expectation value of the energy immediately,

$$(\psi' | H_0 + H' | \psi') = \sum_n [n A_n C_n^2 + 2n^{1/2} G C_n C_{n-1} + 2n^{1/2} (n-1)^{1/2} P C_n C_{n-2}], \quad (2.30)$$

where

$$A_n = \frac{1}{n} \int (\hat{p}_0^n + \hat{p}_0^n + \sum \mathbf{K}_0^n) \prod^n |F(\mathbf{K}^r)|^2 d\mathbf{K} \sim \int K_0 |F(\mathbf{K})|^2 d\mathbf{K} + 2\pi/n, \quad (2.31)$$

$$G = \frac{f'}{4\pi} \int K_0^{-1/2} \{(\sigma_1 \Delta) e^{-i\mathbf{K} \cdot \mathbf{R}_1} + (\sigma_2 \Delta) e^{-i\mathbf{K} \cdot \mathbf{R}_2}\} F(\mathbf{K}) d\mathbf{K} + \text{c.c.}$$

$$\equiv f' \int G(\mathbf{K}) F(\mathbf{K}) d\mathbf{K} + \text{c.c.},$$

$$G(\mathbf{K}) = \frac{\pi}{4\pi} K_0^{-1/2} \{(\sigma_1 \Delta) e^{-i\mathbf{K} \cdot \mathbf{R}_1} + (\sigma_2 \Delta) e^{-i\mathbf{K} \cdot \mathbf{R}_2}\},$$

$$P = \frac{(f')^2}{4\pi} \int \{K_0^{-1/2} \{e^{-i\mathbf{K} \cdot \mathbf{R}_1} + e^{-i\mathbf{K} \cdot \mathbf{R}_2}\} F(\mathbf{K}) d\mathbf{K}\}^2 + \text{c.c.},$$

$$f' = 2^{-1/2} (2\pi)^{-3/2} f. \quad (2.32)^*$$

These values are the same with α , β and γ given by (2.21), (2.22) and (2.24) respectively. We can say, therefore, that the Dyson-Foldy approximations correspond to one pair approximations of Matthews and Salam assuming that the virtual nucleon momenta are very small compared with π .

Then, the minimal equation for $(Y-2\pi)$, is given by

$$\begin{aligned} (n-\epsilon) C_n + G (n^{1/2} \bar{C}_{n-1} + (n+1)^{1/2} C_{n+1}) \\ + \bar{P} (n^{2/2} (n-1)^{1/2} C_{n-2} + (n+2)^{1/2} (n+1)^{1/2} C_{n+2}) = 0, \end{aligned} \quad (2.33)$$

where

$$\epsilon = E/A, \quad G = G/A, \quad \bar{P} = P/A, \quad A = A_n - 2\pi/n. \quad (2.34)$$

This equation has the same form as that obtained by Matthews and Salam, and was solved

*) σ in A , G , and P means the expectation value.

exactly by them using the method of Glauber and Luttinger. Hence we have the spectrum,

$$(1-4P^2)^{-1/2}[1+2\delta^2(1+2\bar{P})+2\epsilon]=2m+1, \quad (2.35)$$

and

$$\langle n|C^m\rangle=K^m\int_{-\infty}^{\infty}H_n^*(x)H_m^*(x)\exp[-1/2(1+r)x^2-i\delta^{1/2}x]dx, \quad (2.36)$$

where

$$\delta=\bar{G}(1+2\bar{P}), \quad r=(1-2\bar{P})(1+2\bar{P})^{-1}, \quad (2.37)$$

and K^m is a normalizing factor.

The energy of the lowest level is given from (2.33) by

$$E^0(\mathbf{R}^1, \mathbf{R}^2)=-\frac{G^2}{A(1+2\bar{P})}-\frac{1}{2}A[1-\sqrt{1-3\bar{P}^2}]. \quad (2.38)$$

In this expression, the term G represents the contribution from the gradient coupling term of eq. (2.26), and P (or \bar{P}) represents the contribution from the pair coupling term of eq. (2.27).

2. *P-wave cloud and S-wave cloud.*

If \bar{P} is very small we can take $F(\mathbf{K})$ approximately, by the discussion given by Matthews and Salam, as

$$F(\mathbf{K})\sim F_p(\mathbf{K})=L_p^{-1/2}K_0^{-3/2}(4\pi)^{-1}[(\sigma_1A_1)e^{i\mathbf{K}R_1}+(\sigma_2A_2)e^{i\mathbf{K}R_2}]. \quad (2.39)$$

For the P -wave cloud, \bar{P} is actually small compared with unity except for the region of very small $\mathbf{R}_1-\mathbf{R}_2$. $L_p^{-1/2}$ is a normalizing factor. It is not clear that this P -wave cloud is a good approximation, however, as the pair coupling term may have considerable effects on $F(\mathbf{K})$. This will be discussed in the calculation of the ratio of S -waves to P -waves in the meson cloud. To derive the long range second order potential where the pair term has not a large effect, we can use (2.39) for $F(\mathbf{K})$. A , G and P are now evaluated as follows.

$$LA\sim L_pA_p=\frac{1}{(4\pi)^2}\int K_0^{-2}|(\sigma_1A_1)e^{i\mathbf{K}R_1}+(\sigma_2A_2)e^{i\mathbf{K}R_2}|^2d\mathbf{K}, \quad (2.40)$$

$$L^{1/2}G\sim L_p^{1/2}G_p=\frac{f'}{4\pi}\int K_0^{-2}|(\sigma_1A_1)e^{i\mathbf{K}R_1}+(\sigma_2A_2)e^{i\mathbf{K}R_2}|^2d\mathbf{K}+\text{c.c.}, \quad (2.41)$$

$$LP\sim L_pP_p=\frac{(f')^2}{(4\pi)^3}\int K_0^{-2}(e^{-i\mathbf{K}R_1}+e^{-i\mathbf{K}R_2})[(\sigma_1A_1)e^{i\mathbf{K}R_1}+(\sigma_2A_2)e^{i\mathbf{K}R_2}]d\mathbf{K}+\text{c.c.}, \quad (2.42)$$

Denoting

$$x\equiv\mu|\mathbf{R}_1-\mathbf{R}_2|, \quad (2.43)$$

and using

$$L_p^{-1/2}=2/\pi, \quad (2.44)$$

we have

$$A_p=x+\mu\left(\frac{\mu}{x}\right)^2(\sigma_1A)(\sigma_2A)\frac{e^{-x}}{x}, \quad (2.45)$$

$$G_p = \frac{1}{2} \sqrt{\frac{f^2}{4\pi}} \left[x + \mu \left(\frac{\mu}{x} \right)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) \frac{e^{-x}}{x} \right], \quad (2.46)$$

$$P_p = \frac{1}{8} \left(\frac{f^2}{4\pi} \right) \left(\frac{\mu}{x} \right)^3 \mu [(\sigma_1 \mathcal{D}) + (\sigma_2 \mathcal{D})] \frac{e^{-x}}{x}. \quad (2.47)$$

The energy $E^0(x)$ is then expressed by

$$E^0(x) \sim E_p^0(x) = \frac{-\frac{1}{4} \left(\frac{f^2}{4\pi} \right) \left[x + \mu \left(\frac{\mu}{x} \right)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) \frac{e^{-x}}{x} \right]}{1 + \frac{\frac{1}{4} (f^2/\pi) (\mu/x)^3 \mu [(\sigma_1 \mathcal{D}) + (\sigma_2 \mathcal{D})] \frac{e^{-x}}{x}}{x + \mu (\mu/x)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) e^{-x}/x}}. \quad (2.48)$$

When the nucleon is separated infinitely, $E_p^0(x)$ gives

$$E_p^0(\infty) = -1/4 (f^2/\pi) x. \quad (2.49)$$

This is twice the energy of one nucleon given by Matthews and Salam. The potential energy between two nucleons is given by

$$\begin{aligned} V_p(x) &= E_p^0(x) - E_p^0(\infty) \\ &= \frac{-\frac{1}{4} \left(\frac{f^2}{4\pi} \right) \mu \left(\frac{\mu}{x} \right)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) e^{-x}/x + \frac{(1/4)^2 (f^2/4\pi)^2 (\mu/x)^3 \mu [(\sigma_1 \mathcal{D}) + (\sigma_2 \mathcal{D})] e^{-x}/x^2}{1 + (\mu/x)^3 \mu [(\sigma_1 \mathcal{D}) + (\sigma_2 \mathcal{D})] e^{-x}/x}}{1 + \frac{(1/4) (f^2/4\pi) (\mu/x)^3 \mu [(\sigma_1 \mathcal{D}) + (\sigma_2 \mathcal{D})] e^{-x}/x^2}{x + \mu (\mu/x)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) e^{-x}/x}} \end{aligned} \quad (2.50)$$

Taking the terms of the order of f^2 from (2.50), we obtain the second order potential,

$$\begin{aligned} V^{(2)}(x) &= -1/4 (f^2/\pi) \mu (\mu/x)^2 (\sigma_1 \mathcal{D}) (\sigma_2 \mathcal{D}) e^{-x}/x \\ &= -\mu (f^2/4\pi) (\mu/2x)^2 \{ (\sigma_1 \sigma_2) + (1 + 3/x + 3/x^2) S_{12} \} e^{-x}/x. \end{aligned} \quad (2.51)$$

This is well known second order $PS(ps)$ potential. If we neglect all the contributions from the pair coupling term in (2.50), $V(x)$ is identical with (2.51). In the case of the gradient coupling only, our approximation is the same with that of the first order approximation in the perturbation theory. The pair coupling term contributes to diminish the potential energy as it appears in the denominator of the eq. (2.50). The pair coupling term has appreciable effects only when the coupling constant is very large or the distance x is very small. The factor containing spins in P -term represents the effects of the interactions between one nucleon and the P -wave meson cloud of the other nucleon through the pair coupling term, and the interference of these effects. We can see from eq. (2.50) that S -wave interactions through the pair coupling, although this is very small, act to diffuse the direction of the spin, whereas P -wave interactions through the gradient coupling act to make the spin vector coincide with the line joining nucleons.

However, we can not expect that the eq. (2.50) gives the fourth order potential. This is because we have used one pair approximations and P -wave cloud that is obtained

in the absence of pair coupling term for the $F(\mathbf{K})$, as in the case of one nucleon problem.⁶⁾ The fourth order potential,⁹⁾¹⁰⁾ arises almost entirely from virtual nucleon-pair formations. Hence we return to the eq. (2.30) and we tentatively assume as a approximation that the distribution of a bound meson is spherically symmetric.

$$F(k) \sim F_s(k) = L_s^{-1/2} (K_0)^{-3/2} (e^{i\mathbf{K}\mathbf{R}_1} + e^{i\mathbf{K}\mathbf{R}_2}), \quad (2.52)$$

as it is suggested from eq. (2.33). $L_s^{-1/2}$ is the normalizing factor, and we take

$$L_s^{-1/2} = 1/2\pi. \quad (2.53)$$

Then, the energy given by eq. (2.38) becomes,

$$E^0(x) \sim E_s^0(x) = -\frac{G^2}{A_s(1+2P_s/A_s)} - \frac{A_s}{2}(1 - \sqrt{1-4P_s/A_s}). \quad (2.54)$$

We omit the second term of (2.54) which is $O(\bar{P})$ and small when the coupling is not so strong, for the sake of simplicity in the following discussions. In the eq. (2.54),

$$A_s = x + \mu \frac{e^{-x}}{x}, \quad (2.55)$$

$$G_s = \frac{1}{4} \sqrt{\frac{f^2}{4\pi}} \left(\frac{\mu}{x} \right) \mu [(\sigma_1 \mathcal{A}) + (\sigma_2 \mathcal{A})] \frac{e^{-x}}{x}, \quad (2.56)$$

$$P_s = \frac{1}{2} (f^2/\pi) (\mu/x) \mu [(x/\mu) + e^{-x}/x]^2. \quad (2.57)$$

Then the potential energy becomes,

$$\begin{aligned} V_s(x) = E_s(x) &= -\frac{G_s^2}{A_s + 2P_s} \\ &= \frac{-(f^2/4\pi)}{1 + (f^2/4\pi)(1 + \mu/x e^{-x}/x)} \frac{1/16(\mu/x)^3 \mu [(\sigma_1 \mathcal{A}) + (\sigma_2 \mathcal{A})] e^{-x}/x^2}{(1 + \mu/x e^{-x}/x)} \end{aligned} \quad (2.58)$$

$V_s(x)$ must be multiplied by the S -wave probability which depend on f . The denominator $1 + (f^2/4\pi)(1 + (\mu/x)e^{-x}/x)$ represents the large damping effect of virtual nucleon pair formations which corresponds to that was discussed by Wentzel,¹¹⁾ Drell and Henley,¹²⁾ and Brueckner, Gell-Mann and Goldberger.¹³⁾ The factor containing spins is the effect of the gradient coupling between one nucleon and S -wave meson cloud of other nucleon and the interference of these effects.

3. The linear combinations of P and S waves.

Next, we take as trial wave functions for the meson cloud the linear combinations of P and S waves,

$$F_{sp}(\mathbf{K}) = F_p(\mathbf{K}) + aF(\mathbf{K}), \quad (2.59)$$

where $F_p(\mathbf{K})$ and $F_s(\mathbf{K})$ are given by eq. (2.39) and eq. (2.52). The coefficient

u of S waves is determined by minimizing the energy. A , G and P in the eq. (2.30) are, in this case,

$$A = A_p + u\hat{A} + u^2\hat{A}_s, \quad (2.60)$$

$$G = G_p + uG_s, \quad (2.61)$$

$$P = P_p + u\hat{P} + u^2\hat{P}_s, \quad (2.62)$$

where A_p , G_p , P_p and A_s , G_s , P_s are the same with that are given by (2.40), (2.41), (2.42) and (2.55), (2.56), (2.57) respectively. \hat{A} and \hat{P} are the interference terms between P and S waves and are given by

$$\begin{aligned} \hat{A} &= \int K_0 F_s^*(\mathbf{K}) F_p(\mathbf{K}) d\mathbf{K} + \text{c.c.} \\ &= \left(\frac{\mu}{x}\right) \mu ((\sigma_1 \mathcal{A}) + (\sigma_2 \mathcal{A})) \frac{e^{-x}}{x}, \end{aligned} \quad (2.63)$$

$$\begin{aligned} \hat{P} &= \frac{(f')^2}{4x} \left[\int K_0^{-1/2} (e^{-i\mathbf{K}R_1} + e^{-i\mathbf{K}R_2}) F_p(\mathbf{K}) d\mathbf{K} \right] \left[\int K_0^{-1/2} (e^{-i\mathbf{K}R_1} + e^{-i\mathbf{K}R_2}) F_s(\mathbf{K}) d\mathbf{K} \right] + \text{c.c.} \\ &= \frac{1}{4} \left(\frac{f^2}{4\pi}\right) \left(\frac{\mu}{x}\right)^2 (x + \mu e^{-x}/x) ((\sigma_1 \mathcal{A}) + (\sigma_2 \mathcal{A})) \frac{e^{-x}}{x}. \end{aligned} \quad (2.64)$$

The energy becomes

$$E^0(x, u) = \frac{-(G_p + uG_s)^2}{(A_p + u\hat{A} + u^2\hat{A}_s) + 2(P_p + u\hat{P} + u^2\hat{P}_s)}. \quad (2.65)$$

The value of u which minimize the energy $E^0(x, u)$ is

$$u = \frac{2G_s(\hat{A}_p + 2P_p) - G_p(\hat{A} + 2\hat{P})}{2G_p(\hat{A}_s + 2P_s) - G_s(\hat{A} + 2\hat{P})}. \quad (2.66)$$

This is expressed as

$$u = \frac{(f^2/4\pi)(\mu/x)^2 S_1 ((S_1^2 - S_0 S_2))}{(4S_0 S_2 - (\mu/x) S_1^2) + (f^2/4\pi)(\mu/x) (4S_0^2 S_2 - 1/2(\mu/x) S_0 S_1^2)}, \quad (2.67)$$

where we have used the abbreviations,

$$\begin{aligned} S_0 &= (x/\mu) + e^{-x}/x, \\ S_1 &= ((\sigma_1 \mathcal{A}) + (\sigma_2 \mathcal{A})) e^{-x}/x, \\ S_2 &= (x/\mu)^3 + (\sigma_1 \mathcal{A})(\sigma_2 \mathcal{A}) e^{-x}/x. \end{aligned} \quad (2.68)$$

When the distance between two nucleons becomes infinite, or the coupling constant becomes zero, the S -wave probability converges to zero. Hence, for the large distance x or for the small f , the long range force $V^{(2)}$ given by eq. (2.51) which is obtained using the P wave cloud is a very good approximation.

Taking leading term in (μ/x) , u is approximately given by

$$u \sim \frac{-(1/4)(f^2/4\pi)(\mu/x)^2 S_1}{1 + (f^2/4\pi)(\mu/x) S_0}. \quad (2.69)$$

The S wave coefficient u is then, for the small f , $(f^2/4\pi) \ll 1$,

$$u \sim -(1/4)(f^2/4\pi)(\mu/x)^2 S_1 \quad (2.70)$$

and for the large f , $(f^2/4\pi) \gg 1$, although our approximations can not be applied to the case $f^2/4\pi \gg 1$,

$$u \sim -(1/4)(\mu/x)^2 S_1. \quad (2.71)$$

The S -wave probability increases as f increases from the weak coupling limit $f=0$, but converges to a certain value in the strong coupling limit. This is the damping character of the pair coupling term in the $P^2S(p_s)$ meson theory which has been discussed by several authors from different standpoints.

The potential energy for the $P+S$ waves cloud is evaluated from the equation,

$$V_{ps} = E^0(x, u) - E^0(\infty, u), \quad (2.72)$$

where $E^0(x, u)$ is given by (2.67) or (2.69). Expanding (2.72) in powers of f , we can get the second order potential, but we can not get the fourth order potential⁹⁾¹⁰⁾ because of our poor approximations for $x < 1$.

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A Generalization of Special Lorentz Transformation in de Sitter Space-time

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The transformations of the full Lorentz group are generalized in de Sitter space-time from the standpoint of the group of motions. When the curvature of the space-time tends to zero, the generalized transformations tend to the well-known forms of the corresponding transformations of the Lorentz group. Some properties of the transformations, especially those of the generalized special Lorentz transformation are studied. In connection with this research the important rôle the group of motions plays in relativity is considered and the difference of the notion of the invariancy in general and special relativities is also studied.

§ 1. Introduction

Special Lorentz transformation plays an important rôle in the special theory of relativity. In this paper we are going to obtain a transformation in de Sitter space-time $[S]$ corresponding to the Lorentz transformation in Minkowski space-time $[M]$. Lorentz transformation as a transformation in $[M]$ has various properties, so we may generalize it from various standpoints such as that of the group of motions, that of the generalized uniform motion*, etc. In this paper we shall take the first standpoint, i.e. we shall obtain the generalized Lorentz transformation from the standpoint of the group of motions in the sense used in Riemannian geometry¹⁾ using the fact that the Lorentz transformation is a motion in $[M]$. As will be seen later, a generalization of uniform motion in $[S]$ is obtained as a byproduct.

First in § 2 and § 3 some considerations are made about the physical meanings of the group of motions and the coordinate systems used in this paper. In § 4 and § 5 the infinitesimal and the finite forms of the generalized special Lorentz transformation L_1 are determined in the coordinate system of (4.2). Then some properties of this L_1 are studied in § 6 and § 7. Lastly in § 8 the infinitesimal and the finite forms of L_1 are obtained in another coordinate system in which (8.6) holds.

*) Since Lorentz transformation was first obtained in relativity in connection with the problem of equivalency of the observers moving uniformly relative to each other, we may first generalize this notion of uniform motion suitably and then may define the generalized Lorentz transformation in $[S]$ as a transformation connecting observers in such a generalized uniform motion relative to each other.

§ 2. Group of motions in a space-time and the equivalency of observers

Mathematically speaking, the fundamental principle of special relativity is that the physical laws are invariant under the transformations of Lorentz group, while one of the leading principles of general relativity is that all the physical laws are invariant under all transformations. However the meaning of the term "invariant" as used in both principles is not necessarily the same. In the following we shall first explain the concept of the invariance in general relativity.*

Now let A, B, \dots be tensors expressing physical quantities, then a physical law in general relativity is expressed by one or several tensor equations of the form:

$$F(A, B, \dots; \nabla_i A, \dots) = 0, \quad (2.1)$$

where ∇_i denotes covariant derivative of the tensor. If we regard this (2.1) as an equation for $A, B, \dots; \nabla_i A, \dots$, it is form-invariant under any coordinate transformation and in this sense all coordinate systems are equivalent. This is the meaning of the invariance in general relativity. (Here it is to be noticed that some of these coordinate systems have no such physical meaning as observation system.) In general, however, this (2.1) is not necessarily invariant in the sense used in special relativity. To clarify the circumstance we shall give an example.

If we regard the fundamental tensor g_{ij} of the space-time as a physical quantity its components are functions of the coordinate variables and their functional forms depend upon the coordinate system. Hence in the equation $ds^2=0$, i.e. $g_{ij}dx^i dx^j=0$ concerning the path of light, ($i, j=1, \dots, 4$), if we regard ds^2 as a function of g_{ij} and dx^i , this law is form-invariant under any coordinate transformation, while on the other hand if we regard this ds^2 as a function of x^i and dx^i the law $ds^2=0$ is form-invariant only under conformal transformations in the space-time and not form-invariant under another transformations, and further the quantity ds^2 is form-invariant only under the group of motions which forms a sub-group of the above stated conformal group.

Therefore from the standpoint of the form-invariance of ds^2 as a function of x^i and dx^i , it is only the observers relatively connected with each other by the motion of the space-time that are equivalent. This is the meaning of invariance in special relativity. In the theories which treat of such invariance it is needless to deal with the coordinate systems which have no physical meaning, since motions are, in general, accompanied by some physical meanings.** Moreover it is not only to be noticed that tensor equations are not necessarily form-invariant, but also that invariant laws are not necessarily to be expressed by tensor equations.***

*) See also the references 2) and 3).

**) Møller laid stress on the motion in a curved space-time and called it a generalized Lorentz transformation⁴⁾. It is to be noticed, however, that his terminology differs from the present one.

***) For instance, if we take $(\gamma^i \partial/\partial x^i + m)\psi=0$ where $\gamma^{(i}\gamma^{j)}=g^{ij}$ as the generalized Dirac equation, this equation is form-invariant under any motion, though it is not of tensor form. If we wish to treat Dirac equation of tensor form, we must take $(\gamma^i \nabla_i + m)\psi=0$ using covariant derivative ∇_i in place of $\partial/\partial x^i$.

From the above consideration we can conclude that when we wish to discuss about the invariancy of a physical law we must first make clear in what sense the invariancy is treated, otherwise the deduction will be meaningless.

It was first emphasized by Robertson in his relativistic cosmology that the concept of group of motions plays an important rôle in the theory of equivalency of observers in a curved space-time. We may rather say that the equivalency of observers relatively connected with each other by motions was the starting point of his theory.⁵⁾

As is well known, the group of motions of $[M]$ is the full Lorentz group with 10 parameters, and in this sense we may regard the special relativity as a theory concerning the physical laws form-invariant under the group of motions of $[M]$. Hence in order to construct in a curved space-time a theory which corresponds to the special theory of relativity in $[M]$, we have to deal with the physical laws form-invariant (in the sense used in the special relativity) under the group of motions of the space-time. For this purpose, we must first make clear the group of motions. In Riemannian geometry, however, it is well known that the number of the parameters of the group is 10 only when the space-time is $[M]$ or $[S]$, and is less than 10 in another space-time. Accordingly from the standpoint of the group of motions, $[S]$ is the curved space-time in which the generalization of the Lorentz transformation is dealt with most naturally.

In this paper, we shall first determine the forms of the transformations of the group of motions in this $[S]$ so that they may tend to the well-known forms of the corresponding transformations of the full Lorentz group when the curvature of the space-time tends to zero, and then we shall study some properties of the transformation corresponding to the special Lorentz transformation.

§ 3. Standard coordinate system in a curved space-time

In Minkowski space-time $[M]$ there exists a standard coordinate system in which the values of the coordinate variables correspond directly to the results of our measurement or experiment. Namely, it is the coordinate system in which the metric ds^2 takes the form:

$$ds^2 = -dx^2 - dy^2 - dz^2 + dt^2, \quad (3.1)$$

where, for brevity's sake, the units of length and time are taken so as the light velocity in vacuum is unity. On the other hand in de Sitter space-time or, generally speaking, in a curved space-time, such a standard coordinate system does not exist. Furthermore it is very difficult to determine theoretically such a coordinate system. This problem will not be solved till we succeed in clarifying the law which connects the values of the coordinate variables and the results of our observations in a curved space-time.

Therefore we can not decide what coordinate system in $[S]$ is the most suitable one for our present investigation. In the following we shall explain this in detail: In the coordinate systems of $[S]$ treated later in which ds^2 takes the forms (4.2) and (8.6) respectively, if the constant k^2 which gives the curvature of the space-time tends to zero, both ds^2 's tend to the same (3.1). And it is impossible to determine theoretically which coordinate system is the fundamental one corresponding to (3.1) of $[M]$. To answer

this it is necessary to compare both systems by experimenting in some way or other. But it seems to the writer that at the present stage even by experiment it is difficult to solve this problem completely, and moreover it should be noticed that there exist many coordinate systems with similar qualifications. Hence in this paper we shall not deal with this problem further and shall content ourselves only with the investigation of the transformation in $[S]$ corresponding to Lorentz transformation in the two coordinate systems stated above.

§ 4. Infinitesimal form

As is well known, $[S]$ is a Riemannian space of constant curvature whose signature of the fundamental form is given by the type $(---+)$. Its curvature tensor K_{ijlm} satisfies

$$K_{ijlm} = k^2 (g_{im} g_{jl} - g_{il} g_{jm}), \quad (4.1)$$

where k is a constant, and its line element is reducible to the form

$$ds^2 = -c^{2kt} (dx^2 + dy^2 + dz^2) + dt^2 \quad (4.2)$$

by a suitable choice of the coordinate system. In (4.2) the units of x , y and z are taken so as the light velocity may become unity when k tends to 0. If k tends to 0 $[S]$ tends to $[M]$ and (4.2) to (3.1).

Transformations which keep (3.1) form-invariant constitute the full Lorentz group $\bar{\mathcal{G}}$ or the group of motions in $[M]$ using the terminology of geometry.* $\bar{\mathcal{G}}$ is a continuous group of 10 essential parameters and in the coordinate system of (3.1) its 10 independent transformations are given by

$$\bar{\mathcal{G}}: T_1, T_2, T_3; R_1, R_2, R_3; \bar{U}; \bar{L}_1, \bar{L}_2, \bar{L}_3, \quad (4.3)$$

using the operator forms of the corresponding infinitesimal transformations, where $T_1 = \partial_x$, $T_2 = \partial_y$ and $T_3 = \partial_z$ give the translations of the space-frame; $R_1 = -z\partial_y + y\partial_z$, $R_2 = -x\partial_z + z\partial_x$ and $R_3 = -y\partial_x + x\partial_y$ give the rotations of the space-frame; $\bar{U} = \partial_t$ gives the translation of the time-frame; and

$$\bar{L}_1 = t\partial_x + x\partial_t, \quad \bar{L}_2 = t\partial_y + y\partial_t, \quad \bar{L}_3 = t\partial_z + z\partial_t \quad (4.4)$$

give the special Lorentz transformations. Finite forms of these transformations are also well known. For later use we shall give the finite forms of \bar{L}_a , ($a=1, 2, 3$):

$$\bar{L}_1: x' = (x + vt) / \sqrt{1 - v^2}, \quad y' = y, \quad z' = z, \quad t' = (t + vx) / \sqrt{1 - v^2}. \quad (4.5)$$

The corresponding forms of \bar{L}_2 and \bar{L}_3 are given from \bar{L}_1 by the cyclic changes of x , y and z .

In $[S]$, the transformations which keep its ds^2 form-invariant (i.e. its motions) also form a group of 10 parameters as in the case of $[M]$ and the concrete forms of the 10

*) We study the problem from the standpoint of the continuous group of transformations so that we do not deal with the transformation which has no infinitesimal transformation such as a reflection $x' = -x$.

independent elements in the coordinate system of (4.2) are given by the following operator forms:*

$$\mathfrak{G}: T_1, T_2, T_3; R_1, R_2, R_3; U_0; S_1, S_2, S_3 \quad (4.6)$$

where T_a and R_a , ($a=1, 2, 3$), are the same as those in \mathfrak{G} , and

$$U_0 = k(x\partial_x + y\partial_y + z\partial_z) - \partial_t, \quad (4.7)$$

$$S_1 = \{e^{-2kt} + k^2(x^2 - y^2 - z^2)\} \partial_x + 2k^2x(y\partial_y + z\partial_z) - 2kx\partial_t, \text{ etc.}$$

Any transformation of \mathfrak{G} is given by a linear combination with constant coefficients of 10 operators given in (4.6). Hence in order to obtain the generalizations of the transformations in (4.3) we have only to determine the linear combinations of (4.6) which tend to the operators of (4.3) respectively. By this method we can easily obtain the following 10 transformations in place of those given by (4.6):

$$\mathfrak{G}: T_1, T_2, T_3; R_1, R_2, R_3; U; L_1, L_2, L_3, \quad (4.8)$$

$$\text{where } U = -U_0 = -k(x\partial_x + y\partial_y + z\partial_z) + \partial_t, \quad (4.9)$$

$$L_1 = \frac{1}{2k}(T_1 - S_1) = \left\{ \frac{1}{2k}(1 - e^{-2kt}) + \frac{k}{2}(y^2 + z^2 - x^2) \right\} \partial_x - kx(y\partial_y + z\partial_z) + x\partial_t, \text{ etc.}^{**} \quad (4.10)$$

Evidently when $k \rightarrow 0$ we have $U \rightarrow \bar{U}$ and $L_a \rightarrow \bar{L}_a$ ($a=1, 2, 3$).

Hence we must conclude that in $[S]$ the transformation I_a defined by (4.10) are the generalized special Lorentz transformations. In the following sections we shall obtain the finite form of I_a which corresponds to (4.5) of \bar{L}_a , and then shall study some of its properties.

U given by (4.9) is the generalization of the translation of the time-frame \bar{U} in $[S]$ and its finite form will be given in the appendix.

§ 5. Finite form of L_1

In this section we shall determine the finite form of L_1 . In order to make clear the correspondence between this process and that of obtaining the finite form (4.5) of \bar{L}_1 in $[M]$, we shall first give a brief description of the latter:

From (4.4) we have the simultaneous ordinary differential equations

$$dx'/d\tau = t', \quad dy'/d\tau = 0, \quad dz'/d\tau = 0, \quad dt'/d\tau = x'. \quad (5.1)$$

Solving (5.1) under the initial condition

$$x' = x, \quad y' = y, \quad z' = z, \quad t' = t \quad \text{for } \tau = 0, \quad (5.2)$$

$$\text{we have } x' = ax + b\tau, \quad y' = y, \quad z' = z, \quad t' = at + bx, \quad (5.3)$$

*) Here we give the results obtained by the present writer.⁽⁷⁾

**) If we use Siba's notations, we have $L_1 = (U_7 - U_8)/2k$. His expressions of U_7 and U_8 were given in the coordinate system of (3.6).⁽⁸⁾

where $\alpha = \cosh \tau$ and $b = \sinh \tau$. Then putting $b/\alpha = \tanh \tau = v$ we get (4.5).

We have only to follow the same process with respect to (4.10) in place of (4.4). From (4.10) we have the following equations corresponding to (5.1):

$$dx'/d\tau = (1/2k)(1 - e^{-2kt'}) + (k/2)(y'^2 + z'^2 - x'^2), \quad (5.4_1)$$

$$dy'/d\tau = -kx'y', \quad dz'/d\tau = -kx'z', \quad dt'/d\tau = x'. \quad (5.4_{2,3,4})$$

From (5.4₂), (5.4₃) and (5.4₄), we have

$$y'/y = z'/z = \exp\{-k(t' - t)\}, \quad (5.5)$$

and from (5.4₄),

$$d^2t'/d\tau^2 + k/2 \cdot dt'/d\tau + Pe^{-2kt'} - 1/2k = 0, \quad (P \equiv 1/2k - k/2 \cdot (y^2 + z^2)e^{2kt}). \quad (5.6)$$

Integrating this we have

$$\left(\frac{dt'}{d\tau}\right)^2 e^{kt'} - \frac{2}{k}Pe^{-kt'} - \frac{1}{k^2}e^{kt'} = e^{kt'}(r^2 - 1/k^2) - 1/k^2 \cdot e^{-kt'} (\equiv Q), \quad (5.7)$$

where $r = \sqrt{x^2 + y^2 + z^2}$, by virtue of (5.2). From (5.7) we can get $dt'/d\tau$ and can integrate this easily.* By using (5.2) and (5.4), we finally obtain

$$\begin{cases} 2Le^{\tau}e^{kt'} = (Le^{\tau} - 1/2 \cdot k^2 Q)^2 - 2kP, & (5.8_1) \\ kx'e^{kt'} = \eta \sqrt{(e^{2kt'} + k^2 Q)e^{kt'} + 2kP}, & (5.8_2) \end{cases}$$

where
$$L = 1/2 \cdot (1 + k^2 r^2)e^{kt} - 1/2 \cdot e^{-kt} + \eta kxe^{kt}, \quad (5.9)$$

and $\eta = \pm 1$. Since the result for $\eta = -1$ is obtained from that for $\eta = +1$ by reversing the orientations of x and x' axes, we shall assume that $\eta = +1$.

(5.5) and (5.8) together give the finite form of L_1 which we are seeking for. In fact we can easily verify that when $k \rightarrow 0$ (5.5) tends to the second and the third equations of (4.5) and that (5.8₁) and (5.8₂) tend to the fourth and the first of (4.5) respectively.

From the mathematical standpoint, the above obtained transformation {(5.5), (5.8)} is somewhat complicated than (4.5) in that y and z are contained in the expressions of x' and t' besides x and t and that y and z are not invariant under the transformation.

§ 6. Finite form of L_1 expressed in a symmetric form

In this section we shall show that we can express the finite form {(5.5), (5.8)} of L_1 obtained in the last section in a more symmetrical form by using three invariants and one relative invariant.

From (5.5) it is easily seen that both

$$A_1 \equiv ye^{kt} \quad \text{and} \quad A_2 \equiv ze^{kt} \quad (6.1)$$

are invariants of L_1 , i.e.

*) $dx'/d\tau$ is determined to within a sign. The result for minus sign, however, becomes identical with that for plus sign by changing the sign of the parameter τ .

$$\Delta_1' \equiv y' e^{kt'} = \Delta_1, \quad \Delta_2' \equiv z' e^{kt'} = \Delta_2. \quad (6.2)$$

Next from (5.8₂) we have

$$e^{2kt'} (k^2 x'^2 - 1) - e^{kt'} \{ e^{kt} (k^2 r^2 - 1) - e^{-kt} \} - 1 + k^2 e^{2kt} (y^2 + z^2) = 0. \quad (6.3)$$

From (6.2) and (6.3), we obtain

$$\Delta_3' = \Delta_3 \text{ where } \Delta_3 \equiv e^{kt} (1 - k^2 r^2) + e^{-kt}, \quad (6.4)$$

and Δ_3' is the corresponding quantity in (x', \dots, t') -system. Hence this Δ_3 is the third invariant of L_1 . Every invariant of L_1 is expressible as a function of these three invariants.

Then using (6.1) and (6.4) we can show that (5.8₁) can be rewritten in the form

$$\Delta_4' = e^\tau \Delta_4 \text{ where } \Delta_4 \equiv 2e^{kt} (1 + kx) - \Delta_3. \quad (6.5)$$

Hence Δ_4 is a relative invariant of L_1 .

Evidently $\Delta_1, \dots, \Delta_4$ are relatively independent, so we can finally express the finite form of L_1 by the equation:

$$L_1: \Delta_1' = \Delta_1, \quad \Delta_2' = \Delta_2, \quad \Delta_3' = \Delta_3, \quad \Delta_4' = e^\tau \Delta_4. \quad (6.6)$$

This is the symmetrical form of L_1 . If we denote (6.6) by $L_1(\tau)$, the identical transformation and the inverse one are given by $L_1(0)$ and $L_1(-\tau)$ respectively and further it holds that:

$$L_1(\tau) L_1(\tau') = L_1(\tau') L_1(\tau) = L_1(\tau + \tau'). \quad (6.7)$$

This is also evident from the theory of the continuous group of transformations.

If we put $k \rightarrow 0$ in order to make clear what quantities in $[M]$ correspond to $\Delta_1, \dots, \Delta_4$ in $[S]$, then we have

$$\Delta_1 \rightarrow y, \quad \Delta_2 \rightarrow z, \quad (\Delta_3 - 2)/k^2 \rightarrow (t^2 - r^2), \quad \Delta_4/k \rightarrow (t + x). \quad (6.8)$$

Evidently the quantities $y, z, (t^2 - r^2); (t + x)$ are three invariants and one relative invariant of \bar{L}_1 respectively.

By the investigation of this section we also know that the finite form (4.5) of the special Lorentz transformation \bar{L}_1 can be rewritten as (6.6) with $\Delta_1 \equiv y, \Delta_2 \equiv z, \Delta_3 \equiv (t^2 - r^2), \Delta_4 \equiv (t + x)$ and $v \equiv \tanh \tau$. As will be seen from the investigation of L_1 in § 8, this form of L_1 is convenient to determine the form of \bar{L}_1 in the coordinate system in which the metric of $[M]$ is not of the form (3.1).

§ 7. Some properties of L_1

Finite forms of \bar{L}_1 obtained in § 5 and § 6 are somewhat complicated compared with that of \bar{L}_1 . In this section we shall obtain some simple properties of L_1 . For brevity's sake we denote the observers whose coordinate systems are (x, y, z, t) and (x', y', z', t') by K and K' respectively. Then main differences between L_1 and \bar{L}_1 are as follows; In

\bar{L}_1 , all points at rest relative to K move uniformly parallel to the x' axis with common constant velocity relative to K' . On the other hand, in L_1 , as is easily seen from (5.5) and (5.8) any point at rest relative to K moves non-uniformly relative to K' on the curves not parallel to the x' axis in general. Since $L_1^{-1}(\tau) = L_1(-\tau)$, the same circumstance holds even if K and K' are interchanged.

The points on the x axis of K are also on the x' axis of K' and both systems have these axes in common. On this common axis, from (5.6), (5.7) and (5.9) we have

$$2kP=1, \quad k^2Q=e^{kt}(k^2x^2-1)-e^{-kt}, \quad 2L=e^{kt}(1+kx)^2-e^{-kt}, \quad (7.1)$$

and the transformation equations for the events on this axis become

$$2Le^{kt'}e^{\tau} = (2Le^{\tau} - k^2Q)^2/4 - 1, \quad kx'e^{kt'} = \sqrt{(e^{2kt'} + k^2Qe^{kt'} + 1)}. \quad (7.2)$$

Now we shall study the motion of the spatial origin O of K relative to K' . O moves on the x' axis of K' . To study the properties of this motion we shall obtain the relation between the coordinates x' and t' of O relative to K' . Substituting $x=0$ into (7.1) we have

$$-1/2 \cdot k^2Q = \cosh kt, \quad L = \sinh kt. \quad (7.3)$$

From (7.2) and (7.3) we have

$$\begin{cases} e^{kt'} = \sinh kt \cosh \tau + \cosh kt, \\ 2 \cosh kt = e^{kt'} + e^{-kt'} - k^2x'^2e^{kt'}. \end{cases} \quad (7.4)$$

Eliminating e^{kt} from (7.4), after some calculations, we finally obtain the equation:

$$kx' = 1/2 \cdot V(1 + k^2x'^2 - e^{-2kt'}), \quad (V \equiv \tanh \tau), \quad (7.5)$$

which gives the motion of O relative to K' .*

When $k \rightarrow 0$, (7.5) becomes $x' = Vt'$ which coincides with the relations obtained from (4.5). Hence V is the parameter corresponding to v of \bar{L}_1 . If we calculate from (7.5) the velocity of O relative to K' , we get

$$v = dx'/dt' = Ve^{-2kt'}(1 - kVx')^{-1}, \quad (7.6)$$

where x' and t' are coordinates of O with respect to K' .** We can rewrite this expression of v by using the relation (7.5), and further by using (7.4) we can also express it as a function of t , the quantity in K , as follows:

$$v = V(1 - k^2T^2)(1 + kT/\sqrt{1 - l'^2})^{-1}(1 + kT\sqrt{1 - l'^2})^{-1}, \quad (7.7)$$

where $kT = \cosh kt$.

Next we shall obtain a generalization of the well-known composition law of velocities concerning \bar{L}_1 . Let K , K' and K'' be three observers relatively connected by

*) Strictly speaking, we must take $\pm kx' = \dots$ in place of (7.5). But we have taken plus sign in order that this equation may coincide with the one obtained from (4.5) when $k \rightarrow 0$. The result for minus sign is identical with the one above obtained if we change the sign of the parameter τ .

**) See Appendix (1).

L_1 . If we assume that they satisfy the relations:

$$K' = L_1(\tau)K, \quad K'' = L_1(\tau')K', \quad (7.8)$$

then from (6.7), we have

$$K'' = L_1(\tau + \tau')K. \quad (7.9)$$

Therefore if we denote the velocities of the spatial origin O of K relative to K' and K'' by v and \bar{v} respectively and that of the spatial origin O' of K' relative to K'' by v' , then by (7.6) we have

$$\begin{aligned} v &= V e^{-2k\tau} (1 - \bar{V} k x')^{-1}, \quad (x' \text{ and } t' \text{ are coordinates of } O \text{ in } K'), \\ v' &= V' e^{-2k\tau'} (1 - V' k x'')^{-1}, \quad (x'' \text{ and } t'' \text{ ,, ,, } O' \text{ in } K''), \\ \bar{v} &= \bar{V} e^{-2k\bar{\tau}} (1 - \bar{V} k \bar{x})^{-1}, \quad (\bar{x} \text{ and } \bar{t} \text{ ,, ,, } O \text{ in } K''), \end{aligned} \quad (7.10)$$

$$\text{where} \quad \bar{V} = \tanh \bar{\tau} = \tanh (\tau + \tau') = (V + V') / (1 + VV'). \quad (7.11)$$

(7.10) and (7.11) give the composition law sought for. When $k \rightarrow 0$ v , v' and \bar{v} tend to V , V' and \bar{V} respectively and (7.11) gives the ordinary composition law of \bar{L}_1 .

§ 8. L_1 in another coordinate system

Hitherto we have made clear the infinitesimal and finite forms of L_1 in the coordinate system of (4.2). From these we can easily obtain the forms of L_1 in another coordinate system. The following method may be the most simple one:

Let*

$$x^i = f^i(x), \quad (|\partial^i x^j / \partial \bar{x}^j| \neq 0), \quad (8.1)$$

be the coordinate transformation from the coordinate system of (4.2) to the new one. If we denote the linear operator corresponding to L_1 by $\hat{\xi}^i \partial / \partial x^i$, then $\bar{\xi}^i$ in the new coordinate system is obtained from the original one by the transformation equation of a contravariant vector:

$$\bar{\xi}^i = \partial^i x^j / \partial x^j \cdot \xi^j. \quad (8.2)$$

Hence the infinitesimal form of L_1 in the new coordinate system is obtained by this method. Next, to obtain the finite form of L_1 in the new system we have only to use the \mathcal{L} 's introduced in § 6, that is, it is given by (6.6) in which the \mathcal{L} 's are those in the new coordinate system.

Now we shall give an example. It is well known that (4.2) is transformed into

$$ds^2 = -(1 - k^2 r^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + (1 - k^2 r^2) dt^2 \quad (8.3)$$

by the transformations:

$$x = r \cdot \sin \theta \cos \phi, \quad y = r \cdot \sin \theta \sin \phi, \quad z = r \cdot \cos \theta, \quad (8.4)$$

*) Throughout this paper we denote the resulting coordinate variables of motion by x'^i and those of the ordinary coordinate transformation which is independent of the motions by x^i .

and

$${}^t r = r e^{kt}, \quad e^{k't'} \sqrt{(1 - k'^2 r'^2)} = e^{kt}. \quad (8.5)$$

In (8.3) primes are omitted for brevity's sake. Further, transforming (8.3) into (x, y, z, t) -system by using (8.4) again, we finally obtain

$$ds^2 = -(dx^2 + dy^2 + dz^2) - k^2 \sigma^{-2} (x dx + y dy + z dz)^2 + \sigma^2 dt^2, \quad (8.6)$$

where $\sigma = \sqrt{(1 - k^2 r^2)}$. We shall determine the forms of L_1 in the coordinate system of (8.6).

By the reason above stated the transformation to (8.6) from (4.2) is given by

$$x e^{kt} = {}^t x, \quad y e^{kt} = {}^t y, \quad z e^{kt} = {}^t z, \quad e^{kt} = e^{k't'} \sqrt{(1 - k'^2 r'^2)}. \quad (8.7)$$

Hence from (8.2) and

$$\begin{aligned} \xi^1 &= (1 - e^{-2kt})/2k + k(y^2 + z^2 - x^2)/2, \\ \xi^2 &= -kx y, \quad \xi^3 = -kx z, \quad \xi^4 = x, \end{aligned} \quad (8.8)$$

which is obtained from (4.10), we have

$$\xi^1 = \sigma/k \cdot \sinh kt, \quad \xi^2 = \xi^3 = 0, \quad \xi^4 = x/\sigma \cdot \cosh kt. \quad (8.9)$$

This (8.9) gives the infinitesimal form of L_1 in the coordinate system of (8.6).^{*} When $k \rightarrow 0$ in (8.9), we have $\xi^1 \rightarrow t$ and $\xi^4 \rightarrow x$ and L_1 tends to the \bar{L}_1 given by (4.4).^{**}

Then we shall determine the finite form of (8.9). From (6.1), (6.4), (6.5) and (8.7), we have

$$A_1 = y, \quad A_2 = z, \quad A_3 = 2\sigma \cosh kt, \quad A_4 = 2(kx + \sigma \sinh kt), \quad (8.10)$$

in the coordinate system of (8.6). Hence the finite form is (6.6) with (8.10). We can also obtain (8.9) from (8.10) by taking τ as an infinitesimal quantity of the first order.

In this coordinate system we can make similar discussions as those made in § 7 with respect to the relative motion of K and K' . For example, the motion and the velocity of O relative to K' is given by

$$k'^2 x'^2 (1 + V^2 \sinh^2 kt') - V^2 \sinh^2 kt' = 0, \quad (8.11)$$

$$\text{and} \quad v = V(1 - k^2 r'^2)^{3/2} \cosh kt' = V(1 - k^2 x'^2) \sqrt{\{1 + k^2 x'^2 (1/V^2 - 1)\}}, \quad (8.12)$$

corresponding to (7.5) and (7.6) respectively.

Appendix

(1) An alternate method of introducing (7.6).

We may use the following method to determine the velocity v of the spatial origin O of K relative to K' : We define v by

^{*}) From (8.9) we have the relation $L_1 = (U_T - U'_K)/2k$ stated in the footnote concerning (4.10).

^{**}) When $k \rightarrow 0$, the transformation (8.7) tends to the identical transformation.

$$v = \left[\frac{dx'}{dt'} \right]_{0, [dx/dt]_0=0} = \left[\left\{ \frac{\partial x'}{\partial t} + \frac{\partial x'}{\partial x} \frac{dx}{dt} \right\} \left\{ \frac{\partial t'}{\partial t} + \frac{\partial t'}{\partial x} \frac{dx}{dt} \right\}^{-1} \right]_{0, [v]_0=0} \\ = \left[\frac{\partial x'}{\partial t} / \frac{dt'}{dt} \right]_0, \quad (\text{A.1})$$

where an index 0 means the value of the parenthesized quantity at $x=y=z=0$. For L_1 defined by (5.5) and (5.8), we have

$$[2kP]_0=1, \quad [L]_0=\sinh kt, \quad -[k^2Q/2]_0=\cosh kt, \\ [\partial/\partial t_0 \cdot 2L]_0=k \cosh kt, \quad [\partial/\partial t \cdot k^2Q]_0=-k \sinh kt, \quad (\text{A.2})$$

using which, from (7.2) we have*

$$[\partial'/\partial t]_0 = \{\cosh kt \cosh \tau + \sinh kt\} \{\sinh kt \cosh \tau + \cosh kt\}^{-1}, \\ [k'x \partial x'/\partial t]_0 = \sinh kt \sinh \tau \{\sinh kt \cosh \tau + \cosh kt\}^{-3}, \quad (\text{A.3}) \\ [k'x]_0 = \sinh kt \sinh \tau \{\sinh kt \cosh \tau + \cosh kt\}^{-1}.$$

From these relations and (7.4) we easily obtain (7.7) and hence (7.6).

(2) Finite form of U .

Finite forms of all elements of \mathfrak{G} except U have been made clear, so we shall here give that of U . Solving the differential equations:

$$\frac{dx'}{d\tau} = -kx', \quad \frac{dy'}{d\tau} = -ky', \quad \frac{dz'}{d\tau} = -kz', \quad \frac{dt'}{d\tau} = 1, \quad (\text{A.4})$$

with the initial condition (5.2), we have

$$x' = e^{-k\tau}x, \quad y' = e^{-k\tau}y, \quad z' = e^{-k\tau}z, \quad t' = t + \tau, \quad (\text{A.5})$$

which is the finite form of U in the coordinate system of (4.2).

Further in the coordinate system of (8.6) its infinitesimal and finite forms are given by ∂_t and $t' = t + \tau$ respectively. Hence it coincides with \bar{U} in (4.3) in this coordinate system.

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*) As to the \pm sign of $k'x$, the same circumstance as in § 7 holds.

On the Wave Theory of Light in General Relativity, I

— Path of Light —

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In the general theory of relativity the problems of optics have been restricted mostly to those concerning the path of light as null geodesic, and the character of light as wave has scarcely been treated. Therefore we intend to investigate the nature of light as wave in general relativity. In this first paper we shall research the relation between the method of wave optics and that of geometrical optics concerning the propagation of light in a curved space-time.

§ 1. Introduction

To treat the optical problems there are two methods, i.e., geometrical and wave optical ones. In the former light is treated as ray and in the latter the nature of light as wave is taken into account. Actually we believe that light is an electromagnetic wave, therefore the characters of light are to be researched consistently by the wave optical method based on the wave equation of the electromagnetic field. The geometrical optics is, however, just a special large scale manifestation of the wave optics. Accordingly the method of geometrical optics has been used when the character of light as ray is concerned, because this method is convenient and approximate in such case.

In the general theory of relativity light is also to be considered as an electromagnetic wave, but the optical problems hitherto treated in the theory have been restricted mostly in those of the geometrical optics. Namely, the problems have been mostly those concerning the path of light as null geodesic, and those concerning the properties of light as an electromagnetic wave have scarcely been taken into consideration. Nevertheless light plays the most important rôle in general relativity, that is, two of the so called three crucial tests of the theory, i.e., the red-shift of the spectral lines and the deflection of light in gravitational field are the problems concerning the properties of light, especially the former relates directly to the wave character of light. And moreover in cosmical problems in which general relativity has found its important applications light is almost only one means with which we can observe the cosmical events, some of which relates directly to the wave properties of light such as the red-shift of the spectral lines of the extragalactic nebulae.

Now we take the point of view that light is an electromagnetic wave in the general theory of relativity, so we consider at first by what equation the electromagnetic fields can

be expressed. Here we generalize the Maxwell equation in Minkowski space-time into a tensor form in Riemannian space-time as is usually done¹⁾, and regard this tensor equation as the field equation of the electromagnetic field in general relativity. Then if the metric of the space-time is given previously we can get the concrete form of the equation. (In this case the electromagnetic term enters in the field equation of g_{ij} ($i, j=1, \dots, 4$), the metric of the space-time, therefore we cannot decide g_{ij} independently of the electromagnetic field strength. But if we are allowed to ignore the effect of the electromagnetic field on the metric, i.e., gravitational field, we can assume that the metric is given previously.)

Now we shall study the wave character of light in a curved space-time from the standpoint stated above. Concerning such problems there have, hitherto, been no researches except some simple ones, in which the concept of the wave length or the frequency of light wave is used only in a formal sense. Therefore at first in this paper the relation between wave theoretical and ray theoretical treatments of the propagation of light will be made clear. That is, the relation between the solution of wave equation of light and the null geodesic as the path of light will be examined.

§ 2. The propagation of light*

As is well known we have the following two ways to study the character of the propagation of light: 1) the one in which the path of light is studied, and ii) the one in which the displacement of the wave front with time is investigated. In the former the path of light is considered as the null geodesic in a four dimensional space-time, then to obtain the path of light we have only to solve the differential equation of the geodesic with the condition that the four dimensional length of the curve is zero.

On the other hand if we consider the light as wave, the character of the propagation of it is represented by the displacement of the wave front with time in three dimensional space, which is also represented by a hypersurface Σ in the four dimensional space-time.** Three dimensionally considered, the path of light as ray is perpendicular to the surfaces of the wave front in three dimensional space. If we consider it in terms of four dimensional space-time this fact is interpreted as follows: a ray of light is on the hypersurface Σ of the wave front and at the same time is orthogonal to the same Σ , because the ray of light is represented by null line.

Next we shall study how to construct the wave front of light, when the ray of light is previously given. Considering from three dimensional point of view if we construct the orthogonal surface elements to the ray of light at every point on it in the sense of the three dimensional space, these surface elements are the wave fronts of the light. But the surfaces

*) The mathematical foundation of this and the following sections will appear in Appendix.

**) Hereafter we shall use the term "three dimensional space", when we lay stress on the distinction of the three dimensional space and the one dimensional time in the four dimensional space-time. But strictly speaking this separation of the space and the time may hold only when the time axis is perpendicular to three dimensional space axes in the four dimensional space-time, i.e., when $g_{14}=g_{24}=g_{34}=0$. See also Appendix.

thus constructed have only infinitesimal areas. Therefore if we want to construct the wave front having the finite area, we must consider the congruence of the rays in three dimensional space, and four dimensionally this congruence is represented as a congruence of ∞^2 null geodesics on the hypersurface of the wave front Σ . If we can construct a surface to which the congruence of the rays in three dimensional space is normal, then these surfaces (four dimensionally the hypersurface Σ) are nothing but the wave front of light. But it is clear that arbitrarily given rays of null lines are not necessarily normal. Therefore we shall research the conditions that such a congruence be normal, and show how to research the wave character of light using the concept of the geometrical optics. We shall also show how to derive the concept of the ray of light from the solution of wave equation of light.

§ 3. Relation between the wave front and the path of light

In this section we shall investigate in detail the relation between the wave front and the path of light. At first we shall consider this relation in the special theory of relativity.

(a) In the special theory of relativity*

For brevity's sake we shall here study the problem using the Young-Fresnel's wave equation for the scalar wave instead of the Maxwell equation for four-vector potential corresponding to the pure radiation field, because the form of the latter equation for each component is the same as that of the former. Taking the cartesian coordinate system in Minkowski space-time, we have

$$g_{ab} = -1, \quad (a=b); \quad g_{ab} = 0, \quad (a \neq b); \quad g_{a1} = g_{1a} = 0, \quad g_{44} = 1, \quad (a, b = 1, 2, 3),$$

and the wave equation takes the form :

$$\square \psi \equiv g^{ij} \psi_{,ij} = \left\{ \frac{\partial^2}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right\} \psi = 0, \quad (i, j = 1, \dots, 4) \quad (x, y, z, t \equiv x^1, x^2, x^3, x^4), \quad (3.1)$$

where $_{,i} = \partial / \partial x^i$. The characters of propagation of light wave can be represented by the motion of the surface with time on which some kind of the discontinuity of the solution of this differential equation appears.

Let

$$\Sigma : \varphi(x, y, z, t) = 0 \quad (3.2)$$

be the four dimensional expression of the above stated moving surface of discontinuity. This hypersurface is the Σ treated in the last section and the following relation holds on this hypersurface :

$$g^{ij} \varphi_{,i} \varphi_{,j} = 0 \quad \text{that is} \quad -(\partial_x \varphi)^2 - (\partial_y \varphi)^2 - (\partial_z \varphi)^2 + (\partial_t \varphi)^2 = 0, \quad \text{on } \varphi = 0. \quad (3.3)$$

*) The content of this subsection (a) is well known,²⁾ but we mention it here to compare it with that of the next subsection (b).

The hypersurface Σ is called the characteristic manifold of (3.1) and, three dimensionally speaking, it represents the motion of the wave in three dimensional space with time. And the characteristic manifold (3.2) can be embedded in a family of manifold, $\varphi = \text{const.}$, where φ is a solution of the differential equation $g^{ij}\varphi_{,i}\varphi_{,j} = 0$. (without the additional condition "on $\varphi = 0$ ".)

Now let us consider the congruence of the curves defined by the vector field $\varphi_i (= \partial_i \varphi)$ obtained from the above φ . The curves are given by solving the following equations:

$$dx^i(s)/ds = g^{ij}\varphi_j, \text{ or } -\varphi_x = dx/ds, -\varphi_y = dy/ds, -\varphi_z = dz/ds, +\varphi_t = dt/ds$$

where $\varphi_x = \partial\varphi/\partial x, \dots (3.4)$

This is called the characteristic curves of (3.1) and in this case it is clear that this congruence gives the null straight lines. These straight lines are all contained in the hypersurface $\varphi = \text{const.}$ and at the same time normal to it. Among these curves those which are contained in (3.2) give the ∞^2 null geodesics on $\varphi = 0$ in four dimensional sense, and three dimensionally considered, these curves represent the fact that light proceeds with the unit velocity on every straight line normal to the surface representing the wave front of light.

Here it must be remarked that the equation (3.4) represents the congruence of null lines instead of a single line. Therefore if we want to treat the path of light in connection with a wave front, it is not sufficient to take into account only one single path of light, but we must treat the congruence of ∞^2 null geodesics on a characteristic manifold in four dimensional sense, or three dimensionally, we must deal with the congruence of the straight lines orthogonal to the wave front. This congruence is embedded in a four dimensional congruence defined by (3.4). The hypersurface Σ i.e. $\varphi = 0$ expresses the propagation of the light wave having the definite initial condition: the value of the initial phase of the wave. Then the remaining members of the hypersurfaces $\varphi = \text{const.}$ express the propagation of the same light with another initial phases. And the null geodesic defined by (3.4) which are not on Σ give the paths of this light. Accordingly, if we ignore the initial conditions, we can consider that a family of hypersurfaces and a congruence of null geodesics normal to it in the four dimensional space-time express a light in this sense.

(b) In the general theory of relativity

In this case if the wave equation of light is given, we can easily find similar relations between the wave front and the path of light. But here we shall take another way. Since the same circumstance as stated above can also be considered in a curved space-time, we assume that a congruence of null geodesics which represents the paths of light is given previously, and that there exists a family of hypersurfaces representing the wave front of light and orthogonal to the paths of light. Then we shall search for the possible form of the wave equation of light using the condition that the characteristic curves of it coincide with the given paths of light.

Now we shall take up the above problem mathematically. Let $\varphi(x, y, z, t) = \text{const.}$ be the family of hypersurfaces representing the wave fronts of light. Since the paths of light are null and normal to these hypersurfaces, then a path of light $x^i = x^i(s)$ is given by

$$\varphi^i = dx^i/ds, \quad (\varphi^i = g^{ij}\varphi_j, \quad \varphi_j = \varphi_{,j} = \partial\varphi/\partial x^j), \quad (3.5)$$

where a comma denotes covariant derivative.* Along the path of light we have $g_{ij} dx^i dx^j = 0$ from which we have

$$g^{ij}\varphi_i\varphi_j = g^{ij}\varphi^i\varphi^j = 0. \quad (3.6)$$

Since each hypersurface of this family is to be the characteristic surface of the wave equation, the equation must be of the form^{†)}

$$g^{ij}\partial_i\partial_j\psi + \bar{d} = 0 \quad \text{or} \quad g^{ij}\psi_{,ij} + \bar{d} = 0 \quad (3.7)$$

under the assumption that this differential equation is quasi-linear and second order for the function ψ . In (3.7) second partial derivatives of ψ are not contained in d or \bar{d} , and furthermore it seems natural to take this d or \bar{d} as tensor quantity in accordance with the principle of covariance. Here it is to be remarked that both of the following tensor equations for the scalar and vector wave

$$\square\psi \equiv g^{ij}\psi_{,ij} = 0 \quad \text{and} \quad \square\psi_\alpha \equiv g^{ij}\psi_{\alpha,ij} = 0, \quad (3.8)$$

which are obtained by generalizing the corresponding wave equations in Minkowski space-time are the form (3.7).**

Conversely the characteristic curves of (3.7) or (3.8) are null geodesics,*** then if we take (3.8) as the wave equation of light, it is clear that the relations between light rays and wave fronts of light are the same as those in special relativity. Thus the fact that the path of light is null geodesic, can be represented wave theoretically by the characteristic curves of the wave equation of the type (3.7). Namely, it has become clear that the paths of light constructed by the wave motion determined by the wave equation of the type (3.7) form a congruence of the null normal curves in the space-time whose metric is given by g_{ij} and that this congruence is geodesic. In other words, paths of light are represented by a null normal vector field φ^i in a four dimensional space-time.

Thus the ray character of light as well as its wave character is represented by the wave equation of the type (3.7). In the next section we shall clarify the conditions that an arbitrarily given congruence of null geodesics represents the wave front as the characteristic manifold of the wave equation.

§ 4. Relation between wave optics and ray optics

By the above consideration we can conclude that it is possible to describe the characters of light consistently from the wave theoretical point of view. Namely, it is shown

*) Since φ is a scalar, we have $\varphi_{,i} = \partial_i\varphi$.

**) We can easily show that Maxwell equation for four potential usually used in general relativity is also of the form (3.7).¹⁾

**) As is proved in Appendix, a null normal congruence is geodesic.

that the main feature of the ray optics, "the path of light is null geodesic", can be completely taken into the theory of wave optics by using the nature of the wave equation. In this section we shall give the conditions that the congruence of null geodesics corresponds to the wave fronts of light in the sense stated in the last section.

When a congruence of null geodesics is given corresponding to the paths of light, there exists a hypersurface Σ normal to this congruence. This hypersurface is nothing but the expression of the wave front of light. But an arbitrarily given congruence of null geodesics is not normal. Therefore we shall give the conditions that the congruence of null geodesics be normal to a hypersurface which corresponds to the wave fronts of light. These conditions are as follows: (We state here the results only, proof will appear in Appendix.)

Let a congruence of curves defined by φ^i be null and geodesic. Then the mathematical expressions of these properties are as follows:

$$\text{i) The condition null: } \varphi_i \varphi^i = 0. \quad (4.1)$$

$$\text{ii) The condition geodesic: } \varphi^i \varphi^j_{;i} = \sigma \varphi^j, \quad (4.2)$$

where σ is a scalar.

Then a necessary and sufficient condition that this φ^i be normal is given by

$$\text{iii) } P^{ij} P_{ij} = 0, \quad (P_{ij} \equiv \varphi_{[i, j]}),$$

when it is normalized. The term "normalize" means that we make the proportional factor σ of φ^j in (4.2) zero by multiplying φ^i suitably.

Thus using this condition iii) we can examine whether an arbitrarily given congruence of null geodesics represents the propagation character of light. And conversely if there is a congruence of null geodesics which satisfy this condition iii), this congruence represents the wave front of light.

§ 5. Conclusion

In this paper we have shown that in general relativity light may be treated from the wave theoretical point of view, and we have investigated the character of propagation of light in a curved space-time. As a result it has been shown that the paths of light can be considered as null geodesics from the wave theoretical point of view taking the wave equation either of Maxwell or of Young-Fresnel. And conversely it has been shown that the general form of the wave equation having the wave front which corresponds to the propagation of light is of the type (3.7). We can derive the paths of light as a congruence of rays normal to the wave front obtained from the wave equation. Using this fact we have studied the relation between wave theoretical treatment and ray optical one concerning the paths of light.

Main phenomena treated in general relativity have been those which occurred in the large scale domain compared with the wave length of light. This is one of the reasons

why in general relativity light has been treated in almost every case ignoring its wave character and yet many successes were attained. And it is shown that the usual method treating the light as ray is allowed when we consider only the single path of light. However in the cases in which the wave characters of light play an important rôle, we must take the wave theoretical treatment. In the following papers we shall treat such problems further.

The writer is grateful to Prof. Y. Mimura and Prof. H. Takeno for their helpful discussions.

APPENDIX

Mathematical research of null congruence in a four dimensional curved space-time

Hyôitirô TAKENO and Yoshio UENO

Here we shall give some mathematical properties of null congruence in a curved space-time. A contravariant vector field v^i in a space-time defines a congruence of curves. Especially when v^i is null the congruence becomes a congruence of null curves and we shall call this a null congruence. The congruence determined by v^i is called normal when its curves are orthogonal trajectories of a family of hypersurfaces $f(x) = \text{const.}$ ⁴⁾ In other words a congruence is normal when and only when there exist two scalar functions μ and f satisfying $v_i = \mu \partial f / \partial x^i$. When v^i is null and normal the curves of the congruence lie in the hypersurface $f = \text{const.}$ (See Lemma 6). A necessary and sufficient condition that the congruence be geodesic is given by $v^j v^i_{,j} = \nu v^i$, where a comma denotes covariant differentiation and ν is a suitable scalar. From these we obtain the following theorem:

Lemma 1. *Normal null congruence is geodesic.*

Proof. From the assumption we have

$$v^i v_i = 0, \quad (1)$$

$$v_i = \mu f_{,i} \quad (\mu \neq 0, f_{,i} = \partial f / \partial x^i). \quad (2)$$

From these equations we easily obtain

$$v^i v_{j,i} = (v^i \mu_{,i}) f_{,j} = \{v^i (\log \mu)_{,i}\} v_j. \quad (3)$$

Hence the congruence is geodesic.

From the above proof it is evident that a non-null normal congruence is not necessarily geodesic.

If the congruence given by v^i is geodesic it holds that $v^i v^j_{,i} = \nu v^j$ and if we put $\bar{v}^i = \sigma v^i$ where σ is any solution of $\nu + v^i (\log \sigma)_{,i} = 0$, then we have $\bar{v}^j \bar{v}^i_{,j} = \sigma v^i (\sigma \nu +$

$\sigma_{,j} v^j = 0$. We shall call a geodesic congruence whose $\nu=0$ *normalized*. The properties "null" and "normal" are kept invariant by this normalization. Then we have

Lemma 2. If v^i defines a normalized normal null geodesic congruence, the following relations hold:

$$(i) \quad \epsilon^{ijkl} P_{ij} P_{lm} = 0, \quad (4)$$

$$(ii) \quad P_{ij} P^{ij} = 0, \quad (5)$$

where ϵ^{ijkl} is the ϵ -tensor and $P_{ij} = v_{[i} f_{j]}$.

Proof. From $v^j v_{,j} = 0$ and $v^j f_{,j} = 0$, we have

$$v^j P_{ij} = 0 \quad (6)$$

from which we have $|P_{ij}| = (P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23})^2 = 0$, i.e., (4) is obtained, where $|P_{ij}|$ is the determinant of P_{ij} . Next from (2) we have $P_{ij} = \mu_{[ij} f_{k]}$ so $v_{[k} P_{ij]} = 0$, i.e.

$$P_{ij} v_k + P_{jk} v_i + P_{ki} v_j = 0. \quad (7)$$

From this and (6) we have (5).

From Lemmas 1 and 2 we know that any normal null congruence is geodesic and by normalization (4) and (5) hold. In order to obtain the condition that a null geodesic congruence be normal we prove the following lemmas.

Lemma 3. A necessary and sufficient condition that an antisymmetric tensor q_{ij} be simple is given by

$$\epsilon^{ijkl} q_{ij} q_{lm} = 0. \quad (8)$$

Proof. If we put $q_{ij} = u_{[i} \beta_{j]}$ it is evident that the condition is necessary. Next assuming $q_{12} \neq 0$ for instance, if we take $\beta_3 = -(\beta_1 q_{23} + \beta_2 q_{31})/q_{12}$, $\beta_4 = -(\beta_1 q_{24} + \beta_2 q_{41})/q_{12}$, $u_2 = (\beta_2 u_1 - 2q_{12})/\beta_1$, $u_3 = (\beta_3 u_1 - 2q_{13})/\beta_1$ and $u_4 = (\beta_4 u_1 - 2q_{14})/\beta_1$ where $\beta_1 (\neq 0)$, β_2 and u_1 are arbitrary, these u_i and β_i satisfy $q_{ij} = u_{[i} \beta_{j]}$. From this the theorem is obvious.

Lemma 4. If $q_{ij} = u_{[i} \beta_{j]}$ and u_i or β_i is non-null vector there exist \bar{u}_i and $\bar{\beta}_i$ satisfying $q_{ij} = \bar{u}_{[i} \bar{\beta}_{j]}$ and $\bar{u}_i \bar{\beta}^i = 0$.

Proof. If $u^i \beta_i = a \neq 0$ and $\beta^i \beta_i = b \neq 0$ we have only to put $\bar{u}_i = u_i - a/b \cdot \beta_i$ and $\bar{\beta}_i = \beta_i$.

Lemma 5. When $q_{ij} = u_{[i} \beta_{j]}$ and $u_i \beta^i = 0$, $q_{ij} q^{ij} = 0$ and the condition that at least one of u^i and β^i be null is equivalent.

Proof. The theorem is evident from $2q_{ij} q^{ij} = (u_i u^i)(\beta_j \beta^j)$.

Lemma 6. If v^i and \bar{v}^i are two null vector fields mutually orthogonal in the space-time whose signature of the fundamental tensor g_{ij} is -2 , it holds that $\bar{v}^i = \rho v^i$.

Proof. Take a coordinate system in which $ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$ holds locally in any point of the space-time. Then the fourth components of v^i and \bar{v}^i cannot be zero in that point. Hence by putting $\bar{v}^i = \rho v^i$, we can easily obtain $\sum_{a=1}^4 (\bar{v}_a - \rho v_a)^2 = 0$. From which we have $\bar{v}_i = \rho v_i$ in any coordinate system and the theorem is evident.

By the above consideration we finally obtain

Theorem. In a space-time whose signature of g_{ij} is -2 , a necessary and sufficient condition that a normalized null geodesic congruence defined by v^i be normal is given by

$$P_{ij} P^{ij} = 0. \quad (9)$$

Proof. From Lemma 2, the necessity is evident. Next, as in Lemma 2, we can prove (4) from (6). Hence we have $P_{ij} = u_{[i} \beta_{j]}$ by Lemma 3. If we assume that $P_{ij} \neq 0$, (when $P_{ij} = 0$, v^i is normal evidently), then u_i and β_i are linearly independent and from $v^i P_{ij} = 0$ we have

$$(v^i u_i) \beta_j - (v^i \beta_i) u_j = 0$$

from which we have $v^i u_i = v^i \beta_i = 0$. Then we can assume that at least one of u_i and β_i is null by virtue of Lemmas 4 and 5. If β_i is null we have $\beta_i = \kappa v_i$ by Lemma 6. Hence (7) holds and v^i becomes normal. Hence the theorem is proved.

If v^i is not normalized and $v^i v^j_{;i} = \kappa v^j$, the condition (9) becomes $P_{ij} P^{ij} = 2\kappa^2$.

Remark. In Minkowski space-time if we take the coordinate system in which $ds^2 = -dx^2 - dy^2 - dz^2 + dt^2$ holds the vector field v^i given by $v^i = (\cos u, \sin u, 0, 1)$ where $u = u(z)$ determines a normalized null geodesic congruence which does not satisfy (9). Hence this congruence is an example of non-normal normalized null geodesic congruence.

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On the Spin of the μ -meson

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The possibility to determine the spin of the μ -meson is shown in the study of the anomalous $\pi-\mu$ decay. This possibility is due to the fact that the ratio $R(=W_r/W_0)$ of the probability of the radiative process $W_r(\pi\rightarrow\mu+\mu_0+\gamma)$ to that of the ordinary one $W_0(\pi\rightarrow\mu+\mu_0)$ for a meson of spin $1/2$ differs from that for the meson of spin 0 . Especially, we are interested in the range of the μ -meson track shorter than 150μ . Because, for such a short μ -track, the contribution from the magnetic dipole of the μ -meson is more effective than the electric current contribution. The above mentioned R (range $< 150\mu$) is $1/4 \times 10^{-4}$ for a muon of spin $1/2$ and $1/13 \times 10^{-4}$ for a spin 0 muon. It is shown that the straggling effect does not disturb the determination of the spin. The ratio has been measured by Fry et al. Of the sampling of 10636 events in the photographic plate, it was found that the μ -track having the range shorter than 150μ is only one. It is desirable to increase the accuracy and the statistical weight of the observation in order that a more precise comparison between theory and experiment may be possible.

§ 1. Introduction

During the past several years, informations on the properties of the μ -meson have been step by step accumulated. But, so far, the knowledges on the μ -meson spin has not been obtained directly. As is well known, the various decay and absorption phenomena in which the μ -meson participates suggest indirectly spin $1/2$ for the μ -meson*. Since, its spin has an important significance to the idea of "the universal Fermi interaction", it should be desirable to inquire whether there is any other type of experiment which yield a more direct determination of the μ -meson spin. Now, it seems possible to decide its spin, in view of the recent considerable progress in the experimental accuracy. In this note, we point out a possibility to determine the μ -meson spin. Our method makes use of the fact that the ratio $R(=W_r/W_0)$ of the probability of the radiative decay $W_r(\pi\rightarrow\mu+\mu_0+\gamma)$ to that of the ordinary one $W_0(\pi\rightarrow\mu+\mu_0)$ for a meson of spin $1/2$ differs from that of a meson of spin 0 . The second section is devoted to a discussion of this difference, in which all possible couplings accompanied with the spin 0 μ -meson are considered. The third section is concerned with the ambiguity coming from the straggling, and it is shown that this ambiguity makes no trouble to determine the μ -meson spin. From these discussion, it seems possible that a careful detection of the anomalous short μ -track would distinguish between the ratio of spin $1/2$ and 0 meson.

§ 2. Short μ -track in $\pi-\mu$ decay

One of the important properties of the π -meson was the possibility of disintegration

* The spin more than 1 was excluded by the consideration of meson burst production, by Christy-Kusaka.

with the emission of a μ -meson having a definite momentum. Indeed, the distribution in range of μ -particles formed by the $\pi-\mu$ decay in Ilford C2 photographic emulsion gave a sharp maximum at about 600 microns, the width of the peak was attributed to the straggling effect. As is well known, this is due to the fact that the emission of the μ -meson accompanies the ejection of a single neutral particle μ_0 in the $\pi-\mu$ decay. The question on the nature of this particle immediately arises: whether it is neutrino or a neutral particle of spin 0. Anyhow, the decay scheme is expressed as follow,

$$\pi \rightarrow \mu + \mu_0. \quad (1)$$

Recently Fry¹⁾ and others found a few cases of very short μ -track in the $\pi-\mu$ decay, which could not be explained by straggling or decay in flight. The possibility of explanation of this phenomena was discussed by Nakano et al, Primakoff, Eguchi and Fiahlo et al²⁾. After all, if we do not demand new unknown particles, the appearance of such low energy μ -meson in the $\pi-\mu$ decay is explained by the radiative process, such as,

$$\pi \rightarrow \mu + \mu_0 + \gamma. \quad (2)$$

The ratio (2) to (1) was computed by the above authors assuming μ -meson spin to be 1/2 and obtained to be about 10^{-4} . This must be compared with Fry's data $(2.8 \pm 1.2) 10^{-4}$.

We calculated the ratio $R = W_r/W_0$ for both cases of the μ -meson spin 0 and 1/2, where W_r and W_0 are the probabilities of the radiative and the ordinary decay process, respectively.

When we assume that the μ -meson is Fermion*, our check is identical with the one given by Fiahlo-Tiomno, and is

$$R \simeq 1.3 \times 10^{-4} **.$$

When we assume that the μ -meson spin is 0, we cannot attach anomalous magnetic moment, but can take the interaction arbitrary. Indeed, we can take the higher derivatives in the $\pi-\mu$ decay interaction, such as

$$\partial_\mu \cdots \partial_\nu \cdots \phi \cdot \partial_\mu \cdots \partial_\nu \cdots \phi \cdot \partial_\nu \cdots \partial_\nu \cdots \phi. \quad (3)$$

In the spin 0 meson theory with the derivative couplings, it should be noted that the radiative process is the amalgam of two types of transition. One of which is the so called "photo-electric" transition and the other results from the requirement of gauge invariance for the theory.

* In this case however the coupling $(\varphi \gamma_5 \phi) \psi$ was used, the equivalent theorem leads $\gamma_5 \gamma_\mu \rightarrow \gamma_5 \frac{\mu + \mu_0}{M}$ for PV coupling, the interaction with derivatives is kept away from our consideration because of the same reason as the difficulty of Konopinsky-Uhlenbeck theory.

** This result is somewhat smaller than the experimental one. Fiahlo-Tiomno already, proposed²⁾ the possibility that the μ -meson possesses anomalous magnetic moment. There is a different opinion on this point. Hasegawa and Ueda, Soryushi-ron Kenkyu Vol. 3, No. 6 (1951). (mimeographed circular in Japanese).

The various ratios coming from the above stated interactions which may be possible when μ -meson is Bose, are classified as follows.

R^B : The case when the interaction contains no derivatives.

R_∂^B : The case in which, the interaction contains derivatives such as (3), and the photon is emitted by the second order process.

$R_{\partial(cat.)}^B$: If the coupling (3) exists, the "catastrophic"³⁾ transition necessarily occurs by the requirement of gauge invariance.

Now, let us try to estimate R^B , R_∂^B and $R_{\partial(cat.)}^B$ to see the difference between R^B and them. It is useful to compare previously the values of R^B , R_∂^B and $R_{\partial(cat.)}^B$.

(1). First of all, we compare R^B and R_∂^B . The interaction (3) is transformed by the partial integral into

$$\psi \underbrace{\partial_\mu \cdots \phi}_n \cdot \partial_\mu \cdots \varphi \quad (3')$$

It can be shown that if the transition proceeds through the second order process by (3')

$$R^B : R_\partial^B \simeq \left| \frac{eg\phi \frac{(ep')}{(p'k)} \varphi\psi}{eg\phi\varphi\psi} \right|^2 : \left| \frac{eg\phi \frac{(ep')}{(p'k)} \varphi\psi}{eg\phi\varphi\psi} \cdot \left\{ \frac{(q', p'+k)}{(pq)} \right\}^n \right|^2$$

where $p^\pi = p + q$, $p^\pi = p' + q' + k^*$. The conservation laws require $((p' + k, q') - (pq)) = -(p'k)$. We then have

$$(p' + k, q') / (pq) < 1.$$

So, we obtain

$$R^B > R_\partial^B.$$

(2). From the requirement of gauge invariance, the interaction with the electromagnetic field becomes as

$$\psi(\partial - ieA)_\mu(\partial - ieA)_\nu \cdots \phi \cdot \partial_\mu \cdots \varphi$$

the first order terms in A are selected** as

$$\psi \sum_{\mu perm} A_\mu \sigma_{\nu\tau} \cdots \phi \underbrace{\sigma_{\mu\nu\tau} \cdots \varphi}_n.$$

Thus we obtain

$$R^B : R_{\partial(cat.)}^B \simeq \left| \frac{eg\phi \frac{(ep')}{(p'k)} \varphi\psi}{eg\phi\varphi\psi} \right|^2 : \left| \frac{eg(ep')\phi\phi\varphi}{eg\phi\varphi\psi(pq)} \cdot n \frac{(q', p'+k)^{n-1}}{(pq)^{n-1}} \right|^2.$$

We further simplify the equation bearing in mind $|(ep')| = |(eq')|$

$$R^B : R_{\partial(cat.)}^B \simeq 1 : \left| \frac{(p'k)}{(pq)} \cdot n \frac{(q', p'+k)^{n-1}}{(pq)^{n-1}} \right|^2$$

* p^π , p , q , k are respectively π , μ , μ_0 and photon 4-momenta.

** The higher order terms in A decrease in the rate of $1/137$,

We pick up the $(p'k)/(pq)$ and $(q', p' + k)/(pq)$, and introduce the new notations A and B as follows,

$$\frac{(p', k)}{(p, q)} < \frac{1}{M^2 - \mu^2} \cdot \frac{M^2 + \mu^2 - 2ME_\mu}{M - E_\mu} \cdot E_\mu \equiv A,$$

$$\frac{(q', p' + k)}{(p, q)} < \frac{2M}{M^2 - \mu^2} \left(M - E_\mu - \frac{M^2 + \mu^2 - 2ME_\mu}{2(M - E_\mu + \sqrt{E_\mu^2 - \mu^2})} \right) \equiv B.$$

Here M and μ are respectively the mass of π -meson and μ -meson, and E_μ is μ -meson energy. A and B is plotted in Fig. 1. On the other hand,

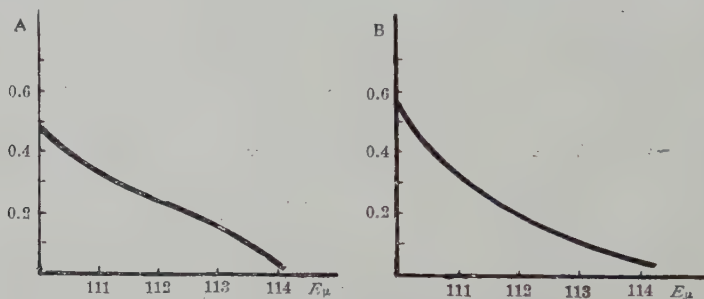


Fig. 1. Variation with μ -meson energy of A and B .

Both A and B has a upper limit < 1 .

the function na^{n-1} ($n=1, 2, \dots$) can not be over 2 if $a < 2/3$, so we have

$$\left| \frac{(p'k)}{(pq)} \cdot n \frac{(q', p' + k)^{n-1}}{(pq)^{n-1}} \right| > 1$$

therefore

$$R^n > R_0^n \text{ (at)}.$$

After all, if μ -meson is assumed Boson, the maximum ratio is R^B .

Fig. 2 gives the $R^n(E_\mu)$ and $R^F(E_\mu)$ referring respectively to spin 0 and $1/2$ μ -meson. It can be seen that

$$R^F(E_\mu) > R^n(E_\mu) \quad (4)$$

and it should be noted that this inequality can not be altered in spite of the arbitrariness causing from the spin 0 μ -meson.

The formula (4) results from the spin flipping of spin $1/2$ meson. Spin flipping is prominent in the domain in which the emitted μ -meson energy is smaller than 2Mev and the prong is shorter than 150μ .

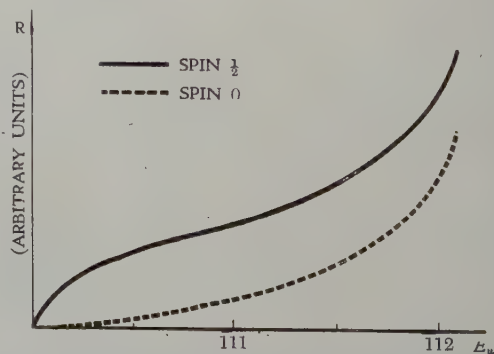


Fig. 2. The ratio of the probability of the radiative π - μ process to that of the ordinary one.

§ 3. Range straggling

Any physical effect which contains the accumulation of the effects of repeated chance events of the same or similar character leads to a dispersion. So, the expected range in emulsion of a μ -meson of a definite energy is not a precise quantities but exhibits a dispersion around the mean range.

Recently, Lewis⁴⁾ shows that the range distribution has a tail in the direction of shorter-than-average range.

It is important to reconsider how many short μ -tracks are attributed to the straggling effect, so we shall touch on this point. If we use Ilford C2 photographic plate, the deviation of the μ -meson from the most probable range (594 μ) is 23 μ . The range distribution of nonrelativistic charged particle is given by Lewis as

$$P(E, y) = h(y) \exp(-y^2/2) \quad (5)$$

where y is the range in unit of standard deviation. This function has a pronounced tail toward shorter than average range. The factor $h(y)$ is shown in Fig. 3. It looks from the figure that the gaussian distribution is deformed, toward the shorter range, but this deformation is practically not so serious.

The probability for the range is given

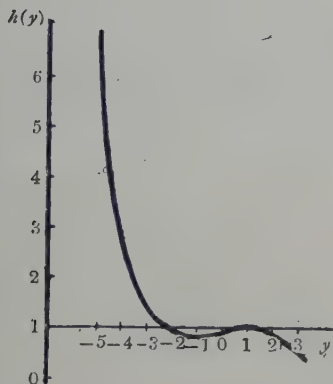


Fig. 3. Relation between $h(y)$ and the standard deviation y .

$$W_r(x) = \int W_r(E) \cdot P(E, x) dE$$

and $W_r(x)/W_0 (= \int P(E, 594) dE)$ is normalized to fit with the Fry's experimental result, W_0 increases rapidly from 480 μ as is shown in Fig. 4. The μ -meson drawing 480 μ prong has the energy about 3.7 Mev, so that the domain in which we are interested is free from the straggling.

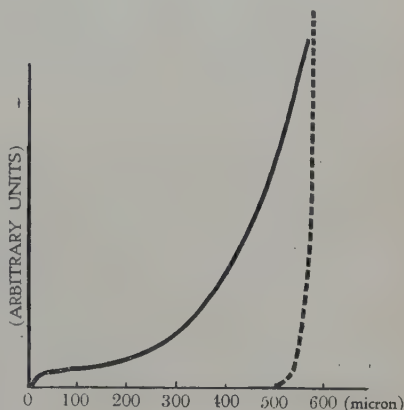


Fig. 4. Range distribution of the anomalous short μ -track (*spin* 1/2), the dotted curve corresponds to the distribution of the μ -meson produced by the ordinary π - μ decay.

§ 4. Conclusion

Now, we have investigated the frequency of the anomalous short μ -track among the ordinary $\pi-\mu$ decay, in the case of μ is Fermi (spin $1/2$) and Bose (spin 0). Especially, we are interested in the range of the μ -meson shorter than 150μ . Because, for such a short μ -track, the contribution from the magnetic dipole of the μ -meson is more effective than the electric current contribution.

The anomalous short μ -track having the range shorter than 150μ , appears once in 13×10^4 ordinary $\pi-\mu$ events for the spin 0 meson, while 4×10^4 for the spin $1/2$ meson. According to the data which have been accumulated¹⁾, the μ -meson having the range shorter than 150μ has been found only one (120μ) in 10686 $\pi-\mu$ decay. So, the chance has not yet arrived to assert its spin. It is important to increase the accuracy, and statistical weight of the observation in order that a more precise comparison between theory and experiment may be possible. After this work was accomplished, Bulletin S. 6, p. 56 (1953) came to our attention which dealt with the measurability of the energy level difference of the μ -mesic atom.

The author is deeply indebted to Professor S. Sakata for his interest, and Messrs. S. Ogawa and H. Okonogi for their helpful discussions.

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On the Baryon-meson-photon System*

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§ 1. General considerations

The last six years have seen a great advance in our understanding of the structure of relativistic field theories through the renormalization program, and at the same time a vastly increased complexity in the observed number and properties of particles which these theories purport to describe. Attempts to come to a better understanding of the existence and properties of these particles by means of a further analysis of the formal possibilities inherent in current theory have had limited success. It is quite clear that much work remains to be done in this direction, especially as regards the description of strongly coupled systems. On the other hand there emerge from the present picture a number of qualitative features which are not logically founded in the premises of the theory as it stands. Parallel with the line of approach just mentioned one may, therefore, ask whether and, if so, how the frame-work of description itself should be enlarged so as to give a rational account of these properties.

In a previous paper¹⁾ (quoted below as I) the following such qualitative questions have been raised and discussed:

- 1) The possibility to have an irreducible wave equation yielding proton and neutron as eigenstates.
- 2) The possibility to incorporate charge independence rationally in our present theories.
- 3) The relation between the newly discovered V_1 -particles and the nucleons.
- 4) The striking stability properties of the V_1 .
- 5) The possibility to derive conservation of heavy particles from first principles.

Experiment tells us that we can no longer talk about conservation of nucleons only but that by heavy particles one has to understand the totality of at least nucleons and V_1 -particles. Without prejudging on the actual nature of the relationship between the V_1 and the nucleon it seems practical to have a collective name for these particles and other which possibly may still be discovered and which may also have to be taken along in the conservation principle just mentioned. It is proposed to use the fitting name "baryon" for this purpose.

In attempts to find unifying views on such questions one can at the present only

* The main content of this paper was presented at the Kyoto Conference. September 1953 and at other lectures given in Japan.

concentrate on general qualitative aspects. In particular it is clear that any guess at a possible mass spectrum of fundamental particles may be as well founded as a divining of a complete atomic spectrum from a few of its lines. Rather it seems more profitable to focus attention on questions connected with the electric charge, where much simplicity exists. In particular one is tempted to inquire whether there is more to the isotopic spin of the nucleon than just a convenient shorthand. Accordingly it was attempted in I to find a way of incorporating the isotopic spin into the foundations of the theory. However, as it happened, the way of approach followed led to an intimate relationship between quantization of charge and of mass.

It has been pointed out in I that a possible way of introducing the isotopic spin in an irreducible manner consists in an appropriate extension of the underlying space time description. The basic idea is to consider the isotopic spin space of the nucleon as a spin $1/2$ representation space of a three-dimensional orthogonal group R_ω acting on a new space, called ω -space. ω -space is defined as the continuous manifold that is carried into itself by any transformation of this group. Thus in ω -space direction is defined, but displacement or distance are not.

As a result, a new vector operator \mathbf{K} is introduced in the formalism. \mathbf{K} (K_1, K_2, K_3) generates the infinitesimal rotations in ω -space and \mathbf{K} takes its place along with the isotopic spin τ in the same way as the ordinary three-dimensional angular momentum operator relates to the ordinary spin.

As a first step in a program of enlargement of the total framework of description we take as a model the direct product of space-time and ω -space. Correspondingly, the group of the ensuing six-dimensional manifold is the direct product of the Lorentz group and R_ω . Now we generalize the Dirac equation for a proton or neutron described by a four component spinor to a "baryon equation" for a wave function $\psi_{\sigma\tau}(x, \omega)$ $\sigma=1, \dots, 4$, $\tau=1, 2$, where ψ is a four component spinor with respect to x -space and a two component spinor with respect to ω -space²⁾:

$$[\gamma_i \partial_i + M_{op}(\tau \cdot \mathbf{K}, \mathbf{K}^2)] \psi(x, \omega) = 0$$

where M_{op} is a mass operator which may depend on the indicated invariants in ω -space.

Regardless of the specific structure of M_{op} , the baryon states are, in addition to the usual quantum numbers for free spin $1/2$ -particle states, specified further by three new quantum numbers. These may be chosen as the eigenvalues of

- 1) $\mathbf{I} = \mathbf{K} + \tau/2$, $\langle \mathbf{I}^2 \rangle = i(i+1)$,
- 2) I_3 ; $\langle I_3 \rangle = n$,
- 3) the ω -reflection operator,

where I_3 is the component of \mathbf{I} in some preferred ω -direction marked 3 (M_{op} is degenerate with respect to I_3) and where the ω -reflection operator has eigenvalues ± 1 corresponding to even or odd " ω -parity". It has been explained in I how each of these new quantum numbers is amenable to a physical interpretation: i and the ω -parity determine the mass levels; n specifies the electric charge of the state concerned, while furthermore the ω -parity plays a decisive role in the discussion of the stability of the higher baryon

levels about which more below. For a further discussion of the quantization and the conservation laws see the next section.

Concerning the coupling of the baryon with other fields the following hierarchy of interactions is assumed:

(a) Interactions invariant with respect to the full group R_w . The baryon meson interaction is of this type. For example if $i g \bar{\psi}(x, w) \gamma_5 \tau^a \psi(x, w)$ is the meson field source, the interaction is $i g \bar{\psi}(x, w) \gamma_5 \tau^a \psi(x, w) \varphi_a(x, w)$ (summed over $a=1, 2, 3$). Hence φ_a , the real representation of the meson field (see Section II, 2) has to be a pseudo-vector with respect to w -space. It has been pointed out in I that this implies a mass spectrum of pseudoscalar mesons and that in general all baryon and meson states are very short lived with the exception of³⁾:

- 1) The $^2S_{1/2}$ -nucleon doublet ground state of the baryon.
- 2) The lowest lying P -state of the baryon. It is proposed, to identify this state with V_1 . (See also Note added in proof).
- 3) The 3S_1 -meson triplet ground state, corresponding to the three charge states of the π -meson.
- 4) Some of the P -states of the meson (" H -mesons") which may be identifiable with some of the new particles in the 1000-1300 mass range. (At any rate, there are indications⁴⁾ that some of the K -particles have spin zero).

The invariance of the interaction with respect to R_w shall be called charge independence. This includes and extends the conventional notion of charge independence applying to the nucleon- π -meson system only.

It is clear that the present ideas do not in any way single out the pseudoscalar nature of the mesons. They could equally well be applied to mesons of other kinds. Nor is the kind of coupling material to the argument; the γ_5 -interaction has been mentioned above and will be used in section II by way of illustration.

(b) Interactions that involve a preferred direction in w -space, and have, therefore, a lower (axial) symmetry. This is the electromagnetic interaction that removes the degeneracy with respect to I_3 . A particle state with $\langle I_3 \rangle = n$ has charge ne , the corresponding anti-particle state has charge $-ne$.

(c) Non- w -reflection invariant interactions. These are needed (see I) to account for the eventual V_1 - and H -decay. They are very weak couplings. As is readily seen such interactions can be conceived as "non-local" couplings with respect to their w -dependence. Thus the three distinct orders of magnitude of coupling are here related to different symmetry properties with respect to w -space.

Let us now consider more specifically the place of V_1 in this scheme. The symbol V_1 comprises the following kinds of particles:

- 1) V_1^0 , so far the best studied one of the group. Most recent data⁵⁾ on V_1^0 decay seem to favor the existence of a unique Q -value and thus of a unique mass of V_1^0 .
- 2) V_1^+ , the positively charged V_1 which has been observed by the Pasadena group.⁶⁾
- 3) V_1^- ; an event has recently been observed by the cosmotron group in Brookhaven⁷⁾ which seems to necessitate a V_1^- for its interpretation.

As far as one knows the masses of the various kinds of V_1 may be considered to be essentially the same (they cannot be expected to be exactly equal due to electromagnetic effects). The existence of the three charge states so far observed makes it impossible to identify V_1 with the ${}^2P_{1/2}$ -baryon state*. Indeed, the latter is a doublet corresponding to particles with charge $+e$ and o respectively. The V_1^- cannot possibly be an anti-particle because there is not sufficient energy available for baryon pair formation. Hence one should identify V_1 with ${}^2P_{3/2}$ which then, according to the considerations of I should be the next lowest baryon state. If this conjecture is correct it implies the existence also of a V_1^{++} .

With respect to the production of V_1 two possibilities have to be distinguished:

1) Production in nucleon-nucleon or π -meson-nucleon collisions. In the present language the initial state has even ω -parity and the same should be true for the final state (here we disregard couplings of the type (c) which play no essential role in the production processes). Thus, besides the V_1 , the production should involve at least one other particle with odd ω -parity. This may be another V_1 or a (short-lived) higher baryon state with odd parity, or it may be a meson state with odd parity, in particular a H -meson. It is interesting to observe that two pictures recently obtained at Brookhaven⁽⁵⁾ as the first instances of artificial V_1^0 -production by $\pi^- - H$ collision may be interpreted as

$$\pi^- + H \rightarrow V_1^0 + (\text{neutral particle of mass } \sim 1300 m_e)$$

which at least fits in well with the present ideas.

Also the V_1 may itself be a decay product of an initially produced parent particle with odd ω -parity but here too a second odd parity particle should be simultaneously produced. A model of this kind involving pairs of unstable parent particles has recently been discussed by Nishijima.⁽⁹⁾ The actual observability of pairs of unstable particles in any of the mentioned processes depends, apart from geometry, on the mode of decay.⁽¹⁾

Thus the copious V_1 -production is here linked to the great strength of the charge independent meson-baryon interaction. It is the interplay of two conservation laws which prohibits a fast decay through this same coupling: first, the conservation of ω -parity for the interaction (a) prohibits $V_1 \rightarrow N + \pi$ (γ -decay is also forbidden, see I and section II, 3). Second, the conservation of energy forbids $V_1 \rightarrow N + H$ (or similar processes). It is thus seen that appropriate relative spacings of a baryon and a meson spectrum are essential to the entire argument. The importance of energy conservation in models of this kind has been emphasized in phenomenological arguments by Nambu et al,⁽¹⁰⁾ Oneda⁽¹¹⁾ and the author.⁽¹²⁾ In the last mentioned paper it has been pointed out that the energy conservation provides a necessary but not sufficient argument for the stability of the V_1 and that in addition one has to distinguish between certain strong ("even") and very weak ("odd") interactions between the particles involved. The distinction between interactions of the types (a) and (c) above just corresponds to these phenomenological considerations.

The production of V_1 together⁽⁵⁾ with V_2^0 , the latter being defined by $V_2^0 \rightarrow \pi^+ + \pi^-$ again is in accordance with the role of energy conservation.⁽¹³⁾ A particle of the kind V_2

* See however the Note added in proof.

cannot be a member of the meson family here discussed which is pseudoscalar. It has been emphasized above, that mesons with different space-time transformation properties could of course find a place in the present scheme, however.

2) Single production of V_1 . This is possible in accordance with the two relevant conservation laws provided the initial state has odd parity. Example: $\pi + \text{nucleon} \rightarrow V_1 + \pi$. Such processes may occur in cosmic ray events in which an odd parity particle is created in a primary event, but they face of course severe competition with spontaneous decay.

It seems to the author that the questions raised at the beginning should constitute an essential part of any program for enlarging the scope of the present theories. It is equally clear that, even with our present limited knowledge, there are further basic questions to be considered for which there is as yet no room for answer in the enlarged framework here discussed. As two major problems I would like to mention:

1) The "light particles" (electron, neutrino, μ -meson and possibly others) and their relation to the baryon. It is impossible to give a full account of the conservation of baryons before this relation is clarified, see I and also sec. II, 3 below.

2) The meaning and relevance of the divergences. This problem becomes even more acute in a theory which tries to connect various kinds of particles than in the current "disjoint" theories. The reason is that in the former there will occur new observables which have to be accounted for. Thus for example the present ideas lead one to ask: what is the mass difference of the baryon levels? Obviously, the baryon equation written down above gives an incomplete answer to the question. Take for instance as a specific case

$$\left[\gamma_i \partial_i + M + \frac{\tau \cdot K}{\Lambda} \right] \psi = 0, \quad (1)$$

(M is the nucleon mass). The V_1 with mass $\approx 1.2 M$ being taken as ${}^2P_{3/2}$ -state gives $\Lambda \approx -0.45 \hbar/Mc$ from (1). But this is not a reasonable procedure as it ignores the mesic mass difference of the levels.¹⁴⁾ The question is, therefore, what happens with the self-energies.

The very fact of the orthogonality of space time and ω -space precludes any new approach to the divergence problems. In fact new divergent aspects appear, because to any intermediate state of the usual theory, specified as fully as possible, there corresponds an infinite set of intermediate states due to the presence of ω -space. The state of affairs is discussed more fully in sec. II, 4.

The present work must, therefore, be viewed as a first step in employing new invariance principles. If this direction of approach proves fruitful it should be followed by further refinement. However this may be, the importance should be stressed of being neither drawn aside from the subject in pursuit of analytical subtleties, nor carried beyond the truth by a favourite hypothesis.¹⁵⁾

The present model leads to multi-mass equations of a different nature than those

obtained by using suitable non-local space-time equations of the type $f(\square)\psi(x)=0$. Non-definite energy difficulties are not encountered here. The present ideas tend perhaps more in the direction of Yukawa's non-local field approach,¹⁰⁾ insofar as the latter also uses "inner variables". In Yukawa's theory the group and structure of the "inner space" is the same as that of space time. It would seem that there is no *a priori* justification for this, however.

In the following section the formal aspects of a theory of the present kind are briefly sketched. By way of example we shall sue (1) as baryon equation.

§ II. Formalism

1. *The baryon.* Equation (1) and its adjoint can be derived from a variational principle

$$\delta \int_V d_4x \int d\omega L(x, \omega) = 0 \quad (2)$$

where $L(x, \omega)$ is a function of $\psi(x, \omega)$ and $\bar{\psi}(x, \omega) = \psi^\dagger(x, \omega)\gamma_4$, V is an arbitrary space time domain, while the ω -integration goes over the whole of ω -space. Apart from the usual boundary conditions, one-valuedness of ψ as a function of ω shall be required. Only one-valued wave functions will occur throughout this paper. If $L(x, \omega)$ depends on no higher than first order x - and ω -derivatives we have as variational equations

$$\frac{\partial L}{\partial \psi} - \partial_i \left(\frac{\partial L}{\partial (\partial_i \psi)} \right) - K \left(\frac{\partial L}{\partial (K\psi)} \right) = 0 \quad (3)$$

and similarly for $\bar{\psi}$. In the present case the appropriate L is

$$L(x, \omega) = -\bar{\psi}(x, \omega) \left(\gamma_i \partial_i + M + \frac{\tau \cdot K}{\Lambda} \right) \psi(x, \omega). \quad (4)$$

The generalization of (3) to the case that higher derivatives are involved is obvious.

From $L(x, \omega)$ an energy momentum tensor (x, ω) -density $T_{ik}(x, \omega)$ may be derived in the usual way by the help of invariance arguments with respect to the Lorentz group only. Consider the canonical tensor $T_{ik}^{can}(x, \omega)$:

$$T_{ik}^{can}(x, \omega) = \sum_\alpha \left\{ \frac{\partial L(x, \omega)}{\partial (\partial_k \psi_\alpha)} \partial_i \psi_\alpha \right\} - L \delta_{ik} \quad (5)$$

where L is supposed to be a function of a set of independent fields $\psi_\alpha(x, \omega)$. In virtue of the field equations (3):

$$\partial_k T_{ik}^{can}(x, \omega) + K \left[\sum_\alpha \frac{\partial L}{\partial (K\psi_\alpha)} \partial_i \psi_\alpha \right] = 0. \quad (6)$$

Thus we get a non-vanishing divergence due to the K -terms in (3). However

$$\partial_k T_{ik}^{can}(x) = 0; \quad T_{ik}^{can}(x) = \int d\omega T_{ik}^{can}(x, \omega). \quad (7)$$

This is in accordance with the prescription given in I that densities of energy and momentum shall be averages over ω -space.

T_{ik}^{can} as defined by (5) is of course in general non-symmetrical. The symmetrization procedure goes in the usual way and for the symmetrical tensor $T_{ik}(x, \omega)$ one has again $\partial_k T_{ik}(x, \omega) + K\text{-derivative} = 0$. Example: corresponding to (4) we have:

$$T_{ik}(\omega) = -\frac{1}{4} [\bar{\psi} \gamma_i \partial_k \psi + \bar{\psi} \gamma_k \partial_i \psi - \bar{\psi} \gamma_i \partial_k | \psi - \bar{\psi} \gamma_k \partial_i | \psi],$$

where $|$ denotes that differential operators to the left of it work backwards. One finds

$$\partial_i T_{ik} = (4I)^{-1} \mathbf{K} [\bar{\psi} \partial_k \tau \psi - \bar{\psi} \partial_k | \tau \psi - (2M - A^{-1}) \bar{\psi} \gamma_k \tau \psi + A^{-1} (\bar{\psi} \mathbf{K} | \gamma_k \psi - \bar{\psi} \gamma_k \mathbf{K} \psi)].$$

ψ can be developed as follows

$$\psi_{\sigma\tau}(x, \omega) = \sum_A \psi_{\sigma A}(x) \chi_{\tau A}(\omega);$$

$$\psi_{\sigma A}(\vec{x}, t) = \sum_m a_{mA} u_{m\sigma A}(\vec{x}) e^{-iE_{mA}t}. \quad (8)$$

Notation: A stands for the set of quantum numbers specifying the mass levels, $\chi_{\tau A}(\omega)$ are an orthonormal set of two-component spinors appropriate to the mass eigenvalue problem, m specifies momentum, spin and sign of energy. The a 's are Jordan-Wigner operators, the u_{mA} are the usual spinors. The quantization is obtained by putting

$$\{a_{mA}, a_{m'A'}^\dagger\} = \delta_{mm'} \delta_{AA'}. \quad (9)$$

One finds

$$\{\psi_{\sigma\tau}(x, \omega), \bar{\psi}_{\sigma'\tau'}(x', \omega')\} = -i S_{\sigma\sigma', \tau\tau'}(x - x', \omega, \omega') \quad (10)$$

$$S_{\sigma\sigma', \tau\tau'}(x, \omega, \omega') = \sum_A (\gamma_i \partial_i - M_A)_{\sigma\sigma'} \mathcal{A}_A(x) \chi_{\tau'A}^*(\omega') \chi_{\tau A}(\omega) \quad (11)$$

where \mathcal{A}_A is the well known \mathcal{A} -function belonging to mass M_A . It is clear that the extension (11) of the usual $S_{\sigma\sigma'}(x)$ to $S_{\sigma\sigma', \tau\tau'}(x, \omega, \omega')$ equally holds for any of the other S -functions provided the corresponding \mathcal{A}_A -function is used. Finally we give a representation of the χ 's. Specify χ fully as $\chi_{k,n}^{(\epsilon)}(\omega)$. Here $\epsilon = i - k = \pm 1/2$. We have

$$\chi_{k,n}^{(1/2)} = \frac{1}{\sqrt{2k+1}} \begin{pmatrix} \sqrt{k+n+1/2} & Y_{k,n-1/2} \\ \sqrt{k-n+1/2} & Y_{k,n+1/2} \end{pmatrix},$$

$$\chi_{k,n}^{(-1/2)} = \frac{1}{\sqrt{2k+1}} \begin{pmatrix} -\sqrt{k-n+1/2} & Y_{k,n-1/2} \\ \sqrt{k+n+1/2} & Y_{k,n+1/2} \end{pmatrix}. \quad (12)$$

As regards the phases of the spherical harmonics, the conventions of Condon and Shortley are used. Note in particular that

$$Y_{k,n}^* = (-1)^n Y_{k,-n}. \quad (13)$$

2. *The meson field.* According to the ideas outlined in I the meson field is a pseudovector in ω -space appropriate to the isotopic spin 1 representation. The isotopic spin vector T of the meson field satisfies

$$[T_\alpha, T_\beta] = iT_\gamma, \alpha, \beta, \gamma = 1, 2, 3 \text{ cycl.}$$

$$T_\alpha T_\beta T_\gamma + T_\gamma T_\beta T_\alpha = \delta_{\alpha\beta} T_\gamma + \delta_{\gamma\beta} T_\alpha. \quad (14)$$

The representation of the meson field which corresponds to T_3 being diagonal is denoted by Φ , where

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_0 \\ \Phi_{-1} \end{pmatrix}$$

is written as a one column matrix. We define the operation of T_α on Φ as

$$T_\alpha \Phi_T = \sum_{T'} T_\alpha^{TT'} \Phi_{T'}; \quad T, T' = 1, 0, -1.$$

or in matrix notation

$$T_\alpha \Phi \rightarrow \tilde{\Phi} T_\alpha, \quad \tilde{\Phi} = (\Phi_1, \Phi_0, \Phi_{-1}). \quad (15)$$

In the representation so defined:

$$T_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Consider a new representation φ of the meson field, obtained from Φ by

$$\Phi = S\varphi, \quad S^\dagger = S^{-1} \quad (16)$$

as a result of which T_α transforms into t_α . Bearing in mind the multiplication (15) which is needed to ensure all group properties, it follows from (16) that

$$T_\alpha = S^* t_\alpha S^{-1}. \quad (17)$$

We are in particular interested in that representation in which φ is real:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \varphi_1^* \\ \varphi_2^* \\ \varphi_3^* \end{pmatrix}. \quad (18)$$

(18) is shown to correspond to

$$S = \begin{pmatrix} -1/\sqrt{2}, & -i/\sqrt{2}, & 0 \\ 0, & 0, & 1 \\ 1/\sqrt{2}, & -i/\sqrt{2}, & 0 \end{pmatrix} \quad (19)$$

provided an additional condition on Φ is imposed, see below, equation (25). Let $\bar{\varphi}$ denote the "adjoint" of φ : $\bar{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$. Then

$$\bar{\varphi}\varphi = \bar{\Phi}\Phi, \quad (20)$$

where

$$\bar{\Phi} = \tilde{\Phi} B, \quad B = S^* S^{-1}. \quad (21)$$

For S given by (19) we have

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}. \quad (22)$$

Consider now the meson field equation which in the Φ -representation is given by

$$[\square - \mu_{op}^2 (KT, (KT)^2, K^2)] \Phi = 0. \quad (23)$$

The general features of the mass spectrum of (23) have been discussed in I, especially in its bearing on the general stability questions. The mass eigenfunctions shall be denoted by $H_{Tkn}^{(\varepsilon)}(\omega)$, $T=1, 0, -1$. $\varepsilon=i-k=1, 0, -1$. k and n are defined as before. We have

$$\Phi_T(x, \omega) = \sum_{\varepsilon, k, n} \phi_{kn}^{(\varepsilon)}(x) H_{Tkn}^{(\varepsilon)}(\omega). \quad (24)$$

The H are given by

$$H_{kn}^{(1)} = \frac{1}{\sqrt{(2k+1)(2k+2)}} \begin{pmatrix} \sqrt{(k+n)(k+n+1)} Y_{k, n-1} \\ \sqrt{2(k-n+1)(k+n+1)} Y_{k, n} \\ \sqrt{(k-n)(k-n+1)} Y_{k, n+1} \end{pmatrix},$$

$$k=0, 1, 2, \dots$$

$$H_{kn}^{(0)} = \frac{i}{\sqrt{2k(k+1)}} \begin{pmatrix} -\sqrt{(k+n)(k-n+1)} Y_{k, n-1} \\ n\sqrt{2} Y_{k, n} \\ \sqrt{(k-n)(k+n+1)} Y_{k, n+1} \end{pmatrix},$$

$$k=1, 2, \dots$$

$$H_{kn}^{(-1)} = \frac{1}{\sqrt{2k(k+1)}} \begin{pmatrix} \sqrt{(k-n)(k-n+1)} Y_{k, n-1} \\ -\sqrt{2(k-n)(k+n)} Y_{k, n} \\ \sqrt{(k+n)(k+n+1)} Y_{k, n+1} \end{pmatrix},$$

$$k=1, 2, \dots$$

The phase conventions are again the same as Condon-Shortley with the exception of a factor i in the expression for $H_{kn}^{(0)}$.

Using (13), (16) and (19) it is readily found that in order to satisfy (18) we still have to require

$$\phi_{kn}^{(\varepsilon)*}(x) = (-1)^n \phi_{k, -n}^{(\varepsilon)}(x). \quad (25)$$

From (25) we obtain

$$\Phi_1^*(x, \omega) = -\Phi_{-1}(x, \omega). \quad (26)$$

(25) implies in particular that neutral particles, which for any ε, k correspond to $n=0$, are described by real fields. The ordinary π -meson field corresponds to $\phi_{0n}^{(1)}(x)$, $n=1, 0, -1$.

The meson field Lagrangian is

$$L(x, \omega) = -\frac{1}{2} (\partial_i \bar{\Phi} \partial_i \Phi + \bar{\Phi} \mu_{op}^2 \Phi) \quad (27)$$

where μ_{op}^2 is the mass operator occurring in (23). Applying the S -transformation we can also write L as;

$$L(x, \omega) = -\frac{1}{2}(\partial_i \bar{\varphi} \partial_i \varphi + \bar{\varphi} \mu_{op}^2 \varphi) \quad (28)$$

with the understanding that here μ_{op}^2 contains t_α instead of T_α . Using (25) and the orthonormality of the $H_{kn}^{(\varepsilon)}$:

$$\begin{aligned} L(x) &= \int L(x, \omega) d\omega \\ &= -\frac{1}{2} \sum_{\varepsilon, k} [\partial_i \phi_{k,0}^{(\varepsilon)2} + \mu_k^{(\varepsilon)2} \phi_{k,0}^{(\varepsilon)2}] \\ &\quad - \sum_{\varepsilon, k} \sum_{n=1}^{k+\varepsilon} [\partial_i \phi_{k,n}^{(\varepsilon)} + \mu_k^{(\varepsilon)2} \phi_{kn}^{(\varepsilon)\tau} \phi_{kn}^{(\varepsilon)}]. \end{aligned}$$

The commutation relations are accordingly

$$[\phi_{kn}^{(\varepsilon)}(x), \phi_{k'n'}^{(\varepsilon')}(x')] = i(-1)^n \delta(\varepsilon - \varepsilon') \delta(k - k') \delta(n + n') \mathcal{A}_{\varepsilon k}(x - x') \quad (29)$$

where $\mathcal{A}_{\varepsilon k}(x)$ is the standard \mathcal{A} -function for mass $\mu_k^{(\varepsilon)}$. It follows from (29) that

$$[\varphi_\alpha(x, \omega), \varphi_{\alpha'}(x', \omega')] = i \sum_{\varepsilon, kn} \eta_{\alpha'kn}^{(\varepsilon)\tau}(\omega') \eta_{\alpha kn}^{(\varepsilon)}(\omega) \mathcal{A}_{\varepsilon k}(x - x') \quad (30)$$

$$\alpha, \alpha' = 1, 2, 3,$$

where

$$\eta_{kn}^{(\varepsilon)} = S^{-1} H_{kn}^{(\varepsilon)}. \quad (31)$$

Vacuum expectation values, P -brackets, etc., are treated in the usual way.

3. *The baryon-meson-photon system.* The total Lagrangian is

$$\begin{aligned} L(x, \omega) &= -\bar{\psi} \left[\gamma_i \left\{ \partial_i - ie \left(I_3 + \frac{1}{2} \right) A_i \right\} + \bar{M} + \frac{\tau K}{\Lambda} \right] \psi - ig \bar{\psi} \gamma_5 \tau_\alpha \psi \varphi_\alpha \\ &\quad - \frac{1}{2} [\bar{\varphi} (\partial_i + ie I_3 A_i) | (\partial_i - ie I_3 A_i) \varphi + \bar{\varphi} \mu_{op}^2 \varphi] - \frac{1}{4\pi} \cdot \frac{1}{4} F_{ik} F_{ik}; \\ \bar{\varphi} I_3 &= -K_3 \bar{\varphi} + \bar{\varphi} t_3. \end{aligned}$$

Here A_i and F_{ik} are the electromagnetic potential and field respectively. It has been shown in I that these quantities must depend on x_i only.

$L(x, \omega)$ is Lorentz-invariant. Its non-electromagnetic part is also invariant for R_ω but, as already mentioned, the very existence of electromagnetic couplings must of needs destroy the invariance for the full group R_ω . In addition L has the following properties:

1) Phase invariance:

$$\psi = e^{ia} \psi'; \quad \bar{\psi} = e^{-ia} \bar{\psi}'; \quad \varphi = \varphi'; \quad A_i = A_i' \quad (32)$$

leaves L invariant. Here a is a constant. The corresponding conservation law is

$$\frac{\partial S_i}{\partial x_i} = 0; \quad S_i = \int d\omega \bar{\psi} \gamma_i \psi \quad (33)$$

which, as discussed in I, corresponds to the conservation of baryons.

2) Gauge invariance. Consider the gauge transformation

$$\begin{aligned}
\psi &= e^{ie(I_3+1/2)b(x)} \psi', \\
\bar{\psi} &= \bar{\psi}' e^{-ie(I_3+1/2)b(x)}, \\
\varphi &= e^{ieI_3b(x)} \varphi', \\
\bar{\varphi} &= \bar{\varphi}' e^{-ieI_3b(x)}, \\
A_i &= A_i' + \frac{\partial b}{\partial x_i},
\end{aligned} \tag{34}$$

as a result of which

$$L'(x, \omega) = L(x, \omega) + eg b(x) K_3 [\bar{\psi}' \gamma_5 \tau_a \psi' \varphi_a']$$

for infinitesimal $b(x)$. As *any* physical density is an ω -average, $L(x, \omega)$ may be said to be gauge invariant. Correspondingly we have the conservation of charge:

$$\begin{aligned}
\frac{\partial J_i}{\partial x_i} &= 0, \quad J_i = -\frac{\partial}{\partial A_i} \int L(x, \omega) d\omega, \\
J_i &= ie \int d\omega \left[\bar{\psi} \left(I_3 + \frac{1}{2} \right) \gamma_i \psi \right. \\
&\quad \left. - \frac{1}{2} (\bar{\varphi} I_3 | \partial_i \varphi - \bar{\varphi} \partial_i | I_3 \varphi - 2ie \bar{\varphi} I_3 | I_3 \varphi A_i) \right].
\end{aligned}$$

Of course in this formalism gauge invariance with x -independent phase is not equivalent to phase invariance. This is just due to the fact that there are now two possibilities for a choice of phase which satisfy algebraic additivity, namely: a constant or a constant times I_3 , corresponding to two distinct conservation laws. There is a marked dissymmetry between (32) and (34) however: in (32) a must be constant whereas in (34) b is not. This is a somewhat unsatisfactory situation which may be connected with the fact that the conservation law (33) is so far insufficiently dealt with from a physical point of view, as was pointed out in section I.

4. *Divergent features of the theory.* Some general comments on this point have already been made in section I and we will here illustrate the situation by means of an elementary example, namely, the g^2 -self energy of a baryon level with mass quantum numbers that will be denoted briefly by A . With the help of (10) and (30) and of similar relations for other S and A -functions one finds for the self energy density

$$\begin{aligned}
\delta H_A(1) &= ig^2 \bar{\psi}_A(1) \sum_{a, A'} |c(A, A', a)|^2 \\
&\quad \cdot \int d^2 \gamma_5 S_{A'}^c(12) \gamma_5 A_a^c(21) \psi(2), \\
c(A, A', a) &= \sum_a \int d\omega \mathcal{K}_A^*(\omega) \tau_a \mathcal{K}_{A'}(\omega) \eta_{aa}(\omega).
\end{aligned}$$

Here $A'(a)$ denotes the mass level of the intermediate baryon (meson) state. S^c and A^c are the causal propagation functions for the masses indicated.

Evaluating the space time integral with the help of a regulator mass one gets for the self mass

$$\frac{\partial M_A}{M} = -\frac{g^2}{16\pi^2} \lim_{\rho \rightarrow \infty} \sum_{A, A'} |c(A, A', a)|^2 \int_0^1 dx [M_{A'} - M_A(1-x)] \\ \cdot [l_g \{M_A^2 x^2 + (M_A^2 - M_{A'}^2)x + \rho^2(1-x)\} \\ - l_g \{M_{A'}^2 x^2 + (M_A^2 - M_{A'}^2)x + \rho_a^2(1-x)\}] ,$$

where all masses on the right hand side have been expressed in units of M . Hence

$$\frac{\partial M_A}{M} = -\frac{g^2}{32\pi^2} \lim_{\rho \rightarrow \infty} l_g \rho^2 \sum (2M_{A'} - M_A) |c(A, A', a)|^2 \\ + \frac{g^2}{16\pi^2} \sum |c(A, A', a)|^2 \int_0^1 dx [M_{A'} - M_A(1-x)] \\ \cdot [l_g(1-x) - l_g \{M_A^2 x^2 + (M_A^2 - M_{A'}^2)x + \rho_a^2(1-x)\}] . \quad (35)$$

The first line of (35) contains the ultraviolet divergence. Using closure on $|c|^2$ one shows that this line can be written as

$$\frac{g^2}{32\pi^2} \text{"}\lim_{\omega \rightarrow 0} \delta(\omega)\text{"} \lim_{\rho \rightarrow \infty} l_g \rho^2 \left\langle 3 - \frac{4\tau K}{AM} \right\rangle_A , \quad (36)$$

where " $\lim_{\omega \rightarrow 0} \delta(\omega)$ " denotes the limit of the value at the origin of a distribution that tends to a δ -function in ω -space. This singularity occurs as a multiplicative factor of the ultraviolet divergence. This means that it does no harm in (36) as far as the treatment of the ultraviolet catastrophe in the mass is concerned. Indeed, (36) implies that, at least in order g^2 , a renormalization of A and of M is possible in such a way that the momentum space divergence gets removed for all baryon levels. However, this is by no means enough to ensure a consistent treatment of the self mass even to this order. Indeed the remaining part of (35) is free of momentum space divergences but not of ω -divergences.

I hope to come back elsewhere to the treatment of these ω -divergences. However I believe that this is a point of at most formal interest. The question should rather be whether a more intimate connection between space time and ω -variables can be established through which the entire problem of the divergences will get a new perspective.

Note added in proof: It turns out that the Q -value for the V_1^- observed at Brookhaven is of the order of 100 Mev. It would not seem reasonable to attribute the large difference between $Q(V_1^-)$ and $Q(V_1^0)$ to electromagnetic effects. Rather one would assume that the lowest baryon level is ${}^2P_{1/2}$ and that the ${}^2P_{3/2}$ is separated from ${}^2P_{1/2}$ by an amount $\sim Q(V_1^-) - Q(V_1^0)$. The ${}^2P_{1/2}$ -states cannot decay into ${}^2P_{1/2} + \pi$ for energy reasons; and they cannot decay into ${}^2S_{1/2} + \pi$ for parity reasons. The members of the ${}^2P_{3/2}$ -quadruplet with charge $+$ and 0 can be seen to decay rapidly into the corresponding ${}^2P_{1/2}$ -states under γ -emission. Thus the only metastable ${}^2P_{3/2}$ -levels are those with charge $-e$ and $+2e$. A cloud chamber picture recently obtained by Ascoli (Phys. Rev. 90 (1953), 1079) gives some evidence for a doubly charged particle of transprotonic mass. The crucial question is whether the V_1^+ has a Q -value of the order of $Q(V_1^0)$ rather than $\sim Q(V_1^-)$. The cosmic ray evidence on the sign of the charge of V_1^\pm together with $Q(V_1^\pm)$ is limited, but some of the results presented at the Bagnères Conference speak against a low Q for V_1^+ .

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Letters to the Editor

On the Universal Fermi Interaction*

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The establishment of the universal Fermi interaction among Fermi particles is perhaps one of the most striking results in the recent progress of the theory of elementary particles. In the current theory, however, the introduction of the mutual interaction of elementary particles is done ad hoc**, and this remarkable fact can not help being regarded as entirely accidental. This situation seems to be an evidence for the phenomenological feature of the current theory in the stage of the development of the theory of elementary particles, and the establishment of the universal Fermi interaction suggests strongly the necessity of overcoming such a limitation.

Recently Yukawa³⁾ and we^{1), 2)} proposed to regard elementary particles as corresponding to various states of the internal motion of a kind of Urmaterie, and discussed the structure of elementary particles. We shall show in this note that the extension of our theory so as to include the interaction of Urmaterie suggests a way of such an approach.

Unfortunately, satisfactory theory of the interaction of non-local fields has not yet been given. Here we adopt Yukawa's formalism⁴⁾ and investigate the interaction of Urmaterie by assuming an appropriate interaction Lagrangian and therefrom constructing the S-matrix according to him. Of course there is no positive base for this procedure, and in this respect our attempt must be said to be of a very provisional nature. As will be seen below, however, the universal Fermi interaction follows as rather a qualitative feature of our theory, not depending upon details of the way of treating the interaction of non-local fields.

Using the same notations as those of Yukawa, the S-matrix is given by

$$S = 1 + i \langle L \rangle + i^2 \langle L D_+ L \rangle + \dots \quad (1)$$

As the interaction Lagrangian density we assume

$$L = g \bar{\psi} O_{\mu\nu} \dots \psi \cdot \bar{\psi} O_{\mu\nu} \dots \psi, \quad (2)$$

where ψ is a spinor non-local field describing spinor

Urmaterie, and $O_{\mu\nu} \dots$ some Dirac matrices whose explicit form is, for example, one of Bethe's five covariants of β interaction. As discussed in (I) and in (II), ψ is a superposition of local fields with various spin and rest mass;

$$\langle x | \psi | x' \rangle = \sum_{s, m} \psi_{s, m}(X_\mu) \varphi_{s, m}(r_\mu), \quad (3)$$

where $\varphi_{s, m}(r_\mu)$ is the eigenfunction of the spin and mass operators which serves as form factors when interaction is introduced. Substituting (3) into (2), (2) is written in the form

$$\begin{aligned} L = g \sum_{\substack{s, m; s', m'; \\ s'', m''; s''', m'''}} \bar{\psi}_{s, m}(X_\mu) O_{\mu\nu} \dots \psi_{s', m'}(X_{\mu'}) \\ \times \bar{\psi}_{s'', m''}(X_{\mu''}) O_{\mu\nu} \dots \psi_{s''', m'''}(X_{\mu'''}) \\ \times (\text{form factors}). \end{aligned} \quad (4)**$$

(4) represents various interactions of Fermi particles according to which eigenvalues of the spin and rest mass the Urmaterie takes. It gives, for example, the interaction of the nucleon and the electron, of the nucleon and the μ -meson, or of the μ -meson and the electron, ... according to the values of s and m . Here the important fact is that they all appear with the same coupling constant g . As form factors are reduced to δ -functions in low energy region (By low energy region we mean the energy region below approximately $(\hbar c/\lambda) \approx 1 \text{ BeV}$ corresponding to λ taken of the order of the Compton wave length of the nucleon. The explicit form of form factors will be given in a separate note.), this explains at once the similarity of coupling constant of various Fermi interactions.

Thus, the universal Fermi interaction follows as an immediate consequence of introducing the interaction of elementary particles in an unified way as the interaction of Urmaterie. This picture suggests more various relations among interaction constants of the local theory. We hope that our theory thus serve to promote a step in elucidating the "structure of the interaction of elementary particles," a problem recently emphasized by Sakata.⁵⁾

In conclusion, we wish to express our sincere gratitude to Prof. S. Sakata for continual encouragement and valuable discussions.

Details will be published in a near future with related problems.

* This note is a direct continuation of our previous ones, "The Theory of the Structure of Elementary

Particles¹⁾” and “The Conservation of Heavy Particles²⁾” which will be referred to as (I) and (II) respectively, and the notations are the same with those used there unless otherwise remarked.

** Of course the introduction of the mutual interaction must be done in accordance with the requirements of Lorentz-invariance or others. These restrictions are, however, far from giving definite guiding principle in doing it.

*** If we make the interaction Lagrangian L invariant under the transformation $\psi \rightarrow \exp(i\theta a + i\beta) \psi$ and $\psi^* \rightarrow \psi^* \exp -i(\theta a + \beta)$ by introducing some projection operators, conservation laws of heavy particles and of electric charge are guaranteed. (θ and e are mesic and electric charges²⁾ and a and β are arbitrary constants.)

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On the Radiative Interaction of Phonon Field with Electrons

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The radiative interaction of the phonon field with electrons yields two effects: one effect is shown in a shift of energy levels of the system consisting of phonons and electrons and the other in a width of levels. The level shift is due to virtual emission and absorption of phonons by electrons and level width is due to the real emission and absorption. These effects can not be correctly treated except in their combination. We shall calculate the level shift and the level width and we shall show that the level shift is identical with the self energies of electrons which Fröhlich¹⁾ calculated and that the use of Cauchy's principal value is justified.

In the Schrödinger representation the wave func-

tion $\Psi(t)$ satisfies the equation of motion

$$i\hbar \cdot \partial \Psi / \partial t = H \Psi, \quad (1)$$

where the Hamiltonian is given by

$$H = -\hbar^2/2m \cdot \int \psi^* \Delta \psi d\mathbf{x} + 1/2 \cdot \int \{\dot{\phi}^2 + s^2 (F\phi)^2\} d\mathbf{x} - \sqrt{4F'\zeta/3n} \int \psi^* \phi \dot{\psi} d\mathbf{x}. \quad (2)$$

We rewrite the Hamiltonian as

$$H = H_0 + H_I + H_{II},$$

$$H_0 = -\hbar^2/2m \cdot \int \psi^* \Delta \psi d\mathbf{x} + 1/2 \cdot \int \{\dot{\phi}^2 + s^2 (F\phi)^2\} d\mathbf{x}, \quad (3a)$$

$$H_I = -\sqrt{4F'\zeta/3n} \int \psi^* \phi \dot{\psi} d\mathbf{x}, \quad (3b)$$

$$H_{II} = -\sqrt{4F'\zeta/3n} \int \psi^* \phi \dot{\psi} d\mathbf{x} + 1/2 (s'^2 - s^2) \int (F\phi)^2 d\mathbf{x}, \quad (3c)$$

where s and F are the renormalized coupling constant and the phonon velocity respectively and $F' = F'/s'/s$, $s = s' + \delta s$.

The wave function $\Phi(t)$ in the interaction representation is given in terms of $\Psi(t)$ by

$$\Psi(t) = \exp(-i/\hbar \cdot H_0 t) \Phi(t). \quad (4)$$

Then $\Phi(t)$ satisfies the equation of motion

$$i\hbar \cdot \partial \Phi / \partial t = \{H_I(t) + H_{II}(t)\} \Phi, \quad (5)$$

where

$$H_I(t) = \exp(i/\hbar \cdot H_0 t) H_I \exp(-i/\hbar \cdot H_0 t). \quad (6)$$

Let Φ_0 be the interaction representation wave function of an unperturbed state. Hence we see from (4) that Φ_0 is independent of time. We shall calculate the variation with time of $(\Phi_0, \Phi(t))$, the amplitude for finding the system still in the unperturbed state at time t a long time after t_0 . We get

$$d(\Phi_0, \Phi(t))/dt = -i/\hbar \cdot (\Phi_0, \{H_I(t) + H_{II}(t)\} \Phi(t)). \quad (7)$$

The solution of (5) valid to the first order in H_I is²⁾

$$\Phi(t) = [1 - i/\hbar \cdot \int_{-\infty}^t H_I(t') dt'] (\Phi_0, \Phi(t)) \Phi_0 + \dots \quad (8)$$

Substituting (8) into (7) we have

$$1/F \cdot dF/dt = 1/i\hbar \cdot (\Phi_0, H_{II}(t) \Phi_0) - 1/\hbar^2 \cdot \int_{-\infty}^t dt' (\Phi_0, H_I(t') H_I(t') \Phi_0), \quad (9)$$

where $F = (\Phi_0, \Phi(t))$.

Let us analyze $\phi(x, t)$ and $\phi(x, t')$ into Fourier components

$$\phi(x, t) = 1/(2\pi)^{3/2} \int a_k \exp(ikx - ik_0 t) dk, \quad (10a)$$

$$\phi(x, t) = \sqrt{\hbar/2s(2\pi)^3} \int 1/v [\delta_{v0} \exp(i\mathbf{w}x - i\mathbf{w}s t) + \delta_{v0}^* \exp(-i\mathbf{w}x + i\mathbf{w}s t)] d\mathbf{w}. \quad (10b)$$

From (3c) and (10) we get

$$(\Phi_0, H_{11}(t) \Phi_0) = -\delta s \int \hbar v (N_w + 1/2) d\mathbf{w}. \quad (11)$$

The second term on the right hand side of (9) gives

$$\begin{aligned} & -1/\hbar^2 \cdot \int_{-\infty}^t d\epsilon' (\Phi_0, H_1(\epsilon) H_1(\epsilon') \Phi_0) \\ & = -4F\zeta/3n\hbar^2 \int_{-\infty}^t d\epsilon' \int d\mathbf{x} \int d\mathbf{x}' \langle \psi(x, \epsilon) \psi(x', \epsilon') \rangle_0 \\ & \quad \times \langle \psi^*(x, \epsilon) \psi(x, \epsilon) \psi^*(x', \epsilon') \psi(x', \epsilon') \rangle_0 \\ & = 4F\zeta/2(2\pi)^3 3n\hbar^2 \left[\int_{-\infty}^t d\epsilon' \exp i/\hbar (\epsilon_k - \epsilon_{k-w} - \hbar v s) \right. \\ & \quad \times (\epsilon - \epsilon') \int \hbar v s (1 + N_w) n_k (1 - n_{k-w}) d\mathbf{w} d\mathbf{k} \\ & \quad + \int_{-\infty}^t d\epsilon' \exp i/\hbar (\epsilon_k - \epsilon_{k+w} + \hbar v s) (\epsilon - \epsilon') \\ & \quad \times \int \hbar v s N_w n_k (1 - n_{k+w}) d\mathbf{w} d\mathbf{k} \left. \right]. \quad (12) \end{aligned}$$

Using the relation

$$\int_{-\infty}^0 e^{i\alpha t} dt = \pi \delta(\alpha) + P \cdot 1/i\alpha = 2\pi \delta_+(\alpha),$$

we can reduce (12) to the following form

$$\begin{aligned} & 4F\zeta/(2\pi)^3 6n \left[\pi \left\{ \int \hbar v s (1 + N_w) n_k (1 - n_{k-w}) \right. \right. \\ & \quad \times \delta(\epsilon_k - \epsilon_{k-w} - \hbar v s) d\mathbf{w} d\mathbf{k} + \int \hbar v s N_w n_k \\ & \quad \times (1 - n_{k+w}) \delta(\epsilon_k - \epsilon_{k+w} + \hbar v s) d\mathbf{w} d\mathbf{k} \left. \right\} \\ & \quad + 1/i \left\{ P \int \frac{\hbar v s (1 + N_w) n_k (1 - n_{k-w})}{(\epsilon_k - \epsilon_{k-w} - \hbar v s)} d\mathbf{w} d\mathbf{k} \right. \\ & \quad \left. \left. + P \int \frac{\hbar v s N_w n_k (1 - n_{k+w})}{(\epsilon_k - \epsilon_{k+w} + \hbar v s)} d\mathbf{w} d\mathbf{k} \right\} \right], \end{aligned}$$

where N_w and n_k are the occupation numbers of phonons and electrons and P indicates the principal value. Finally we find

$$1/F \cdot dF/dt = -1/2 \cdot \Gamma/\hbar - i/\hbar \cdot \Delta E, \quad (13)$$

where

$$\Gamma = -4F\zeta\pi/(2\pi)^3 3n \left\{ \int d\mathbf{w} d\mathbf{k} \cdot \hbar v s (1 + N_w) \right.$$

$$\begin{aligned} & \times n_k (1 - n_{k-w}) \delta(\epsilon_k - \epsilon_{k-w} - \hbar v s) + \int d\mathbf{w} d\mathbf{k} \cdot N_w \\ & \times n_k (1 - n_{k+w}) \delta(\epsilon_k - \epsilon_{k+w} + \hbar v s) \left. \right\}, \quad (14a) \end{aligned}$$

$$\begin{aligned} \Delta E & = 4F\zeta/(2\pi)^3 6n \left[1/2 \cdot P \int d\mathbf{w} d\mathbf{k} \cdot \hbar v s n_k \right. \\ & \times \{ 1/(\epsilon_k - \epsilon_{k-w} - \hbar v s) - 1/(\epsilon_k - \epsilon_{k+w} + \hbar v s) \} \\ & \left. - P \int d\mathbf{w} d\mathbf{k} \cdot \hbar v s n_k n_{k-w} / (\epsilon_k - \epsilon_{k-w} - \hbar v s) \right], \quad (14b) \end{aligned}$$

$$\begin{aligned} \hbar v \delta s & = 4F\zeta/(2\pi)^3 6n \int \hbar v (n_k - n_{k-w}) / (\epsilon_k - \epsilon_{k-w} \\ & \quad - \hbar v s) \cdot d\mathbf{k}. \quad (14c) \end{aligned}$$

The integration over \mathbf{k} includes the summation over the spin coordinates. The same result with ours has been obtained by S. Nakajima.³⁾

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Note on the Light Absorption in the Insulating Crystals

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Concerning the shift with temperature of the long wave length limit of absorption line in the crystal, two representative approaches have been known. One is the method of level broadening with temperature¹⁾ and the another is the view point of level shift originated from the electron-phonon interaction.²⁾ The both show that the energy difference between the top of valence band and the bottom of conduction band decreases with temperature. But the latter is seemed to be more useful, especially in the case of non-polar crystals, referring to the experimental data. However, the shift with temperature of absorption line due to the exciton band, which was proposed by Mott,³⁾ has not been theoretically considered. And this is the task of this short note.

When an insulating crystal is hit by the photon of which the energy is a little smaller than the difference energy between the top of valence band and the bottom of conduction band, the Frenkel's exciton is made. Wannier's expression for the exciton state is⁴⁾

$$\Gamma(k_\alpha, \beta) = N_0^{-1/2} \sum_{\beta} \sum_n U_\tau(\beta) \exp i(k_\alpha, n) \\ \times A(n-1/2 \cdot \beta, n+1/2 \cdot \beta), \quad (1)$$

$$A[n-1/2 \cdot \beta, n+1/2 \cdot \beta] = (N_0!)^{-1/2} \\ \left| \frac{a_g(1, 1) \cdots a_g(1, n+1/2 \cdot \beta) a_g(1, n+1) \cdots a_g(1, N_0)}{a_g(N_0, 1)} \right|,$$

where β is the space vector between the electron and its hole, U_τ is the hydrogenic wave fn. with the quantum number τ and a_g, e stand respectively for the atomic fn. of ground state and excited state.

The state of total crystal containing the exciton is described by

$$(q, N; \beta, k_\alpha) = \prod_q \phi_{Nq}(a_q) \Gamma(k_\alpha, \beta), \quad (2)$$

where a_q and N_q denote respectively the amplitude variable of phonon with wave vector q and the quantum number of the phonon.

$$H = H(+) + H(-) \\ = - \sum_q e_q \text{Grad } V(r-n) (a_q \exp i(q, n) \\ + a_q^* \exp -i(q, n)) \quad (3)$$

is taken as the phonon perturbation, here $V(r)$ is an atomic potential for an exciton and e_q is the unit vector of phonon with longitudinal mode.

According to the current view point (Fröhlich *et al.*), the interaction energy between the exciton and the phonon is given by

$$\Delta \epsilon = \sum_{k_\alpha} \sum_q \sum_{\beta} \frac{|H(\pm)(N_q \mp 1, k_\alpha', \beta'; N_q, k_\alpha, \beta)|^2}{\epsilon(k_\alpha, \beta) - \epsilon(k_\alpha', \beta') \pm \hbar \omega_q} \quad (4)$$

Substituting (1), (2) and (3) into (4), we can get $\Delta \epsilon$ as the fn. of $N_q(T)$. Matrix elements in eq. (4) were calculated by the author,⁵⁾ but the adaptation of the result to eq. (4) is hampered by the serious mathematical difficulties. However, as far as the calculation of light absorption is concerned, fortunately, these difficulties are quite removed and the very simple calculation is enough to clarify the shift of the absorption line.

This situation is due to the following two reasons.

(I). Only the special exciton of $k_\alpha=0$ plays the essential role in the light absorption.

When an electron makes transition from the valence band to the exciton state, absorbing the light of which wave vector is σ , the selection rule,

$$k_\alpha + \sigma = 0,$$

is gained.³⁾ As $|\sigma|$ is so small compared with the

reciprocal of lattice constant, we may put approximately

$$k_\alpha \approx 0. \quad (5)$$

This shows that the bottom of exciton band plays an important role in the light absorption. Experimental data show an absorption maximum corresponding to eq. (5). Shift with temperature of it is considered here.

(II). The hydrogenic motion of an electron (or hole) in the exciton state is adiabatic to the phonon perturbation.

This will reasonably be agreed, because the ratio of period of phonon motion to that of electronic rapid motion is at least of the order of 10^3 . According to this supposition, the distance between the electron and the hole is kept constant, which should be determined by $|\sigma|$, at any temperature. i.e.,

$$\beta = \beta_\sigma = \text{const.} \quad (6)$$

By the conditions (I) and (II), the motion of our exciton must be restricted by the rule

$$k_\alpha \rightarrow \pm q. \quad (7)$$

The change of β is responsible for the light absorption of longer wave length and this is shown as the tail part of long wave length limit in the absorption spectrum.

Thus the matrix element is very simplified and is got by (1), (2), (3), (5) and (6). It is

$$H(\pm)(\pm q, \beta_\sigma) = \mp (MN_0)^{-1/2} \bar{V} \left(\frac{\hbar}{2\omega_q} \right)^{1/2} \\ \begin{cases} N_q^{1/2}, \\ (N_q+1)^{1/2}, \end{cases} \quad (8)$$

where \bar{V} is the averaged value of V and stands for the interaction constant between the exciton and the phonon. This equation should be valid for the both cases of polar and non-polar crystals, since the effects of the polarization field on the electron and the hole would tend to cancel. The character of the potential V will be discussed at the next opportunity.

The calculation of shift with temperature of the line is from here quite similar to the Fan's treatment²⁾ and it is repeated here.

The author is grateful to Prof. K. Ariyama for his continual encouragement and also to Prof. H. Shoji for his giving convenience to this study.

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On the Atom-exciton Interaction

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The study on the interaction potential between an atom and an exciton was not done in the previous paper¹⁾ where the light absorption in the insulating crystal was treated. And any proposal on the potential has not been done yet. However, recently it was required strongly to clarify the interaction in the various problems of the ionic crystal.²⁾ In this paper an idea of the interaction mechanism is proposed.

The exciton which is composed by an electron and a hole would behave itself like a rotating dipole, i.e. the exciton-dipole makes a field in the electronic shell of the nearest neighbouring atom. This time dependent field would produce a polarization since it can be perfectly followed by the electron cloud in the shell. Now, we consider about the interaction between the induced dipole and the exciton dipole. The period of the rotation of the exciton-dipole is not so small that the Dirac radiation process is not important and only the electro static interaction should be taken into account which is

$$V_{\ell}(\mathbf{r}) = \frac{a}{(4\pi\epsilon)^2} \left\{ \text{Grad} \frac{(\boldsymbol{\mu} \cdot \mathbf{r})}{r^3} \cdot \text{Grad} \right\} \frac{(\mathbf{u} \cdot \mathbf{r})}{r^3}$$

at given time t . Where a , ϵ and $\boldsymbol{\mu}$ denotes respectively the polarizability of the ion, the dielectric constant of the medium and the exciton-dipole moment. The direction of the induced dipole moment is nearly parallel (or anti-parallel) to that of the distance vector (\mathbf{r}) between the atom and the exciton as far as r is large compared with the exciton radius. Accordingly the interaction energy between the both dipoles becomes large in the absolute value only when the direction of the exciton-dipole is nearly parallel (or anti-parallel) to that of \mathbf{r} and the exciton has the tendency to rotate in the plane involving \mathbf{r} . Then the above expression is simplified and give $9a\mu^2 \cdot \cos^2 \omega t / (4\pi\epsilon r^3)^2$, where we take only the \mathbf{r} -parallel

component of the exciton dipole moment and ω is the angular velocity of the exciton. Replacing $\cos^2 \omega t$ by $\overline{\cos^2 \omega t} = 1/3$, we get the representation of the interaction potential. More detailed treatise by the use of the second order perturbation theory will give the equivalent result. Thus it is of the van der Waals type, weak, attractive and of short range.

An application of this interaction potential will be published shortly in anywhere.

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On the Radiation Damping and the Decay of an Excited State

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According to Hellund¹⁾, the excited state of an atom never shows the decay of Weisskopf-Wigner type, i.e., the exponential type. His conclusion is of course correct mathematically, but we can hardly imagine that Weisskopf-Wigner's result is entirely meaningless. Perhaps it would be an asymptotic expression for the extreme values of certain parameters. We here ascertain this expectation from the damping theory.

We have to solve the equation for the transformation function $S(t)$,

$$i \partial S(t) / \partial t = (H_0 + H) S(t) \quad (t > 0), \quad (1)$$

under the conditions $S(t) = 0$ ($t < 0$) and $S(+0) = 1$. H_0 and H are the free and the interaction Hamiltonian, respectively. In order to reformulate the damping theory as the initial value problem, it is convenient to use the complex Fourier transformation²⁾ by means of which $S(t)$ is transformed into the upper half plane of the energy variable as

$$\tilde{S}(E) = \frac{1}{i} \int_0^{\infty} e^{iEt} S(t) dt \quad (\text{Im } E > 0).$$

The inverse transformation is given by

$$S(t) = \frac{i}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} \bar{S}(E) dE \quad (\delta > 0). \quad (2)$$

From the Fourier transform of (1) we get

$$(E - H_0 - H) \bar{S}(E) = 1. \quad (1')$$

It is to be noted that the unity on the right hand side of (1') comes from the initial value of $S(t)$. In the representation in which H_0 is diagonal, the formal solution³⁾ of (1') is

$$\bar{S}(E) = \left(1 + \frac{1}{E - H_0} U(E) \right) \frac{1}{E - H_0 + i/2 \cdot \Gamma(E)}, \quad (3)$$

where

$$U(E) = \left(H + H \frac{1}{E - H_0} U(E) \right)_{nd}, \quad (4)$$

$$\Gamma(E) = 2i \left(H + H \frac{1}{E - H_0} U(E) \right)_a.$$

Now, if we regard the initial state as an eigenstate of H_0 with the eigenvalue E_0 and write the probability amplitude of the state in the form $a(t) \cdot e^{-iE_0 t}$, then we obtain

$$a(t) = \frac{i}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} \frac{e^{-i(E-E_0)t}}{E - E_0 + i/2 \cdot \Gamma(E)} dE \quad (5)$$

from (2), (3), and (4). $\Gamma_0(E)$ is the value of $\Gamma(E)$ in the initial state. We shall here assume, following Weisskopf-Wigner and Hellund, that there are only two atomic states; an excited state 0 (energy E_0) and the ground state G (energy E_G) plus one photon (energy k). Since $H_a = 0$, we can put Γ_0 into the form,

$$\Gamma_0(E) = 2i \int_0^\infty \frac{f(k) k^2 dk}{E - E_G - k}, \quad (6)$$

in which $f(k)$ is the positive real function of k . Although the integral is generally divergent, we suppose that it could become uniformly convergent by the renormalization or the out-off procedure. As Hellund has shown, the integrand of (5) has no singularities other than the branch point $E = E_G$ of Γ_0 . This fact excludes the exponential decay.

We put $k = E - E_G$ (> 0) and $\gamma = \text{Re } \Gamma_0(E_0)$ and assume $\gamma \sim |\Gamma_0(E_0)|$. The case $k_0 \gg \gamma$ alone is interesting here. It is then easily found from (6) that $a(t) \simeq 1$ and $\dot{a}(t) \simeq 0$ for $k_0 t \gg 1$. The latter, which confirms Hellund statement, is essentially due to $H_a = 0$.⁴⁾ However there is little physical significance in the behavior at $t \simeq 0$. We have rather to deal with the time $1/k_0 \leq t \leq 1/\gamma$. We divide this time interval into two regions: $t \leq 1/\sqrt{k_0 \gamma}$. For the most important interval $1/\gamma \gg t \gg 1/\sqrt{k_0 \gamma}$,⁵⁾ we have

$b \equiv (k_0 t) (\gamma t) \gg 1$. Putting $(E - E_0)t = -b\xi$, (5) becomes

$$a(t) = \frac{1}{2\pi i} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{e^{\xi b \gamma}}{\xi - i \cdot \Gamma_0(E) t / 2b} d\xi. \quad (7)$$

We are going to find the asymptotic form of (7) for $b \gg 1$ and $\gamma t \gg 1$. As is clear from the functional form and the uniform convergency of the integral (6), $\Gamma_0(E)$ is regular at $E = E_0$ on the Riemannian plane and we have

$$\lim_{|E| \rightarrow \infty} |1/E \cdot \Gamma_0(E)| = 0 \quad (8)$$

and

$$\left| \frac{1}{(E_0 - E_G)^n} \Gamma_0(E_0) \right| \geq \left| \frac{1}{n!} \Gamma_0^{(n)}(E_0) \right|,$$

which permit us to expand $\Gamma_0(E)$ as follows:⁶⁾

$$\Gamma_0(E) = \Gamma_0(E_0) (1 + O(\gamma t)) \quad (|\xi| \gtrsim 1). \quad (9)$$

We now take a positive real number c_1 of order 1 and divide the integral (7) into three parts:

$$a(t) = \int_{-c_1}^{c_1} + \int_{-\infty}^{-c_1} + \int_{c_1}^{\infty}. \quad (7')$$

Integrating by parts and using (9), the last two integrals can be easily estimated to be $O(1/b)$. On the other hand, the right hand side of (9) may be inserted into the first integral and thus the integrand has the sole simple pole at $\xi = i/2b \cdot \Gamma_0(E_0)t (1 + O(\gamma t))$. We can replace that integral by the contour integral around the pole minus the integral R along the upper semi-circle with the radius c_1 . Since there is another positive number c_2 of order 1 such that

$$|\xi - i \cdot \Gamma_0(E_0)t / 2b| < 2c_2 \quad (|\xi| \sim 1),$$

$|R|$ is found to be

$$|R| < \frac{2c_1 c_2}{\pi} \int_0^{\pi/2} e^{-b \cdot c_1 \sin \theta} d\theta < \frac{c_2}{b},$$

Therefore we obtain

$$a(t) = e^{-1/2 \cdot \Gamma_0(E_0)t (1 + O(\gamma t))} + O(1/b). \quad (10)$$

This shows that the initial state indeed decays exponentially for the time interval $1/\gamma \gg t \gg 1/\sqrt{k_0 \gamma}$. For $1/\sqrt{k_0 \gamma} \gg t \gg 1/k_0$, $a(t)$ would show the more complicated behaviors as a transient to the case we have just studied. We can say nothing definite about the time variation of $a(t)$ for $t \gg 1/\gamma$ without regard for the details of $f(k)$ because the integral (7) is then sensitive to the features of $\Gamma_0(E)$ near the branch point $\xi = 1/\gamma t$, and really there is not so much signi-

fiance in this interval that we do not discuss further. But it is expected from several examples that $\alpha(\ell)$ decreases more slowly than the exponential decay.

The authors wish to express their gratitude for the financial aids from the Yukawa Fellowship of Osaka University.

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- 6) We make use of this specification alone about $\Gamma_0(E)$, so that the result (10) is fairly general.

A Test of Charge Independence by the Nuclear Reactions

Yoichi Fujimoto* and Yoshio Yamaguchi**

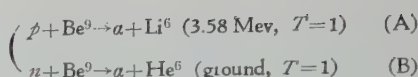
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October 5, 1953

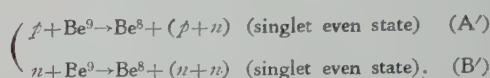
Since Adair¹⁾ pointed out the isotopic spin to be a good quantum number, the charge independence has become one of the most interesting topics in nuclear physics. However, most of the examples, which are regarded as strong support for charge independence, are interpreted equally well by the lesser restriction of charge symmetry. In fact there seems to be rather few definite arguments:

- a) correspondence of levels in such triad nuclei as C^{13} , N^{13} and O^{13} ,
- b) ft -values of allowed β -decays in light nuclei, etc.,

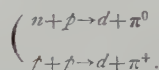
for charge independence. Therefore, it seems to us noteworthy to remark the possible test of charge independence based on nuclear reactions. If we use the triad (or more generally, polyad) nuclei, we can test charge independence as follows. Let us consider the reactions:



or



Assuming the charge independence, the ratio of differential cross sections (A) and (B) or (A') and (B') at any angle is turned out to be 1:2 (of course, discarding a deviation due to the differences of Coulomb forces and of the energy levels). This offers a nice test of charge independence. Also (A) and (B) are analogous to the famous pion reactions:



Consulting the "telephone book", we can easily write down many other useful reactions, but may not necessitate any further discussions on them.

One of other possibilities to test the charge independence is given by the nuclear reaction, which goes through one particular level of compound nucleus. In this case one can assign to the compound nucleus a characteristic isotopic spin. If the compound nucleus has several decay modes, their branching ratios depend quite critically on isotopic spin values of both compound state and final nuclear states. This fact again gives the valuable test of charge independence.

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Effect of Electromagnetic Radiation on Lamb Shift

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October 5, 1953

The Lamb-shift is an effect associated with the coupling of an electron to the vacuum. Electro-

magnetic field, if present, will be a source of additional perturbation, which also influences the value of the line-shift.

There are two ways in which an electron in a radiation field is transferred to the intermediate energy state, E_i from its ground state, E_n :-

(i) It absorbs a photon \vec{k} and goes to the state i . Starting with N_k photons, the number of photons in the intermediate state will be $N_k - 1$.

The electron then goes back to its original state, n , emitting the photon \vec{k} in the process.

(ii) The transition can also take place in the reverse order. In this case, the number of photons in the intermediate state will be $N_k + 1$.

The absorption transitions do not arise in the case of vacuum.

Recently, Auluck and Kothari discussed the effect of e.m. radiation on the self-energy of a free and bound-electron. However, the expressions for the shift as given by them do not contain the contribution from the additional transitions of the type (i). This effect is, however, of the same order, and is duly taken into account in this present work, which also gives the results of numerical evaluation of the shifts for the $2s$ and $2p$ levels of Hydrogen.

The second-order matrix element for the transverse self-energy arising due to weak interaction with black-body radiation at temperature, T , gives for this additional interaction-energy a value,

$$W = \pi/3 \cdot a \cdot RT/E \cdot RT \quad (1)$$

for a free-electron with energy E ; R is the Boltzmann gas-constant, while $a(=e^2/\hbar c)$ is the Sommerfeld's fine-structure constant. We get identical results on Dirac's one-electron theory and on 'hole' theory (negative energy states, all occupied), as one would rather expect in this particular case. One gets similar matrix elements on 'hole' theory because of the anti-symmetry of the wave-functions in the coordinates of the original electron and the one supplied from the vacuum.

We have considered above, the interaction of a free electron with black-body radiation. A more general case will be to consider the intensity of each spectral element to be an unspecified function of its frequency. This leads us to a significant result. It is found that the electron-photon self-energy is the same for two isotropic photon-fields of frequency ν_1 and ν_2 , provided their intensities are equal.

The perturbing effect of radiation on a bound electron will manifest itself in a slight shift of the energy levels. In the non-relativistic approximation,

the energy-state (n, l) of the Hydrogen atom is shifted by an amount

$$W'(T) = -\frac{2a}{\pi\mu} \int_0^\infty \frac{x dx}{e^{2\pi x} - 1} \sum_i \frac{(E_i - E_n)^2 f(n, i)}{q_i^2 - x^2} \quad (2)$$

with

$$\mu = mc^2$$

and

$$q_i = (E_i - E_n)/2\pi\hbar T \quad (3)$$

In the above expression,

$$f(n, i) = (3m/\hbar) \nu_{ni} \left| \int \psi_n^* \vec{r} \psi_i dV \right|^2 \quad (4)$$

is the oscillator-strength for the transition, $n \rightarrow i$, of frequency ν_{ni} of the atom, characterized by the wave-functions, ψ_n and ψ_i in the initial and final energy state E_n and E_i respectively. The summation, i , extends over the whole spectrum, discrete as well as continuous. Expression (2) has been obtained by assuming that the photon distribution is governed by the relation

$$N_k = 1/(e^{ck\hbar/RT} - 1). \quad (5)$$

The only interesting case is when $RT \ll 1$. We obtain for the shift the expression

$$W'(T) = -\frac{\pi a}{3\mu} (RT)^2 \left[1 - \frac{2\pi^2}{7} \times \sum_i \frac{f(n, i)}{(E_i - E_n)^2} (RT)^2 + \dots \right]. \quad (6)$$

It is generally recognised that the sun emits radiation which has an energy-density and spectral-composition different from that of a black-body at the same temperature. We can assume that the low-frequency part at the temperature of the sun, corresponds to a black-body curve at much higher temperature. If we cut off the frequency spectrum at $k=k_0$, the shift due to the low-frequency quanta is given by

$$W'(T) = -\frac{a}{\pi\mu} \left[RT(E_i - E_n) f(n, i) \times \log \left| \frac{E_i - E_n + ck_0}{E_i - E_n - ck_0} \right| + (E_i - E_n)^2 f(n, i) \times \log \left| \frac{(E_i - E_n)^2 - c^2 k_0^2}{(E_i - E_n)^2} \right| + \dots \right] \quad (7)$$

for $RT \gg 1Ry$.

Assuming that k_0 is equal to one Rydberg, we get for the difference between, the shifts in the $2s$ and $2p$ levels,

$$W'(T, 2\lambda) - W'(T, 2s) \cong 58(RT/Ry) \text{ megacycles. (8)}$$

The various terms in E_t and E_n are evaluated numerically in a manner, almost similar to that indicated by Bethe, Brown and Stehen. It is interesting to note that the additional shift is in a direction opposite to that for the Lamb shift.

The details of the work, along with the discussion of the results will be published elsewhere. My thanks are due to Professor D. S. Kothari for his interest and to Dr. F. C. Auluck for guidance; also due, to the Atomic Energy Commission, Government of India, for the grant of a fellowship and for sponsoring this scheme.

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Note on the Meson Theory of Nuclear Force

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It is necessary to make clear how far we can explain experimental results about nuclear forces assuming the present meson theory. Our purpose here is to investigate into the characteristic features of the meson potential in the intermediate and high energy regions.

Near and outside of the range of nuclear force, i.e., in the outside region, we have adopted the symmetrical fs meson potential of the second and fourth order perturbation calculation with pv coupling in the static approximation calculated by M. Taketani et al.¹⁾ In this region, the higher order contributions do not alter the features of this potential so severely.^{2,3)} In the inside region where the static meson potential becomes meaningless, we have adopted phenomenological potential, i.e., hard core or square well, which may well be energy dependent, but what has been

adopted here fits to the low energy data. We emphasize that in the state where S wave plays an important role, this inside phenomenological potential is determined, though not uniquely, so as to fit the low energy scattering parameters, whereas the phase shifts of P-, D-, ... waves are almost independent of the features of the potential in the region $x \lesssim 0.3$,⁴⁾ where x is the distance between two nucleons in the unit of meson Compton wave length.

The coupling constant $g^2/4\pi$ is taken as 0.08. Calculations are performed by the numerical integration. Our results are not the final ones but serve only as a starting point for future works.

Here we should like to point out:

(I) Some of the values in the Table 3 of reference I are not correct. The central force in the triplet even state is so strongly repulsive for the deuteron to be bound, that we have to cut off this potential at $x \lesssim 0.5$. As an example, in the region where $x \lesssim 0.6$, the central attractive well of the depth of about 125 Mev and zero tensor potential give the correct binding energy. These phenomenological potentials of the inside region together with the outside meson potential give also nearly good value of the effective range, the D state probability and the quadrupole moment. However, we do not regard it to be the best one. At any rate, $fs(pv)$ meson potential has to be cut off at $x \lesssim 0.5$ for the deuteron to be bound. The main part of sixth order $fs(pv)$ potential calculated by Machida et. al.²⁾ gives the inside potential similar features to our phenomenological one. This is not inconsistent with the results of Brueckner and Watson.³⁾

(II) The potential by the fs meson theory with either pv or fs coupling can give the characteristic features of neutron-proton and proton-proton scattering up to energy of 100 Mev, because:

i) As has already been shown in reference I and by Brueckner and Watson,³⁾ fs meson potential fits to the low energy data. Therefore, S-wave phase shifts at high energies are not so much different from those due to many phenomenological potentials.⁵⁾

ii) For the neutron-proton scattering, the meson potential in the triplet odd state is, near and outside of its range, rather small. Moreover, tensor potential is stronger than central one in this region. Therefore, the effects of 3P_0 -, 3P_1 - and 3P_2 -phase shifts cancel out as a whole and consequently the angular distribution is almost symmetric about 90° .

iii) For the proton-proton scattering, the potential in the singlet even state is, near and outside of its range, every small. Therefore, the phase shifts of 1D -wave is also small and consequently the destructive

interference does not break the isotropy of angular distribution so severely.

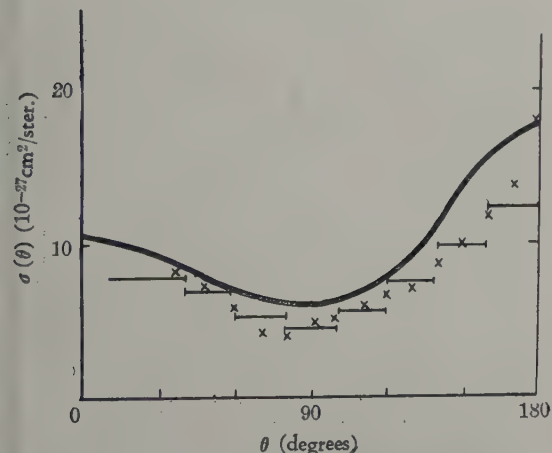
We show the angular distribution of n-p and p-p scattering in the center of mass system in Figs. 1, 2 and 3. The total cross section of n-p scattering at 90 Mev which is larger by about 25% than the experimental value, and the larger differential cross section of p-p scattering at 40 Mev are due to the large phase shift of triplet a wave (that reduces to triplet S wave in the absence of tensor force) and of singlet S wave respectively. But these are somewhat reduced by taking another way of cutting off in the inside region.

As has been stated before, the potential near and outside of its range, which determines the P- and D-wave phase shifts, is not altered so severely by the higher order corrections, so that our remarks above will keep its validity. At the present stage of meson theoretical approach to nuclear force, we consider it necessary to find these small corrections to the meson potential near and outside of its range phenomenologically from experiments.

More detailed discussion will soon be published in this journal.

We wish to express our cordial thanks to the members of group of nuclear force, especially S. Onuma for their illuminating discussion and guidance. We are also grateful to Professor M. Kobayasi for his encouragement. We are indebted to Professor Brueckner for his discussion during his stay in Kyoto in Sept. 1953.

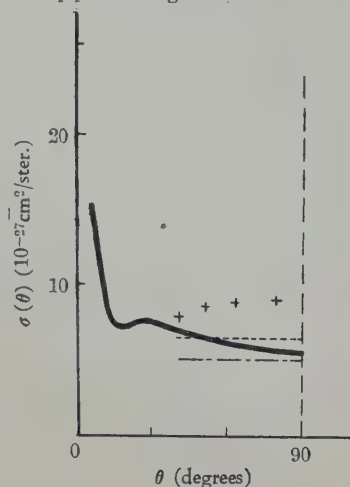
Fig. 1
n-p scattering at 90 Mev.



The crosses are the counter data; the horizontal lines are the cloud chamber data. (see reference 5.)

Fig. 2

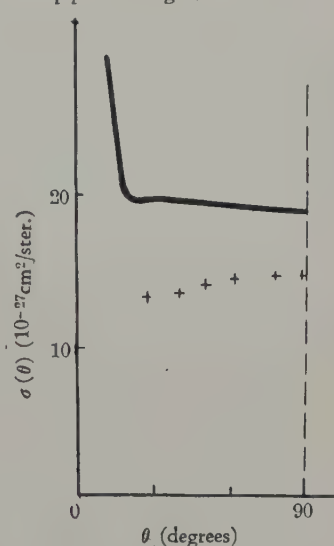
p-p scattering at 90 Mev.



..... Experimental data at 75 Mev.⁶⁾
----- Experimental data at 105 Mev.

Fig. 3

p-p scattering at 40 Mev.



• Experimental data at 32 Mev.⁷⁾

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Effect of Collective Motion on Beta Decay

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The collective properties, especially the surface distortion, of a nucleus tend to reduce the nuclear transition probability as a result of a partial orthogonality of the states. We call such effect an unfavoured factor (F -factor).

We define the experimental F -factor and the theoretical F -factor according to the following expressions respectively;

$$F_{\text{exp}} = t_{\text{exp}}/t_{\text{sp}}; \quad F_{\text{theo}} = t_{\text{coll}}/t_{\text{sp}}, \quad (1)$$

where t_{exp} is the experimental half life time, t_{coll} and t_{sp} is the theoretical half life time for the collective model and the single particle model respectively. We compared F_{theo} with F_{exp} for unfavoured allowed β -decay of odd nuclei.

The main contribution to the unfavoured factor will be a effect of the nuclear surface distortion, so that we make use of the collective model in A. Bohr's strong coupling approximation²⁾. Then, we can see that the F -factor is composed of following two parts;

- (1) surface distortion effect,
- (2) rotational effects, $\begin{cases} \text{(i) core,} \\ \text{(ii) odd nucleon.} \end{cases}$

As to the superallowed transition, we did not treat them in this paper. Nuclear spin and parity are determined according to j - j shell scheme, and according to A. Bohr³⁾ we suppose that $K=Q=I=j$ to

the ground state of a nucleus. Then the nuclear wave function of ground state is given by

$$\Psi_{I, I', 0} = \frac{1}{\sqrt{2}} [\varphi^{I, 0}(\beta\gamma) [x_I D_{M, I'}(\theta_i) + x_{-I} D_{M, -I'}(\theta_i)], \quad (2)$$

hereafter we use the same notations with the reference 2.

In the allowed β -decay, there are only two essentially different possibilities for the interaction, that is, Fermi and Gamow-Teller interaction. We must pay some attention to interaction operators, because the states of the decaying odd nucleon, x_{Ω}^I , are quantised in the relative coordinate (λ) to the nuclear core, so that the operators for a odd nucleon must be represented in this coordinate. On the other hand, as our observation is done in the fixed system (x) in space, we must use the operator $\sigma_x = \sum \sigma_{x\lambda}$ to $G-T$ interaction and in this expressions, $\sigma_{x\lambda}$'s are the transformations from coordinate (λ) to (x). Thus, we think that the odd nucleon transforms by $\sigma_{x\lambda}$ and the rotational state of nucleus as a whole by $\sigma_{x\lambda}$'s. As the results, we find the following expressions to F_{theo} ;

$$F_{\text{Fermi}} = 1/(\varphi^{I, 0}|\varphi^{I, 0})^2 \equiv 1, \quad (3)$$

$$F_{I, I'-T=0} = [1/(\varphi^{I, 0}|\varphi^{I, 0})^2] 3[(I+1)/I]^2, \\ I' = I \neq 1/2, \quad (4.1)$$

$$F_{G-T=1} = [1/(\varphi^{I, 0}|\varphi^{I, 0})^2] [3/2] \\ \times [(2I+3)/(2I+1)], \quad I' = I+1, \quad (4.2)$$

$$F_{G-T=1} = [1/(\varphi^{I, 0}|\varphi^{I, 0})^2] [2/3] \\ \times [(2I+1)/(2I-1)], \quad I' = I-1. \quad (4.3)$$

In these expressions, the first term represents the distortion effect and next terms the rotational effect. In our model of the odd nucleus, if $I=0$, the nuclear core does not change its states at all before and after a transition, so that in this case the distortion effect is identically equal to 1.

In the distortion effect, we must take a special attention to the case of spin change ($1/2 \rightarrow 3/2$). Namely, if $I=1/2$, then the interaction between an odd nucleon and a nuclear surface oscillation vanishes and moreover if we take into account a correction term indicated by Davidson and Feenberg, β_1 of surface deformation parameter β which makes the effective potential for the surface oscillation minimum is equal to 0, whether the nucleus has a closed core or not. This result is not inconsistent with Ford's formula of nuclear quadrupole moment derived from semi-quantum mechanical method, but in his formula it is not necessary that $\beta_1=0$ when $I=1/2$,

The numerical values of rotational terms and distortion terms are given in the Table I and II. The distortion effects do not depend appreciably on mass number A and have values about 1~1.3. If we neglect the $D-F$ correction term, $1/(\varphi'|\varphi)^2$ depends appreciably on A , only when the spin change is $(1/2-3/2)$, (Table II). When we take into account the $D-F$ correction term, the wave function of the nucleus of spin $1/2$ does not depend on deformation parameter γ , so that we cannot take a region of the integration for $\gamma=\gamma_1$, ($\gamma_1=\pi$), from

Table I. Rotational effects

$I' \backslash I$	1/2	3/2	5/2	7/2	9/2
I	—	25/3	147/25	243/49	363/81
$I+1$	3	9/4	2	15/8	9/5
$I-1$	—	3	9/4	2	15/8

Table II. Unfavoured factors for the odd nuclei in allowed transition.

Nu- cleus	Spin	$1/(\varphi' \varphi)^2$	F_{theo}	F_{exp}
Ne ²³	5/2-3/2	1.38	3.11	6.2
S ³⁵	3/2-3/2	1	8.33*	12
Ca ⁴⁵	7/2-7/2	1	4.96*	100
Zn ⁶⁹	1/2-3/2	1.24(4.99)	3.72(14.56)	20
Ge ⁷⁵	1/2-3/2	1.26(5.61)	3.78(16.82)	51
Se ⁸¹	1/2-3/2	1.24(5.51)	3.72(16.54)	32
Br ⁷⁷	3/2-1/2	1.25(4.81)	3.75(14.44)	26
Sn ¹²¹	3/2-5/2	1.3	2.6	46
Te ¹²⁷	3/2-5/2	1.25	2.51	180

*; the values calculated as $G-T$, $\Delta l=0$ transition,
(); the values neglected the $D-F$ correction term.

Table III. Theoretical β_1 and experimental β_Q . We used β_Q of Ford and data of Mack⁶⁾

	Ne ²³	S ³⁵	Ca ⁴⁵	Zn ⁶⁹	Ge ⁷⁵	Se ⁸¹	Br ⁷⁷	Sn ¹²¹	Te ¹²⁷
β_1	0.515	0.348	0.388	0	0	0	0.23	0.183	0.18
β_Q		0.12							

	Na ²³	Cl ³⁵	Sc ⁴⁵	Ga ⁶⁹	As ⁷⁵	Br ⁸¹	Se ⁷⁷	Sb ¹²¹	I ¹²⁷
β_1	0.44	0.348	0.388	0.24	0.232	0.24	0	0.217	0.212
β_Q		0.15		0.16	0.17	0.13		0.05	0.09

$-\infty$ to $+\infty$. The important region of γ is $|\gamma-\pi| \lesssim 1/\sqrt{2t}$ for $I=3/2$ and $1/\sqrt{2t} \sim \pi/4$ for our nuclei. Then $1/(\varphi'|\varphi)^2 \sim 1.3$. The F_{theo} -factor does not change its value so sensitively to these cut-off values of γ . And we used $\beta^1 d\beta$ and $(1+x)^4 dx$, ($\beta_1 x = \beta - \beta_1$), for spin change $(1/2-3/2)$ and others respectively.

The rotational effects are given by Table I. It is important and interesting result that the rotational effect and distortion effect have the same order contribution to F -factor, but the former is larger by factor 2 than the latter.

If we compare the theoretical values of β_1 with β_Q expected from Ford's formula, we obtain such values as Table III. We suppose rather that β_Q gives a lower limit of β_1 but β_1 is larger by factor 2~4 than β_Q . On the other hand, F_{theo} is smaller than F_{exp} , and even if we neglect the $D-F$ correction term, it is about 75% at best, and there is a group that F_{exp} is larger than 100, but it is difficult to find reason such large unfavouredness.

Generally, it seems that F_{exp} does not depend on the numbers of odd nucleons appreciably. As a result, we shall not be able to explain the unfavoured factor exactly only by the surface distortion effect, so that it is difficult to discuss with mixing of interaction types.

It is author's pleasure to acknowledge his indebtedness to Professor J. A. Wheeler for his interest, to Professor M. Kobayasi and other members in our laboratory for their valuable discussions and continual encouragement.

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On a Five Dimensional Representation of the Electromagnetic and Electron Field Equations in a Curved Space-time

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We propose a five dimensional representation of the electromagnetic and electron field equations in a curved space-time which is immersible in a flat space of five dimensions. This representation is simplified by restricting basic coordinate transformations to the transformations that connect the "equivalent" frames of reference. In the de Sitter space-time our results are compared with the formalism proposed by Dirac some twenty years ago. We point out the gap of Dirac's formalism and show that the gap automatically disappears in our theory. Particularly we obtain a new representation of the electron wave equation other than that given in Dirac's paper. The present investigation is a generalization of Dirac's formalism to a more general space-time.

§ 1. Introduction

A space-time in general relativity is a four dimensional Riemannian space having the fundamental form of the type $(-, -, -, +)$. Some of them can be represented by a hypersurface of a five dimensional flat space; that is, they are immersible in a five dimensional flat space. For example, the de Sitter space-time which is interpreted as a stationary universe can be represented by a pseudo-hypersphere

$$-(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2 - (x^5)^2 = -R^2, \quad (1.1)$$

R being the radius of the universe, of a five dimensional flat space having the fundamental form

$$ds^2 = -(dx^1)^2 - (dx^2)^2 - (dx^3)^2 + (dx^4)^2 - (dx^5)^2. \quad (1.2)$$

And the Einstein universe can be considered as a pseudo-hypercircular-cylinder of a five dimensional flat space. By making use of these facts some properties concerning the gravitational field can be expressed in an advantageous form, as will be illustrated in the following.

Let us consider the motion of test particles in the de Sitter space-time whose fundamental tensor is given by g_{ij} in coordinates x^i .^{*} As stated above, this space-time can

^{*} For instance, g_{ij} is given by

$$g_{11} = -(1 - r^2/R^2)^{-1}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2 \theta, \quad g_{44} = 1 - r^2/R^2$$

in a coordinate system $x^i = (r, \theta, \phi, t)$ suitably chosen. As for indices of a tensor Latin ones take the values 1, ..., 4 and Greek 1, ..., 5 in this paper.

be represented by a pseudo-hypersphere (1.1) of a five dimensional flat space given by (1.2). Test particles in the de Sitter universe move along the geodesic world lines in time-like directions given by

$$d^2x^i/ds^2 = -\left\{ \begin{matrix} i \\ st \end{matrix} \right\} (dx^s/ds)(dx^t/ds), \quad g_{st}dx^sdx^t > 0, \quad (1.3)$$

where $\left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ are the Christoffel symbols formed with respect to g_{ij} . It is usually not easy to solve (1.3) in an arbitrarily given coordinate system. However, if we treat the cartesian coordinates x^i 's in (1.2) as a particular type of coordinates, five in number and satisfying (1.1), of the de Sitter space-time, the first equation of (1.3) reduces to

$$d^2x^\lambda/ds^2 = x^\lambda/R^2. \quad (1.4)$$

The general solution of this can easily be obtained and is given by

$$x^\lambda = x_0^\lambda \cosh(s/R) + R(dx^\lambda/ds)_0 \sinh(s/R), \quad (1.5)$$

where x_0^λ and $(dx^\lambda/ds)_0$ are initial values of x^λ and dx^λ/ds at $s=0$ respectively.¹⁾ In this integration the second equation of (1.3) has been used.

Thus regarding the x^i 's as coordinates of the de Sitter space-time, we can describe the motion of test particles in a simple form. Similar situations hold not only for the motion but for some properties concerning the metric tensor of the de Sitter space-time. For instance, using the above five coordinates we can also obtain the finite form of transformation leaving g_{ij} invariant,¹⁾ which is interpreted as the one connecting the frames of reference equivalent to each other.²⁾ It can also be shown that we encounter similar situations also in the Einstein universe, etc.* Interpreting these results physically we may state as follows: in a curved space-time which is representable by a hypersurface of a five dimensional flat space we can give a simple form to the expressions of some properties concerning the gravitational field alone, if the five cartesian coordinates of the enveloping space are looked upon as a particular type of coordinates of the four dimensional space-time.

The above results are concerned with the gravitational field alone. In addition to this, if one considers the electromagnetic or electron field, is it possible to draw similar conclusions? Or, can any equations concerning the electromagnetic or electron field in a curved space-time be described in an advantageous form by using the above five coordinates? The present paper will give an answer to this question for the electromagnetic and electron field equations in a curved space-time which can be represented by a hypersurface of a five dimensional flat space. In detail will it be shown that these field equations can be represented in a favourable form adopting a particular type of five coordinates of the four dimensional space-time and further, if necessary, by using the unit vector normal to the hypersurface representing the given space-time.

For this purpose we now consider the electromagnetic and electron field equations³⁾ which are usually treated in general relativity. They are of tensor form

* See Appendix.

and invariant for arbitrary transformations of space-time coordinates. Therefore, ordinary differentiations being replaced by covariant ones, the field equation involves terms containing the Christoffel symbols and its individual component is really not so simple as it looks in tensor form. In the following, assuming that the space-time is immersible in a flat space of five dimensions, we represent the field equations in a five dimensional form by using the five coordinates of the enveloping flat space and, when necessary, by using also the unit vector normal to the hypersurface representing the given curved space-time (§§ 3 and 4). The five dimensional representation of the field equation thus obtained is of tensor form in the five dimensional space and each component of it is as complex as the original four dimensional form.

Further to simplify this representation, we shall pay attention to the concept of relativistic invariancy. In general relativity the physical laws are invariant for arbitrary transformations of space-time coordinates. On the other hand, in special relativity, the invariancy of physical equations is considered only with respect to the transformations which connect the equivalent frames of reference to each other. These are the so-called Lorentz transformations and are characterized by the property: that they preserve the fundamental tensor η_{ij} of the Minkowski space-time invariant. In a curved space-time there exist also transformations of the same character in general and we may consider that these transformations connect the frames of reference "equivalent" to each other in the space-time.* So we generalize the concept of Lorentz invariancy to a curved space-time and assume in the latter part of this paper that the electromagnetic and electron field equations may be invariant only for the transformations which leave the form of the gravitational potential g_{ij} invariant. On this assumption it will be shown that the five dimensional representation obtained in §§ 3 and 4 can be simplified further (§ 5).

Next we shall examine how the above representation becomes in the de Sitter space-time and shall compare it with the formalism proposed by Dirac about twenty years ago⁽⁶⁾ (§ 6).

Lastly it is to be noted that our study is entirely different from the various investigations hitherto made concerning the five dimensional unified field theory. That is, in the present paper the electromagnetic and electron fields are unrelated to the geometric structure of the space and therefore the five dimensional space is only a means to the purpose.

§ 2. Mathematical preliminaries**

Consider a four dimensional Riemannian space which has the fundamental form

* Here, the meaning of the term "equivalent" is somewhat different from that commonly used in general relativity. Speaking exactly, this means "equivalent with respect to some physical laws (i.e. the electromagnetic and electron field equations in the case of the present investigation)". Of course, even in this sense all the frames of reference are equivalent with respect to the Einstein's gravitational law. See Reference 4).

The equivalent world points in the gravitational field have been considered to be transformable to each other by the point transformations leaving the form of the gravitational potential g_{ij} invariant. See Reference 5).

** For this section see Reference 1).

$$ds^2 = g_{st} dx^s dx^t \quad (2.1)$$

of the type $(-, -, -, +)$.^{*} In order that this space-time be immersible in a five dimensional flat space with the line element^{*}

$$ds^2 = a_{\alpha\beta} dz^\alpha dz^\beta, \quad (2.2)$$

it is necessary and sufficient that a system of differential equations

$$a_{\alpha\beta} (F_i z^\alpha) (F_j z^\beta) = g_{ij} \quad (2.3)$$

have five independent solutions^{**}

$$z^\lambda = z^\lambda(x^1, \dots, x^4). \quad (2.4)$$

The symbol F_i denotes the covariant derivative with respect to g_{ij} . Particularly it holds that $F_i z^\lambda = (\partial z^\lambda / \partial x^i)$, because the z 's are scalars for transformations of coordinates x^i . In the flat space of five dimensions (2.4) defines a hypersurface, which is a representation of the given four dimensional space-time. We have from (2.4)

$$f(z^1, \dots, z^5) = 0, \quad (2.5)$$

which is another equation defining the same hypersurface.

If η^λ is the unit vector normal to the hypersurface,^{***} we have

$$a_{\alpha\lambda} \eta^\alpha F_i z^\lambda = 0, \quad a_{\alpha\lambda} \eta^\alpha \eta^\lambda = e, \quad (2.6)$$

where the quantity e is plus or minus one according to the character of the four dimensional space-time. Then it is readily shown that

$$a^{\lambda\mu} = g^{st} (F_s z^\lambda) (F_t z^\mu) + e \eta^\lambda \eta^\mu. \quad (2.7)$$

Here $a^{\lambda\mu}$ and g^{ij} are the conjugate tensors of $a_{\lambda\mu}$ and g_{ij} respectively. z^λ and η^λ must satisfy

$$\left. \begin{aligned} F_i F_j z^\lambda &= e b_{ij} \eta^\lambda - \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}_a (F_i z^\alpha) (F_j z^\beta), \\ F_i \eta^\lambda &= -b_{is} g^{st} F_t z^\lambda - \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}_a \eta^\alpha F_i z^\beta, \end{aligned} \right\} \quad (2.8)$$

where $\left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\}_a$ are the Christoffel symbols formed with respect to $a_{\lambda\mu}$ and the tensor b_{ij} is defined by these equations.^{****} The integrability conditions of (2.8) are the equations of Gauss and Codazzi for the hypersurface, which are not necessarily satisfied by any space-time. For instance, they do not hold for the space-time of Schwarzschild's exterior solution.¹⁾

Field quantities or wave functions are usually functions of the x 's. This means, from the five dimensional standpoint, that physical quantities are defined only on the hypersurface.

* Throughout this paper the coordinates x 's and z 's are assumed to be real.

** In this case the curved space-time is called to be of class one.

*** The existence of such a vector is a consequence of the assumption $\det. g_{ij} \neq 0$.

**** b_{ij} is the second fundamental tensor of the given curved space-time,

When one represents the field equations in a five dimensional form, it is needed to differentiate physical quantities with respect to z^λ . Nevertheless such a process of differentiation cannot be defined generally, because physical quantities are not defined outside the hypersurface. In order to give a meaning to such a process, the domain of definition of physical quantities in a five dimensional representation is not only the hypersurface, but must be extended to the vicinity of it, such as they are differentiable on the hypersurface with respect to z^λ .^{*} Then the operation of differentiating a physical quantity with respect to x^i is replaced by a linear combination of those with respect to z^λ , that is

$$\partial/\partial x^i = (F_i z^\alpha) (\partial/\partial z^\alpha). \quad (2.9)$$

If a field quantity is expressed by a tensor $v^{i_1 \dots i_n}$ in the four dimensional space-time, the five dimensional representation of the same quantity is given by^{**}

$$V^{\lambda_1 \dots \lambda_n} = v^{i_1 \dots i_n} (F_{i_1} z^{\lambda_1}) \dots (F_{i_n} z^{\lambda_n}). \quad (2.10)$$

We get easily from this

$$v_{i_1 \dots i_n} = V_{\alpha_1 \dots \alpha_n} (F_{i_1} z^{\alpha_1}) \dots (F_{i_n} z^{\alpha_n}), \quad (2.11)$$

$$V^{\lambda_1 \dots \lambda_n} \eta_\alpha = 0, \quad (2.12)$$

where the indices of the four dimensional component of a tensor are raised and lowered by means of g^{ij} and g_{ij} and those of the five dimensional representation by means of $a^{\lambda\mu}$ and $a_{\lambda\mu}$. (2.12) involves that a tensor $V^{\lambda_1 \dots \lambda_n}$ lies on the hypersurface and accordingly belongs to the curved space-time. Therefore, the five dimensional representation of a physical quantity must always satisfy such equations as (2.12). It also holds that

$$\left. \begin{aligned} F_j v_{i_1 \dots i_n} &= (F_j V_{\alpha_1 \dots \alpha_n}) (F_{i_1} z^{\alpha_1}) \dots (F_{i_n} z^{\alpha_n}), \\ v_{i_1 \dots i_n} w_{j_1 \dots j_m} &= V_{\alpha_1 \dots \alpha_n} W_{\beta_1 \dots \beta_m} \\ &\quad \times (F_{i_1} z^{\alpha_1}) \dots (F_{i_n} z^{\alpha_n}) (F_{j_1} z^{\beta_1}) \dots (F_{j_m} z^{\beta_m}), \end{aligned} \right\} \quad (2.13)$$

where the symbol F_λ denotes the covariant derivative with respect to $a_{\lambda\mu}$. These two equations give the relations of the covariant derivative and contraction of a tensor to their five dimensional representations respectively.

In the following two sections we are going to research into a five dimensional representation of the electromagnetic and electron field equations by using the mathematical preliminaries given in this section.

* The method of this extension is arbitrary in general, but sometimes it is replaced under restriction. See § 3.

** Throughout this paper the usual four dimensional component of a physical quantity is written by a small letter and its five dimensional representation by the corresponding capital letter.

§ 3. The electromagnetic field equations

If A^λ is the five dimensional representation of the four-vector potential a^i , we have

$$A^\lambda = a^i \nabla_i \mathcal{E}^\lambda \quad \text{and} \quad A_\alpha \eta^\alpha = 0. \quad (3.1)$$

Then the electromagnetic field strength f_{ij} , being the curl of the four-vector potential, is reducible to

$$f_{ij} = F_{\alpha\beta} (\nabla_i \mathcal{E}^\alpha) (\nabla_j \mathcal{E}^\beta), \quad \text{where} \quad F_{\lambda\mu} = \nabla_\mu A_\lambda - \nabla_\lambda A_\mu, \quad (3.2)$$

in consequence of (2.13). In order that the field strength be expressed by $F_{\lambda\mu}$,

$$\text{we must have} \quad F_{\lambda\alpha} \eta^\alpha = 0, \quad (3.3)$$

which is reducible by means of (3.1) to

$$\eta^\alpha \nabla_\alpha A_\lambda = -A_\alpha \nabla_\lambda \eta^\alpha. \quad (3.4)$$

By making use of (2.7) we have

$$\nabla_s f_i^s = (\nabla_\tau F_{\alpha\beta}) (a^{\beta\tau} - c \eta^\beta \eta^\tau) \nabla_i \mathcal{E}^\alpha.$$

So, if we assume that

$$\eta^\alpha \eta^\beta \nabla_\beta F_{\lambda\alpha} = 0, \quad (3.5)$$

the equation $\nabla_s f^{is} = j^i$, the first of the Maxwell equations, is written in the form

$$\nabla_\alpha F^{\lambda\alpha} = j^\lambda, \quad (3.6)$$

where j^λ is the five dimensional representation of the charge-current density vector j^i . This is justified if j^λ belongs to the given four dimensional space-time, which can be shown as follows.

Combining (3.5) with the covariant derivative of (3.3) with respect to $a_{\lambda\mu}$,

$$\text{we have} \quad F_{\lambda\alpha} \eta^\alpha \nabla_\beta \eta^\beta = 0.$$

If this holds good for any $F_{\lambda\mu}$, $\eta^\alpha \nabla_\alpha \eta^\lambda$ must be proportional to η^λ .^{*} Therefore, η^λ being a unit vector, it follows that

$$\eta^\alpha \nabla_\alpha \eta^\lambda = 0. \quad (3.7)$$

Thus the assumption (3.5) can be replaced by (3.7), which is no more related to the field strength. By (2.8) and (3.7) we have

$$\nabla_\mu \eta^\lambda = -a_{\mu\alpha} (\nabla_s \mathcal{E}^\alpha) (\nabla_i \mathcal{E}^\alpha) \delta^{st}, \quad (3.8)$$

from which and (3.3) we know that $\eta_\alpha \nabla_\beta F^{\alpha\beta}$ vanishes. Thus from (3.6) j^λ lies on the hypersurface, which is consistent with the statement that j^λ is the five dimensional representation of the four-vector of charge and current.

Furthermore we have from (3.4) and (3.7)

^{*} In this case η^λ defines the geodesics in the enveloping space.

$$\eta^\alpha \nabla_\alpha F_{\lambda\mu} = -F_{\alpha\mu} \nabla_\lambda \eta^\alpha - F_{\lambda\alpha} \nabla_\mu \eta^\alpha. \quad (3.9)$$

The proof of this is given by remarking that the order of covariant differentiations is immaterial in a flat space.

At first sight the equations (3.4), (3.7) and (3.9) seem to impose a new restriction upon the four-vector potential, the unit vector normal to the hypersurface and the field strength respectively. However, they define the differentiation of these quantities along the normal to the hypersurface. Therefore these equations do not restrict actually the values of the quantities on the hypersurface. In § 2 we have stated the necessity of extending the domain of definition of physical quantities and the above equations show how the extension must be done.

The conservation equation

$$\nabla_\alpha J^\alpha = 0 \quad (3.10)$$

for the charge and current is a direct consequence of (3.6). This is also the five dimensional representation of the ordinary conservation equation $\nabla_s j^s = 0$, as follows from (3.7).

Next, because we have from (3.9)

$$\eta^\alpha (\nabla_\alpha F_{\lambda\mu} + \nabla_\lambda F_{\mu\alpha} + \nabla_\mu F_{\alpha\lambda}) = 0,$$

the equation $\nabla_i f_{jk} + \nabla_j f_{ki} + \nabla_k f_{ij} = 0$, the second of the Maxwell equations, can be represented by

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = 0. \quad (3.11)$$

The five dimensional representation of the Lorentz condition $\nabla_s a^s = 0$ is given by

$$\nabla_\alpha A^\alpha = 0, \quad (3.12)$$

which is obtained in the same way as (3.10). In order that the gauge transformation $a_i \rightarrow a_i + \nabla_i \lambda$ be represented by

$$A_\lambda \rightarrow A_\lambda + \nabla_\lambda A, \quad \text{where } A = \lambda, \quad (3.13)$$

$$\text{it must hold that} \quad \eta^\alpha \nabla_\alpha A = 0. \quad (3.14)$$

This does not restrict the values of A on the hypersurface, as is seen by a similar consideration as (3.4), etc. On account of the Lorentz condition the function λ satisfies the equation $\square \lambda \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta \lambda = 0$, which can be written in a five dimensional form

$$a^{\alpha\beta} a^{\gamma\delta} (\eta_\alpha \nabla_\gamma - \eta_\gamma \nabla_\alpha) (\eta_\beta \nabla_\delta - \eta_\delta \nabla_\beta) A = 0. \quad (3.15)$$

When A satisfies (3.14), this reduces to

$$\square A \equiv a^{\alpha\beta} \nabla_\alpha \nabla_\beta A = 0.$$

Thus we can state that the five dimensional representation of the d'Alembertian operator is not given by \square , but generally by the operator acting on A in (3.15).

The energy-momentum tensor is written as

$$t_{ij} \equiv f_{is} f_j^s - g_{ij} f_{st} f^{st} / 4 = \bar{T}_{\alpha\beta} (F_i{}^\alpha) (F_j{}^\beta),$$

where

$$\bar{T}_{\lambda\mu} = F_{\lambda\alpha} F_\mu{}^\alpha - \alpha_{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4. \quad (3.16)$$

The right-hand member of (3.16) does not lie on the hypersurface, because it holds that

$$\gamma_i{}^\alpha \gamma_j{}^\beta \bar{T}_{\alpha\beta} = -e F_{\alpha\beta} F^{\alpha\beta} / 4 \neq 0.$$

Therefore as the five dimensional representation of the energy-momentum tensor we must take

$$T_{\lambda\mu} = \bar{T}_{\lambda\mu} - \eta_{\lambda\mu} (\gamma_i{}^\alpha \gamma_j{}^\beta \bar{T}_{\alpha\beta}) = F_{\lambda\alpha} F_\mu{}^\alpha - (\alpha_{\lambda\mu} - e \eta_{\lambda\mu}) F_{\alpha\beta} F^{\alpha\beta} / 4 \quad (3.17)$$

and not $\bar{T}_{\lambda\mu}$, which is a formal generalization of t_{ij} to five dimension. Then we have without any difficulty

$$T_{\alpha\beta} (F_i{}^\alpha) (F_j{}^\beta) = t_{ij}, \quad T_{\alpha\beta} (F_i{}^\alpha) \gamma_j{}^\beta = 0, \quad T_{\alpha\beta} \gamma_i{}^\alpha \gamma_j{}^\beta = 0,$$

which justify the representation (3.17).

In a similar way the equation of motion $F_s t^{ts} = j_s f^{ss}$ can be represented by

$$F_\alpha T^{\lambda\alpha} - \gamma^\lambda (\eta_\alpha F_\beta T^{\alpha\beta}) = j_\alpha F^{\alpha\lambda}, \quad (3.18)$$

which is also obtained directly from (3.17). Comparing (3.17) with the form of the ordinary energy-momentum tensor in special relativity we have an additional term in $T_{\lambda\mu}$, i.e. $e \eta_{\lambda\mu} F_{\alpha\beta} F^{\alpha\beta} / 4$. On account of this term the electromagnetic field contributes nothing to the invariant density as in the ordinary four dimensional theory, since we have from (3.17) $\alpha^{\alpha\beta} T_{\alpha\beta} = 0$.* There appears an additional term also in the equation of motion (3.18). This will be found to vanish in the de Sitter space-time (§ 6).

§ 4. The electron field equations

In this section we shall investigate the five dimensional representation of the Dirac equation³⁾

$$(\gamma^s \nabla_s + \kappa) \psi = 0, \quad \text{where} \quad \kappa = mc/\hbar \quad (4.1)$$

in a curved space-time, where ψ is a four-component spinor and the γ 's are four-rowed square matrices satisfying

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij}. \quad (4.2)$$

The covariant differentiation of ψ is given by

$$\nabla_s \psi = (\partial/\partial x^s + \Gamma_s) \psi, \quad (4.3)$$

where the Γ 's are four-rowed square matrices determined by

$$F_j \gamma_i \equiv \partial \gamma_i / \partial x^j - \left\{ \begin{smallmatrix} s \\ ij \end{smallmatrix} \right\} \gamma_s + \Gamma_j \gamma_i - \gamma_i \Gamma_j = 0 \quad (4.4)$$

to within arbitrary multipliers of unit matrix as additive terms.⁷⁾ In (4.4) $\left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\}$ are the Christoffel symbols formed with respect to g_{ij} .

*) This corresponds to $t_{st} g^{st} = 0$ for t_{st} , while $\alpha^{\alpha\beta} \bar{T}_{\alpha\beta} \neq 0$ for $\bar{T}_{\lambda\mu}$.

When the five dimensional representation of (4.1) comes into question, we must determine at first the relation between the γ 's and the ω 's which satisfy

$$\omega^\lambda \omega^\mu + \omega^\mu \omega^\lambda = 2\omega^{\lambda\mu}. \quad (4.5)$$

It is known that such ω 's can be represented also by four-rowed square matrices. If we put

$$\gamma_0 = \sqrt{e} \gamma_{[1\dots 4]} / \sqrt{g}, \quad \text{where } g = \det. g_{ij}, \quad (4.6)$$

it follows that

$$\gamma_0^2 = e, \quad \gamma_0 \gamma_i + \gamma_i \gamma_0 = 0. \quad (4.7)$$

Then (4.5) is satisfied by

$$\omega^\lambda = \gamma^s \nabla_s \varepsilon^\lambda + e \eta^\lambda \gamma_0, \quad (4.8)$$

which gives ω^λ in terms of the γ 's. Conversely the γ 's are expressed in terms of the ω 's by

$$\gamma_i = \omega_\alpha \nabla_i \varepsilon^\alpha, \quad \gamma_0 = \omega_\alpha \eta^\alpha. \quad (4.9)$$

Corresponding to (4.4) if we define four-rowed square matrices A_s by

$$\nabla_\mu \omega_\lambda \equiv \partial \omega_\lambda / \partial \varepsilon^\mu - \left\{ \begin{matrix} \mu \\ \lambda \mu \end{matrix} \right\}_\alpha \omega_\alpha + A_\mu \omega_\lambda - \omega_\lambda A_\mu = 0, \quad (4.10)$$

then from (4.4) and (4.9) matrices $A_\alpha \nabla_i \varepsilon^\alpha - \Gamma_i$ are determined (to within arbitrary multipliers of unit matrix as additive terms), that is:

$$A_\alpha \nabla_i \varepsilon^\alpha - \Gamma_i = e b_{is} \gamma_0 \gamma^s / 2. \quad (4.11)$$

By means of (4.9) and (4.11) the Dirac equation is represented by

$$\{\omega^\alpha \eta^\beta (\eta_\alpha \nabla_\beta - \eta_\beta \nabla_\alpha) - B \gamma_0 / 2 - e x\} \Psi = 0, \quad (4.12)$$

where we have put $B = b_{st} g^{st}$ and $\Psi = \psi$. The covariant differentiation of a spinor in the five dimensional space is given by

$$\nabla_\lambda \Psi = (\partial / \partial \varepsilon^\lambda + A_\lambda) \Psi, \quad (4.13)$$

corresponding to (4.3).

Now there exists a four-rowed square matrix A such as

$$A^\dagger = -eA, \quad (A\omega^\lambda)^\dagger = A\omega^\lambda, \quad (4.14)$$

where the symbol “+” denotes the Hermitian conjugate of a matrix⁷⁾. Multiplying the equation (4.12) by $e\Psi^\dagger A$ on the left and the Hermitian conjugate of (4.12) by $A^\dagger \Psi$ on the right and summing, we have

$$e\Psi^\dagger A \omega^\alpha \eta^\beta (\eta_\alpha \nabla_\beta - \eta_\beta \nabla_\alpha) \Psi + \{(\eta_\alpha \nabla_\beta - \eta_\beta \nabla_\alpha) \Psi^\dagger\} A \omega^\alpha \eta^\beta \Psi - (1 + e) \Psi^\dagger A \gamma_0 \Psi B / 2 = 0.$$

by means of $\nabla_\lambda A = \partial A / \partial \varepsilon^\lambda - (A_\lambda^\dagger A + A A_\lambda)$.⁷⁾

In the case of $e=1$ this is reducible to

$$\nabla_\alpha \{\Psi^\dagger A (\omega^\beta \eta^\alpha - \omega^\alpha \eta^\beta) \eta_\beta \Psi\} - \Psi^\dagger A \omega^\alpha \eta^\beta (\nabla_\beta \eta_\alpha) \Psi - \Psi^\dagger A \gamma_0 \Psi (B + \nabla_\alpha \eta^\alpha) = 0 \quad (4.15)$$

by virtue of (4.10) and $\nabla_\lambda A = 0$. If we now assume (3.7), the third term of (4.15) vanishing by means of (3.8), (4.15) reduces to

$$\nabla_\alpha \{ \Psi^+ A (u^\beta \eta^\alpha - u^\alpha \eta^\beta) \eta_\beta \Psi \} = 0. \quad (4.16)$$

This can be considered as the conservation equation for the charge-current density

$$S_{(+)}^\lambda = -\Psi^+ A (u^\alpha \eta^\lambda - u^\lambda \eta^\alpha) \eta_\alpha \Psi. \quad (4.17)$$

As $\eta_\alpha S_{(+)}^\alpha$ vanishes, the vector $S_{(+)}^\lambda$ belongs to the four dimensional space-time and its four dimensional components are given by $\psi^+ A \gamma^i \psi$.

In the case of $e = -1$ it seems impossible to obtain such a conservation equation as (4.16) in the usual manner. The reason for this is as follows. For the four matrices γ 's there exists an Hermitian matrix A such that $A\gamma^i$ are Hermitian and also exists an anti-Hermitian matrix of the same character. For the five matrices u 's, however, a matrix A which makes Au^λ Hermitian is either Hermitian or anti-Hermitian according to the signature of the fundamental form (2.2), as shown by (4.14). Speaking in detail, such a matrix is always anti-Hermitian and not Hermitian when e is plus one. On the contrary if e is minus one, such a matrix is necessarily Hermitian. Now we shall return to the starting point to research into a new representation of the Dirac equation from which we can deduce the conservation equation in the case of $e = -1$.

In the above representation of the Dirac equation the u 's have been determined in terms of the γ 's by (4.8) with (4.6). However, we need not confine ourselves to this relation (4.8) and could determine the u 's in terms of the γ 's by other equations. Different from this, we may change the γ 's into another form preserving (4.8) as it is. For instance, because $\sqrt{-e} \gamma_{0i} \gamma^i$ satisfy (4.2) as well as γ^i themselves do, the Dirac equation can be written also in the form

$$(\sqrt{-e} \gamma_{0i} \gamma^i \nabla_s + \kappa) \psi = 0. \quad (4.18)$$

Starting from this and using (4.8) we can give a new representation of the Dirac equation.

Actually in this case (4.18) is represented by

$$\{ \sqrt{-e} u^\alpha u^\beta (\eta_\alpha \nabla_\beta - \eta_\beta \nabla_\alpha) + \sqrt{-e} B + 2\kappa \} \Psi = 0 \quad (4.19)$$

and, contrary to (4.12), only in the case of $e = -1$ we can deduce the conservation equation for charge and current. That is, by a similar method as before we have

$$\nabla_\alpha \{ \Psi^+ A (u^\beta u^\alpha - u^\alpha u^\beta) \eta_\beta \Psi \} + \Psi^+ A u^\alpha u^\beta \Psi (\nabla_\beta \eta_\alpha - \nabla_\alpha \eta_\beta) = 0,$$

which reduces to

$$\nabla_\alpha \{ \Psi^+ A (u^\beta u^\alpha - u^\alpha u^\beta) \eta_\beta \Psi \} = 0, \quad (4.20)$$

by assuming

$$\nabla_\lambda \eta_\mu = \nabla_\mu \eta_\lambda. \quad (4.21)$$

As is known from (2.8) this condition does not restrict the values of η^λ on the hypersurface. If η^λ defines the geodesics in the enveloping space, (4.21) is automatically satisfied. (4.20) shows that the charge-current vector is given by

$$S_{(-)}^{\lambda} = i\Psi^{\dagger} A (a^{\alpha} a^{\lambda} - a^{\lambda} a^{\alpha}) \gamma_{\alpha} \Psi / 2, \quad (4.22)$$

whose four dimensional components are $i\psi^{\dagger} A \gamma_0 \gamma_i \psi$.

When the electromagnetic field is present, we have only to replace Γ_{λ} by $\Gamma_{\lambda} + (ie/\hbar) A_{\lambda}$ in (4.12) or (4.19).

The operator $\gamma_{\lambda} \Gamma_{\mu} - \gamma_{\mu} \Gamma_{\lambda}$ multiplied by $ic\hbar$ can be interpreted as the five dimensional representation of the momentum operator to within additive terms depending on b_{ij} , because it holds that

$$(\gamma_{\alpha} \Gamma_{\beta} - \gamma_{\beta} \Gamma_{\alpha}) (\Gamma_i \mathcal{E}^{\alpha}) (\Gamma_j \mathcal{E}^{\beta}) = 0, \quad (\gamma_{\alpha} \Gamma_{\beta} - \gamma_{\beta} \Gamma_{\alpha}) (\Gamma_i \mathcal{E}_{\beta}) \gamma_{\alpha} = c \Gamma_i + \hbar_{is} \gamma_0 \gamma^s / 2.$$

Further we have

$$\gamma_{\lambda} (\partial / \partial x^{\mu}) - \gamma_{\mu} (\partial / \partial x^{\lambda}) = e \{ \gamma_{\lambda} a_{\mu\alpha} (\Gamma_i \mathcal{E}^{\alpha}) - \gamma_{\mu} a_{\lambda\alpha} (\Gamma_i \mathcal{E}^{\alpha}) \} g^{st} (\partial / \partial x^s)$$

and therefore the left-hand member of this equation is a differentiation along the hyper-surface.

§ 5. Restriction of the basic transformations

In the preceding two sections we have obtained a five dimensional representation of the electromagnetic and electron field equations in a curved space-time which is immersible in a flat space of five dimensions. The obtained representation is of tensor form and its individual component is not less complicated than the original four dimensional form. For instance, the Christoffel symbols are contained still in the representation.

In order to simplify this further, we shall reflect on the notion of invariancy in the theory of relativity, as already mentioned in § 1. In special relativity the physical laws are invariant for the Lorentz transformation which transforms an inertial system to another one, that is, a frame of reference equivalent to the former. As to the concept of invariancy of equations a bold jump is seen in the transition of special into general relativity. That is, in general relativity the laws are invariant for arbitrary transformations of space-time coordinates, and accordingly all the frames of reference are equivalent from the standpoint that a frame of reference is expressed by a coordinate system. The representation in §§ 3 and 4 is invariant in the latter sense. However, when the electromagnetic or electron field is treated in a curved space-time, we may adhere to a generalization of Lorentz invariancy, discarding the above stated invariancy in the sense of general relativity. To that end we shall define in the following the transformations connecting frames of reference "equivalent" with respect to the electromagnetic or electron field equations in a curved space-time* and then shall assume that the physical laws concerning these fields may be invariant only for such transformations.

The Lorentz group consists of the special Lorentz transformation, rotation in the ordinary space of three dimensions and translation of the space-time origin, and is charac-

* Hereafter the term "equivalency" concerning frames of reference will be used in this sense. From a similar point of view the equivalent observers with respect to "some" definite physical laws have been discussed by Ueno and Takeno.⁴⁾

terized by the property of leaving γ_{ij} invariant. So we may state also for a curved space-time that the transformation which leaves the fundamental tensor g_{ij} invariant transforms a frame of reference to another one equivalent to the former. This has been discussed in § 1 and similar standpoints have been taken by Robertson,⁵⁾ Møller⁸⁾ and Takeno.²⁾ In the following we shall assume that *the electromagnetic and electron field equations in a curved space-time may be invariant only for the above restricted transformations*, as a generalization of the Lorentz invariance in special relativity. In spite of such a restriction of the basic coordinate transformations, the ordinary four dimensional formalism remains complicated. That is, the Christoffel symbol leaves as it is and elements of the matrices γ 's are not constant, while those of the ordinary Dirac's γ 's are constant.

The \mathcal{S} 's have hitherto been considered as coordinates of a five dimensional flat space in which the given curved space-time is immersed. However, adopting another point of view we may treat the \mathcal{S} 's as a particular type of coordinates, five in number and satisfying (2.5), of the four dimensional space-time. Then it will be realized more clearly that the five dimensional space is only a means to the end and the results obtained in §§ 3 and 4 are another representation of the field equations in the curved space-time. By taking such a particular type of coordinates in the de Sitter space-time, it can be shown that the finite form of the transformation between equivalent frames of reference and the equation of motion of test particles assume advantageous forms. Similar results have been obtained by the present author in the Einstein universe* and in the general universe in relativistic cosmology.** We encounter also a similar situation for the above obtained representation of the electromagnetic and electron field equations as will be shown in the following.

Interpreting the \mathcal{S} 's as the above, the transformation connecting the equivalent frames of reference in the given curved space-time can be represented by the one between the \mathcal{S} 's which leaves $a_{\lambda\mu}$ and (2.5) invariant.**** Therefore, when the invariance of physical laws is concerned only with the above restricted transformations, it is convenient to adopt the cartesian coordinates in the enveloping flat space. Then it is possible to replace the covariant derivative by the ordinary one and to omit the Christoffel symbols, etc. in the above representation. For instance, the Maxwell equations (3.6) and (3.11) reduce to

$$\left. \begin{aligned} \partial F^{\lambda\alpha}/\partial \mathcal{S}^\alpha &= f^\lambda, \\ \partial F_{\lambda\mu}/\partial \mathcal{S}^\nu + \partial F_{\mu\nu}/\partial \mathcal{S}^\lambda + \partial F_{\nu\lambda}/\partial \mathcal{S}^\mu &= 0. \end{aligned} \right\} \quad (5.1)$$

which are of similar form as the Maxwell equations in special relativity to within the values which the indices take. The Dirac equations (4.12) and (4.19) become

$$[u^\alpha \gamma^\alpha \{\eta_\alpha (\partial/\partial \mathcal{S}^\beta) - \eta_\beta (\partial/\partial \mathcal{S}^\alpha)\} - \kappa] \Psi = 0, \quad \text{for } e=1 \quad (5.2)$$

$$\text{and} \quad [u^\alpha u^\lambda \{\eta_\alpha (\partial/\partial \mathcal{S}^\beta) - \eta_\beta (\partial/\partial \mathcal{S}^\alpha)\} + 2\kappa] \Psi = 0, \quad \text{for } e=-1 \quad (5.3)$$

* See Appendix.

** The result has not been published as yet.

*** The second equation of (5.1) is of tensor form on account of anti-symmetry of $F_{\lambda\mu}$ and accordingly (3.11) is equivalent to the second of (5.1) without restricting basic coordinate transformations.

**** This has been proved by the present author but it is omitted here.

respectively, omitting terms containing the Christoffel symbols, B and the A 's.* Now the a 's can be represented by constant matrices in the same way as the ordinary Dirac's γ 's. Further we can easily show from (4.14) that (5.2) and (5.3) satisfy the Hermitian condition. That is, the operator in (5.2) and (5.3) can be made Hermitian by multiplying it on the left by some factor.

§ 6. The de Sitter space-time

It is merely in the Minkowski and de Sitter space-times that the group of transformations leaving the fundamental tensor invariant is of parameters of the maximum number, i.e. ten parameters. Further some properties concerning such transformations of the de Sitter space-time have been investigated.²⁾ Therefore, when we deal with the field equations which are invariant only for the transformations leaving the fundamental tensor invariant, it is worth while to see how our representation becomes in the de Sitter space-time. In this section we shall carry out such a plan and then shall clarify the connection of the results with the formalism which has been proposed intuitively by Dirac⁶⁾ and systematized theoretically by Watanabe.⁹⁾

The de Sitter space-time is represented by a pseudo-hypersphere

$$a_{\alpha\beta}z^\alpha z^\beta = cR^2 \quad (6.1)$$

of a five dimensional flat space with the line element (2.2), where R is the radius of the universe. For the de Sitter space-time in relativistic cosmology the a 's and c take the values

$$a_{11}=a_{22}=a_{33}=-a_{44}=a_{55}=-1, \quad \text{other } a_{\lambda\mu}=0; \quad c=-1. \quad (6.2)$$

Besides this another one may be considered, for which the values of the a 's and c are

$$a_{11}=a_{22}=a_{33}=-a_{44}=-a_{55}=-1, \quad \text{other } a_{\lambda\mu}=0; \quad c=+1.** \quad (6.3)$$

In both cases b_{ij} and η^λ are given by

$$b_{ij}=-g_{ij}/R \quad \text{and} \quad \eta^\lambda=z^\lambda/R \quad (6.4)$$

respectively.

Substituting these in (3.4) we have

$$z^\alpha(\partial A_\lambda/\partial z^\alpha)=-A_\lambda, \quad (6.5)$$

which shows that $A_\lambda=A_\lambda(z)$ are homogeneous functions of degree -1 . Thus it has been clarified that *our representation is equivalent to Dirac's formalism as regards the electromagnetic field except the energy-momentum tensor*. In Dirac's formalism this tensor has been given by $\bar{T}_{\lambda\mu}$, and not by $T_{\lambda\mu}$. As was shown in § 3, since $\bar{T}_{\lambda\mu}$

* In this case we may omit the Γ 's in (4.3) and therefore B and the A 's disappear from (4.12) or (4.19).

** These two space-times are of the same constant curvature with opposite signs. We have taken these two on purpose to compare our results with Dirac's formalism.

does not belong to the de Sitter space-time and $a_\alpha \bar{T}^{\alpha\beta}$ does not vanish, Dirac's formalism is not suitable as regards this tensor. However, the equation of motion (3.18) in our theory coincides with that in Dirac's formalism, because $\gamma_\alpha \Gamma_\beta T^{\alpha\beta}$ vanishes in the de Sitter space-time by means of (3.8), (3.9) and (6.4).

Using (6.4) our representations of the electron wave equation (5.2) and (5.3) are reducible to

$$[u^\alpha z^\beta \{z_\alpha (\partial/\partial z^\beta) - z_\beta (\partial/\partial z^\alpha)\} - \kappa R^2] \Psi = 0, \quad \text{for } \ell = 1, \quad (6.6)$$

$$[u^\alpha u^\beta \{z_\alpha (\partial/\partial z^\beta) - z_\beta (\partial/\partial z^\alpha)\} + 2\kappa R] \Psi = 0, \quad \text{for } \ell = -1, \quad (6.7)$$

where we have put $z_\lambda = a_{\lambda\alpha} z^\alpha$. The equation (6.7) is the one proposed by Dirac before. When Dirac was convinced that the Hermitian condition of (6.7) is satisfied in the case of *

$$a_{11} = a_{22} = a_{33} = -a_{44} = a_{55} = 1, \quad \text{other } a_{\lambda\mu} = 0; \quad \ell = 1, \quad (6.8)$$

κ was assumed to be a complex number, which, however, is unsatisfactory from the physical point of view**. From our investigation it has become clear that *one must consider two sorts of representation of the electron wave equation, according to the character of the given space-time*. Further (6.6) satisfies the Hermitian condition in the case of (6.8). Therefore we can conclude that as electron wave equation Dirac should have taken (6.6) in the de Sitter space-time represented by (6.1) with (6.8).

We should notice a methodological difference between Dirac's formalism and our representation. That is, Dirac has made a research into a new formalism of the field equations by generalizing directly the equations in the Minkowski space-time to those in the de Sitter one. On the other hand, we have attempted to obtain a representation of the known field equations in general relativity. In Dirac's theory the obtained equations have been compared with those in special relativity, in the neighbourhood of the point $z^\lambda = (0, 0, 0, 0, R)$ of the de Sitter space-time, which shows that the comparison has been performed in the local inertial system at the point. The field equation from which we have started is already known in general relativity and already acknowledged for its coincidence with the equation in special relativity in the local inertial system at any point of the curved space-time.

At last it is also to be noted that the tensor

$$m_{\lambda\mu} = -i\hbar \{z_\lambda (\partial/\partial z^\mu) - z_\mu (\partial/\partial z^\lambda)\} \quad (6.9)$$

has been interpreted by Dirac as the angular momentum operator, including the linear momentum. We have shown that *it is not the angular but the linear momentum operator that is represented by this tensor* (§4). This can be clarified also by the method of Dirac as follows. In the local inertial system at the point $(0, \dots, 0, R)$ the

* If we assume that the fundamental form of the de Sitter space-time (1.3) is of signature +2, it is readily known that (6.2) reduces to (6.8).

** From (6.7) Dirac has obtained the conservation equation for charge-current density in the case of (6.8), but in deriving this equation he has neglected the reality condition of the z^λ .

components m_{ij} are small quantities of the higher order compared with m_{i5} .* Therefore in the local inertial system at the point $(0, \dots, 0, R)$ the non-vanishing components of $m_{\lambda\mu}$ are only m_{i5} , because small quantities of higher orders do not come to the front in a local inertial system. Thus the statement in italics has been assured, as the components m_{i5} was already interpreted as the linear momentum operator by Dirac himself.

Concluding remarks

Our representation is a generalization of Dirac's formalism in the de Sitter space-time to that in a more general curved one. At the same time, in the de Sitter space-time, our investigation has supplied a gap in Dirac's formalism and clarified its connection with the equations in general relativity. In particular as to the electron field, it has been made clear that one must consider two sorts of representation of wave equation in order to obtain the charge-current conservation equation, different from Dirac's formalism. Recently Goto⁽¹⁰⁾ and Raje⁽¹¹⁾ have studied the meson wave equation in the de Sitter space-time by generalizing the electron wave equation (6.7). In a similar way if we generalize the new representation (6.6) obtained in this paper, we shall be able to give a new representation of meson wave equation in the de Sitter space-time.

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Appendix Five dimensional expressions of some equations in the Einstein universe

The Einstein universe can be represented by a pseudo-hypercircular-cylinder

$$(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = R^2 \quad (\text{A} \cdot 1)$$

of a five dimensional flat space with the line element

$$ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 + (dz^5)^2. \quad (\text{A} \cdot 2)$$

In this case η_λ and b_{ij} are given by

$$\eta^i = z^i/R, \quad \eta^5 = 0 \quad \text{and} \quad b_{ij} = -(g_{ij} - \varphi_i \varphi_j)/R \quad (\text{A} \cdot 3)$$

respectively, where we have put $\varphi_i = \nabla_i z^5$.

The transformation between equivalent frames of reference is easily obtained and given by

$$'z^i = a^i_s z^s, \quad 'z^5 = z^5 + c, \quad (\text{A} \cdot 4_1)$$

where the a 's and c are constants satisfying the conditions

$$\sum_{s=1}^4 a^s_i a^s_j = \delta_{ij}. \quad (\text{A} \cdot 4_2)$$

* It is assumed that the field does not vary so rapidly in space and time.

The finite form of motion of test particles is given by

$$z^i = z_0^i \cos(As) + \sin(As) (dz^i/ds)_0 / D, \quad z^5 = (dz^5/ds)_0 s + z_0^5, \quad (A.5)$$

where

$$D^2 = \{(dz^5/ds)^2 - 1\} / R^2.$$

The proof of (A.3) and (A.5) has been obtained by the present author, but it is omitted here for brevity's sake.

Next substituting (A.3) into (3.4) we have

$$z^s (\partial A_i / \partial z^s) = -A_i, \quad z^s (\partial A_5 / \partial z^s) = 0, \quad (A.6)$$

which shows that in the Einstein space-time the representation of the electromagnetic potential must be functions of the z^s 's, homogeneous with respect to z^1, z^2, z^3 and z^4 and of degree -1 for A_i and 0 for A_0 . Of course, this representation is invariant for the transformation (A.4). There occurs now a somewhat difficult circumstance. That is, the Lorentz condition

$$\partial A^\alpha / \partial z^\alpha = 0 \quad (A.7)$$

reduces to

$$\partial A^s / \partial z^s = 0, \quad \partial A^5 / \partial z^5 = 0 \quad (A.8)$$

by comparing the degree of each term. In a suitable coordinate system of the Einstein space-time the second equation of (A.8) becomes $\partial a^4 / \partial t = 0$. Therefore the Lorentz condition (A.7) seems to impose a stronger restriction on the electromagnetic potential than the ordinary four dimensional theory. But this defect can be remedied by taking

$$[(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2] / R^2 (\partial A^s / \partial z^s) + (\partial A^5 / \partial z^5) = 0 \quad (A.9)$$

as the Lorentz condition instead of (A.7). Of course, this condition (A.9) coincides with the old condition (A.7) on the hypersurface representing the Einstein universe, because the equation (A.1) holds good on this hypersurface.

The electron-wave equations (5.2) and (5.3) reduce to

$$[a^s z^t \{z_s (\partial / \partial z^t) - z_t (\partial / \partial z^s)\} - a^5 R^2 (\partial / \partial z^5) - \kappa R^2] \Psi = 0 \quad (A.10)$$

and

$$[a^s a^t \{z_s (\partial / \partial z^t) - z_t (\partial / \partial z^s)\} + 2a^s a^5 z_s (\partial / \partial z^5) + 2\kappa R] \Psi = 0 \quad (A.11)$$

respectively. The conservation equation for charge-current is deducible only in the case of (A.11).

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On the Fundamental Equation for the Nucleon*

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Although we have succeeded qualitatively in the meson theory, we cannot obtain satisfactory results quantitatively. We can mention the inadequacy of the model of mesons and of the approximation method** as its cause. However, there is another question: the validity of the assumption that nucleons satisfy the Dirac's wave equation. And that we cannot find any investigations about it. It is probably due to the situation that we have no clue to the modification of Dirac's equation for nucleons. In this note, a possible direction for this modification is proposed.

§ 1. Re-examination of the degree of freedom of τ -spin

Proton and neutron are considered as the two states of the same particle, which is called nucleon, and nucleon is described by a wave function with two components, $\begin{pmatrix} \psi_N \\ \psi_P \end{pmatrix}$, (the degree of freedom of τ -spin). On the other hand, electron has the degree of freedom of σ -spin, and it is described by a wave function with four components. The comparison*** of the treatment of the degree of freedom of τ -spin for nucleons with that of σ -spin for electrons is the foundation of the following consideration.

Usually, the wave equation of the nucleon field is considered to be the Dirac's equation, *i.e.*

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + m \right) \begin{pmatrix} \psi_N \\ \psi_P \end{pmatrix} = 0. \quad (1)$$

However, this equation does not involve any τ -spin as the operator, so this is only the equation combining two independent equations for ψ_P and ψ_N respectively in a single form. In the conventional treatment, the idea that proton and neutron are the two states of the same particle is expressed only by adding terms, which change the one to the another by

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** The improvement of approximation method (more satisfactory treatment of meson-cloud) seems to be very difficult owing to the algebraic properties of γ -matrices and the implicit non-linearity involved in the tri-linear interaction terms.

*** To take the inversion of the ordinary space into account, we must describe the degree of freedom of σ -spin by the wave-function with four components, not with two components¹⁾. On the other hand, we need not to consider the inversion of τ -spin space, so the wave function with two components is sufficient to describe the degree of freedom of τ -spin. This comparison seems to be meaningful.

emission (or absorption) of charged meson, as interaction terms^{*)}. On the other hand, for electron (the degree of freedom of σ -spin), the Dirac's equation is the simultaneous equations in the genuine sense, and not four independent equations.

But this comparison may not be so essential, because, as for the degree of freedom of σ -spin, we can transform this simultaneous equation into four independent equations by the Tani-Foldy-Wouthuysen²⁾ transformation. However, there is a more interesting comparison as following.

As the Hamiltonian of the free nucleon field does not contain any τ -spin operator, τ -spin operators commute with the Hamiltonian, and are the constant of motion. But, for electron, σ -spin operators do not commute with the Hamiltonian of the free electron field, and are not the constant of motion; that is, there is a phenomenon of Zitterbewegung³⁾.

From these considerations, equation (1) seems to be insufficient as the wave equation of nucleons, that is, equation (1) seems not to describe sufficiently the above mentioned idea. Further, if we give play to our imagination, it seems that nucleon field must satisfy such a wave equation that τ -spin also makes Zitterbewegung as σ -spin does. (The physical effects of this situation is discussed in the next section).

In the author's opinion, the present treatment of the degree of freedom of τ -spin seems to be in the stage where we described the new degree of freedom of electrons by the Pauli's phenomenological σ -matrix before the Dirac's discovery of relativistic wave equation.

§ 2. A possible correspondence

In this section, we discuss what clue can be considered to formulate concretely the wave equation of nucleons.

In the case of electrons, the transformation property of the wave function of electrons, involving the degree of freedom of σ -spin, can be determined by the group theoretical consideration, having nothing to do with the wave equation¹⁾. In the same way, for the wave function of nucleons, involving the degree of freedom of τ -spin, we can determine its transformation property by the consideration of the rotation group of the abstract charge space and the Lorentz group. But it must be noticed that there is a important difference: While the rotation in the σ -spin space and the rotation in the coordinate space are mutually connected, the rotation in the abstract charge space is not connected with them in the present concept.

Now, the Dirac's wave equation was obtained by linearizing the relativistic wave equation, that is, making the following equation in the form of the eigenvalue problem

$$(\square - m^2)\psi = 0.$$

Thus it will be impossible to find the new wave equation of nucleons without finding an

*) Such processes can be described in the field theory without the above mentioned idea, and then the use of the τ -spin operators and wave functions with two components in the interaction terms seems to be insufficient to formulate the idea.

equation in the present formalism, which is valid in some approximation (if there is any other wave equation of nucleons, except the Dirac's one). It would be the first important problem to find such an equation.

If we confine our consideration within the coordinate space and the spin space, the conventional wave equation seems to admit no generalization. However, according to the consideration mentioned in the preceding section, what is to be considered in the description of nucleons by the wave function with two components is the role of the degree of freedom of the τ -spin in the wave equation of the free field. But since the τ -spin operator does not appear in the wave equation of the free field in the conventional formalism, we must consider nucleons interacting with the electromagnetic field in order to obtain the approximate wave equation as mentioned above.

The conventional equation of nucleons interacting with the electromagnetic field is written as

$$\left\{ \gamma_\mu \left(\frac{\partial}{\partial x_\mu} - ie \frac{1 + \tau_3}{2} A_\mu \right) + m \right\} \psi = 0.$$

(Usually, we consider the charge of neutrons to be zero, and do not use $(1 + \tau_3)/2$, but we use the above form to take the parallelism between protons and neutrons into account).

Now, the intrinsic magnetic moment obtained from this equation is one nuclear magneton for proton, and zero for neutron. The experimental value is $\mu_p = 2.7896$, $\mu_n = -1.9103$, respectively, and the rest is considered to be obtained by the interaction between the meson field and the nucleon field. Then, the correction due to the meson field becomes larger than the intrinsic one. To expect such a large value from the correction due to the meson field seems not to be natural from the point of view of the perturbation theory, and further, the quantitative agreement has not been obtained by the conventional meson theory.

While it is the phenomenological point of view to introduce this anomalous magnetic moment in the form of the Pauli's spin, we take here the point of view that a portion of it (which may be large) is to be obtained from the wave equation of free nucleons, and the rest is to be due to the interaction with the meson field.

Then, we are to consider the equation

$$\left\{ \gamma_\mu \left(\frac{\partial}{\partial x_\mu} - ie \frac{1 + \tau_3}{2} A_\mu \right) + m - i\mu\tau_3 \gamma_\mu \gamma_\nu \left(\frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \right) \right\} \psi = 0 \quad (2)$$

as the approximate equation of the wave function of nucleons interacting with the electromagnetic field (taking into account that the eigen-value of τ_3 is $+1$ and -1 for proton and neutron, respectively).

The portion of the anomalous magnetic moment 2μ owing to the wave equation is to be determined by the correct equation automatically. For instance, if μ becomes 1 nuclear magneton, then the another portion due to the interaction with the meson field becomes -0.2104 and $+0.0897$, for proton and neutron, respectively. (This asymmetry of two values for proton and neutron is interesting from the perturbation-theoretical point of view⁵⁾).

Further, we may also consider a term $\gamma_\mu \square A_\mu$ describing e-n, e-p interaction⁴⁾, but the terms containing the higher derivative than this term are excluded from the consideration about the transformation function. We are not concerned at the moment with this term.

Now, we call the correct wave equation of free nucleons X -equation, and expect that we could obtain approximately the above equation (2) when we make the gauge transformation to the X -equation, and that the Hamiltonian corresponding to the X -equation would contain some operators in the τ -spin space and τ_3 would not then commute with it. That is, we expect that the phenomenon of Zitterbewegung of the τ -spin appears in the correct theory.

§ 3. Vectors in the charge space

Zitterbewegung of the τ -spin means that the expectation value of τ_3 is smaller than one. This fact can be interpreted statistically as follows: the probability that the nucleon observed as the proton (neutron) becomes the neutron (proton) is not zero, even if the nucleon is in the free state³⁾. Then, some other vector in the charge space which makes up this fluctuation of charge of the nucleon will be required. For instance, in the case of electrons (the degree of freedom of σ -spin), the σ -spin does not conserve, and the conserving quantity is the total angular momentum, and the above mentioned vector, which is denoted by \mathbf{t} , can be corresponded to the angular momentum vector, and $\mathbf{t} + \tau/2$ seems to become the constant of motion.*)

Now, we propose to take three-dimensional representation of the rotation group in the three-dimensional abstract charge space as the vector \mathbf{t} . The reason why we propose this assumption is the following.

In the present stage, we have the meson field operator as the possible vector corresponding to the vector \mathbf{t} (more exactly, \mathbf{u} defined in the following). But the meson field which runs away from the nucleon cannot be accepted, so the field must be such a field as so-called meson cloud. Further, we must treat this as some operator (concerning to the new degree of freedom of nucleon), and the relation between this operator and the τ -matrix seems to be corresponding to the one between the differential operator and the σ -matrix.

In the conventional treatment of the meson cloud, we consider the observed nucleon as the closed system which consists of the bare nucleon and the virtual mesons balancing with the former⁵⁾. Then, we consider the charge wave functions of nucleon and of meson as the base of the two-, and three-dimensional representation of the rotation group in the three dimensional abstract charge space, and give the isotopic spin τ and the isotopic angular momentum \mathbf{t}_j to them respectively (where j is the index distinguishing the mesons). The total isotopic angular momentum of the system is given by

$$\mathbf{T} = \sum_j \mathbf{t}_j + 1/2 \cdot \tau.$$

Now, the total isotopic angular momentum of the total system can take the value

*) In the following, we denote the vectors in the three dimensional abstract charge space by bold letters.

1/2, only when we take the three-dimensional invariant subspace in the product space of the charge wave functions of mesons. So $\mathbf{Y} = \sum \mathbf{t}_j$ must be the three-dimensional representation. The above mentioned vector \mathbf{t} corresponds to \mathbf{Y} .

At this point it is to be noticed that the direct product of the two- and three-dimensional representation can be decomposed into the direct sum of the two- and four-dimensional representation, and that the isotopic spin of the nucleon can take the values 1/2 and 3/2. This situation will be connected to the *nucleon isobar*.

As well-known, the vector \mathbf{t} can be expressed as

$$\mathbf{t} = \mathbf{u} \times \mathbf{v}$$

by the new vectors \mathbf{u} and \mathbf{v} satisfying the commutation relation

$$[u_i, v_j] = i\delta_{ij},$$

because \mathbf{t} is the odd-dimensional representation. This vector \mathbf{u} corresponds to the operator of the bound meson field, but, in the present method, this is introduced *operationally* and is the new variable describing the state of nucleons. (The simplest representation of \mathbf{v} satisfying the above commutation relation is $v_j = -i \partial/\partial u_j$).

In this way, we can introduce the new degree of freedom. The next problem is to investigate the question: in what form this new degree of freedom can be contained in the fundamental equation of nucleons? For this purpose, we consider the possible combination of the vectors \mathbf{t} and $\boldsymbol{\tau}$ which can be involved in the Hamiltonian, in order that $\mathbf{t} + \boldsymbol{\tau}/2$ is to be conserved. We can see at once that the above requirement is satisfied, if the Hamiltonian contains the vectors of the charge space in the form of such combinations as $(\boldsymbol{\tau} \cdot \mathbf{t})$, $(\boldsymbol{\tau} \cdot \mathbf{u})$, and $(\boldsymbol{\tau} \cdot \mathbf{v})$. For instance, owing to the relations

$$[\mathbf{t}, (\boldsymbol{\tau} \cdot \mathbf{v})] = i[\boldsymbol{\tau} \times \mathbf{v}]$$

and

$$[\boldsymbol{\tau}, (\boldsymbol{\tau} \cdot \mathbf{v})] = -i[\boldsymbol{\tau} \times \mathbf{v}],$$

we obtain

$$[\mathbf{t} + \boldsymbol{\tau}, (\boldsymbol{\tau} \cdot \mathbf{v})] = 0.$$

There are many possible forms in which these factors are contained in the Hamiltonian or the fundamental equation of nucleons. Thus to solve this problem, it seems necessary to enter into the deeper investigation of the charge space. On this connection, it may be noticed that the σ -matrix played the important role to make the corresponding spinor to the arbitrary tensor, when we rewrite the Dirac's equation in the spinor-form, while, in the conventional concept, we have no connection between the coordinate space and the charge space, as we noticed in the preceding section. It is also noticed that the strong coupling theory gives a correlation between the σ - and τ -spin of the nucleon isobar^{*)},

*) However, this correlation may not be essential because of the following reason: First it seems to be plausible that this correlation is meaningless in the relativistic treatment⁷⁾. Second, the present experimental knowledge does not require this correlation⁸⁾.

and that the masses of the proton and the neutron are different.

§ 4. Concluding remarks

In conclusion of this introductory note, we point out some features that this theory would bring into the conventional meson theory when the fundamental equation of nucleons is formulated in complete form.

(i) The dissociation probability, all of which is considered to be due to the interaction with the meson field in the conventional theory, will be diminished in consequence of Zitterbewegung. Then, the perturbation theory may give the satisfactory results, even in the meson theory.

(ii) The new degree of freedom (of charge) may increase the number of the selection rules.

(iii) This degree of freedom may throw some light on the problem of the heavy particle conservation and of the nucleon isobar.

(iv) The situation of the divergence difficulty will be changed by the introduction of the new degree of freedom.*)

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* In the case of electrons, the degree of divergence was diminished on the passage from the non-relativistic description to the hole theory. This situation is closely connected with the increasing of the number of degree of freedom. But, at this point, it must be noticed again that the space of operator is the same as the coordinate space. So we cannot expect the diminishing of the degree of divergence, unless there is some correlation between the charge space and the coordinate space. Now, the fine-structure constant plays the essential role in the Nambu's law concerning the mass spectrum. This situation is difficult to be understood from the stand-point of explaining the mass spectrum as the self-energy, because there are many fields contributing to the latter. Of course, at the present stage, we cannot conclude the existence of the correlation mentioned above from that fact. This situation, however, might be considered to suggest some intrinsic correlation between these spaces.

Note on a j - j Coupling Shell Model

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Magnetic moments of two odd-odd nuclei, first excited states of light nuclei and nuclear quadrupole moments are treated from the standpoint of the charge symmetric j - j coupling shell model.

It is shown that the theoretical magnetic moments calculated according to the assumption given in my previous paper are consistent with the experimental magnetic moments. However, it seems that the quadrupole moments of middle and heavy nuclei might not be explainable by j - j coupling of nucleons in nuclei.

§ 1. Introduction

Hitherto, in the case of odd-odd nuclei except symmetric nuclei, few magnetic moments have been determined in experiments. Therefore it is important to investigate the assumption on the ground state of odd-odd nuclei by calculating the nuclear magnetic moments of Na^{24} and K^{42} which are determined in experiment recently.

It has been pointed out that some of the quadrupole moments of middle and heavy nuclei are several times as large as the maximum value of the quadrupole moment of one nucleon, and so it is impossible to explain the nuclear quadrupole moments by the extreme one particle model. However, according to the j - j coupling shell model all nucleons in the unclosed outermost orbit contribute to the nuclear quadrupole moment. Therefore it is very important to investigate if the nuclear quadrupole moments are explainable by the j - j coupling shell model.

§ 2. Magnetic moments of odd-odd nuclei

In the previous paper¹⁾ I proposed an assumption on the ground state of odd-odd nuclei. After that Flowers kindly informed me the experimental magnetic moments of Na^{21} and K^{42} . In this section the magnetic moments of Na^{21} and K^{42} are calculated by the charge symmetric j - j coupling shell model, and it is shown the results agree with the assumption given in the previous paper¹⁾.

The method of the calculation is shown in the reference 1. Na^{24} has two kinds of primitive structures, that is, $(s.t) = (2.1)$ and (4.1) . First, one can easily calculate the nuclear magnetic moment of the state of the primitive structure $(s.t) = (2.1)$ by constructing the nuclear wave function. The result is shown in Table I.

In order to obtain the nuclear wave function of the primitive structure $(s.t) = (4.1)$ first we construct a suitable nuclear wave function $\phi_{m'}^{j'j'}(1.2.3)$ of three particles system, and

then we construct the following nuclear wave function of the four particles system which has total angular momentum J .

$$\phi(4.1:2.1)=\sum \epsilon_p \phi k_{mm}, \phi_{m,}^{j_1} (1.2.3) \phi_m^{j_2/2} \hbar_{\zeta \zeta}, \tau_{\zeta \zeta}^{3/2} (1.2.3) \tau_{\zeta \zeta}^{1/2} \tag{4}$$

This wave function includes in general, both wave functions of the primitive structures $(s.t)=(2.1)$ and $(s.t)=(4.1)$. By subtracting the wave function of the primitive structure $(s.t)=(2.1)$ from the above wave function $\Psi(4.1:2.1)$, we obtain the required nuclear wave function $\Psi(4.1)$ which corresponds to the primitive structure $(s.t)=(4.1)$:

$$\Psi(4.1)=N\{\Psi(4.1:2.1)-C\Psi(2.1)\}$$

where $C=\{\Psi(4.1:2.1)\Psi(2.1)$,

and N is a normalization constant.

The magnetic moment of this nuclear wave function is shown in Table 1.

Table 1. Calculated magnetic moments of N_{α}^{24} and K^{42} .

Cofiguration		J	T	T'	Primitive structure cal	exp
N_{α}^{24}	$(d_{5/2})^{-4}$	4	1	1	(2,1) 3.21	1.688
					(4,1) 1.80	
K^{42}	$(d_{5/2})^{-1}(f_{7/2})^3$	2	2	2	(2,1) -1.73	-1.137
					(4,1) -1.0	

We can also calculate the nuclear wave function and nuclear magnetic moment of K^{42} in the same manner and the result is shown in Table 1.

It is clearly seen that these results are consistent with my assumption for the odd-odd nuclei, in which $(s.t)=(\lambda_p+\lambda_N, |\lambda_p-\lambda_N|/2)$.

§ 3. Quadrupole moment

Quadrupole moments of nuclei are obtained by calculating the expectation values of the following operator.

$$Q=6\times 1/e\ Q_{33}=\int \rho(3x_3^2-r^2)dv.$$

In the case of closed shell+one proton nuclei, it can be written as follows

$$Q_{jm}=-\frac{3m^2-j(j+1)}{2j(j+1)}avr^2.$$

The above value depends on the total momentum j , but is independent on the orbital angular momentum $l=j\pm 1/2$ except through the unknown part an avr^2 .

In the case of closed shell+three nucleon nuclei, the quadrupole moment of the wave function of the total isotopic spin $T=T_z=\pm 1/2$ and of the primitive structure $(s.t)=(1.1/2)$ is written as follows :

$$Q=Q_{jj}\times \frac{2j+7}{6(j+1)}=-1\frac{(2j-1)(2j+7)}{12(j+1)^2}avr^2\text{ for }T_z=-1/2\text{ (Proton excess)}$$

$$Q = Q_{jj} \times \frac{4j+5}{6(j+1)} = -\frac{(4j+5)(2j-1)}{12(j+1)^2} \text{avv}^2 \text{ for } T_z = +1/2 (\text{Neutron excess}).$$

Quadrupole moments of the other nuclei can be calculated in the same manner. The quadrupole moments of nuclei which have been measured are calculated and the results are shown in Table 2.

Table 2. Calculated quadrupole moments of light nuclei.

Configuration	T	(s, ℓ)	J	$q = k \times q_{jj}$	Q_{exp}	Q_{exp}/Q_{cal}
$\text{Li}^6 \quad (p_{3/2})^2$	0	(2,0)	1	$-2/5 \times (-2/5)$	0	
$\text{Li}^7 \quad (p_{3/2})^3$	1/2	(1,1/2)	3/2	$11/15 \times (2/5)$	0.02	-1.1
$\text{B}^{10} \quad (p_{3/2})^{-2}$	0	(2,0)	3	$-1 \times (-2/5)$	0.06 4	1.6
$\text{B}^{11} \quad (p_{3/2})^1$	1/2	(1,1/2)	3/2	$-1 \times (-2/5)$	0.03 2	0.8
$\text{N}^{14} \quad (p_{1/2})^2$	0	(2,0)	1	(0)	0.02	
$\text{O}^{17} \quad (d_{5/2})^1$	1/2	(1,1/2)	5/2	(0)	-0.005 2	
$\text{Al}^{27} \quad (d_{5/2})^{-1}$	1/2	(1,1/2)	5/2	$1 \times (-4/7)$	0.156	1.5
$\text{S}^{29} \quad (s_{1/2})^{-1}$	1/2	(1,1/2)	1/2	(0)	0	
$\text{S}^{33} \quad (d_{3/2})^{-1}$	1/2	(1,1/2)	5/2	(0)	-0.0795	
$\text{Cl}^{35} \quad (d_{3/2})^3$	1/2	(1,1/2)	3/2	$11/15 \times (-2/5)$	-0.017	1.3
$\text{Cl}^{37} \quad (d_{3/2})^{-3}$	3/2	(1,1/2)	3/2	$1 \times (-2/5)$	-0.0621	0.7
$\text{K}^{39} \quad (d_{3/2})^{-1}$	1/2	(1,1/2)	3/2	$-1 \times (-2/5)$	-0.06	-0.7

In Table 2, the quadrupole moment of S^{33} is of the same order of magnitude as that of Cl^{35} etc, and can not be explainable by j - j coupling of nucleon. The sign of quadrupole moment of Li^7 and K^{39} opposite to that expected from the j - j coupling model. However, in all the other cases, the sign and magnitude of the calculated quadrupole moment agrees with experiment.

In the case of nuclei which have more than 28 protons or neutrons, it is difficult to decide its configuration, thus it is impossible to calculate theoretical quadrupole moment of these nuclei. However quadrupole moment in j - j coupling of nucleons has the following two features.

- 1) The quadrupole moments of the configuration $(j)^n$ and $(j)^{-n}$ have the same magnitude but opposite sign.
- 2) In closed shell + one (-one) nucleon, nuclei has a negative (positive) quadrupole moment.

The middle and heavy nuclei have large positive experimental quadrupole moments in many cases but have small negative quadrupole moment only in the vicinity of closed shell nuclei and have no large negative quadrupole moment. However according to the j - j coupling shell model, if some nuclei have large positive quadrupole moments, there must be other nuclei which have large negative quadrupole moment.

Moreover it seems these large quadrupole moment is too large to be explainable by

the j - j coupling shell model. For example Lu^{175} has the following large quadrupole moment.

$$Q = 5.9 \times 10^{-26}.$$

This is about six times as large as the maximum value of the quadrupole moment of one nucleon in the orbit of radius $r = A^{1/3}r_0$.

The various kind of nuclear wave function are tried to calculate the quadrupole moment, and some of the example are shown in Table 3. However, it was found no nuclear wave function has the quadrupole moment several times as large as the maximum quadrupole moment of one nucleon.

Table 3. Examples of the calculated quadrupole moments.

Configuration	T' T_z (s, t)	J	k	q_{33}
$(5/2)^5$	$1/2 \ -1/2 \ (3, 3/2)$	$3/2$	$15/157$	$-4/7$
	$1/2 \ +1/2 \ (3, 3/2)$	$3/2$	$34 \times 4/157 \times 5$	$-4/7$
$(7/2)$	$3/2 \ -3/2 \ (3, 3/2)$	$5/2$	$13/14$	$-2/3$

Na^{23} has the configuration $(d_{5/2})$, therefore quadrupole moment of the nucleus is given by changing the sign from the above value.

Thus it seems that the quadrupole moment of middle and heavy nuclei can not be explained by the j - j coupling shell model.

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Static Spherically Symmetric Space-times in General Relativity

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The term 'static' used in general relativity in connection with spherically symmetric space-times is usually used in order to qualify the forms of their line elements. In this paper we introduce the concept of the 'staticness' of a spherically symmetric space-time as an intrinsic property of the space-time from the standpoint that the mathematical aspect of general relativity is a theory of analytical invariants. Then some properties of static spherically symmetric space-times are made clear. The method used is based on the theory of characteristic systems of spherically symmetric space-times developed by the writer. Lastly some examples are given.

§ 1. Introduction

As is well known a spherically symmetric space-time S_0 in general relativity is a four dimensional Riemannian space whose line element is reducible to the form

$$ds^2 = -A(r, t) dr^2 - B(r, t) (d\theta^2 + \sin^2 \theta d\phi^2) + C(r, t) dt^2, \quad (1.1)$$

where A , B and C are positive functions of r and t by taking the coordinate system suitably. The present writer called the coordinate system in which (1.1) holds spherically symmetric.*

Various investigations have been made concerning the physical properties of S_0 's, especially of those defined by static spherically symmetric line elements.⁹⁾ Usually this notion of 'static' is used to qualify not the space-time but the form of its line element in some coordinate system. For example, it is seen in Tolman's book that in the coordinate system in which

$$ds^2 = -A(r, t) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + C(r, t) dt^2 \quad (1.2)$$

holds if $A=A(r)$ and $C=C(r)$ this line element is called static, while it is also seen in the same book that in the coordinate system in which

$$ds^2 = -A(r, t) (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + C(r, t) dt^2 \quad (1.3)$$

holds, (i.e. the isotropic coordinate system in polar form), if $A=A(r)$ and $C=C(r)$ the line element (1.3) is also called static. In general, however, static line elements of

* Recently the writer published a series of papers concerning the spherically symmetric space-times and obtained some properties of them.^{1), ..., 8)}

the form (1.2) are not necessarily transformed into static ones of the form (1.3) and vice versa.* Hence from this point of view the 'staticness' is the concept concerning the form of the line element and depends on the choice of the coordinate system. (Here and throughout the present paper we use this new term *staticness* to denote the quality of being static.) Accordingly we can conclude that it is not the concept concerning the intrinsic property of the space-time.

Now we shall explain the circumstances above stated by showing an example: The line element of the form (1.2) of the famous de Sitter space-time [A]** is given by

$$ds^2 = -(1 - k^2 r^2)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + (1 - k^2 r^2) dt^2, \quad (1.4)$$

where k is a constant, and this (1.4) is transformed into the isotropic polar form

$$ds^2 = -e^{2kt} (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + dt^2, \quad (1.5)$$

by the transformation

$$r e^{kt} = r_1, \quad e^{kt} = e^{kt_1} \sqrt{1 - k^2 r_1^2}. \quad (1.6)$$

Hence according to the usual terminology (1.4) is static and (1.5) is non-static in spite of the fact that both are the line elements of the same space-time [A]. Furthermore (1.4) or (1.5) is transformable into various forms such as

$$ds^2 = \{1 - k^2/4 \cdot (t^2 - r^2)\}^{-2} (-dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2), \quad (1.7)$$

$$ds^2 = (1 + kt)^{-2} (-dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + dt^2), \quad (1.8)$$

etc. Both (1.7) and (1.8) are of the isotropic polar form, and are again the line elements of the same space-time [A] and moreover they are non-static by the usual terminology.*** Thus it is evident that the notion of being static, i.e. staticness commonly refers to the form of the line element and also depends on the choice of the coordinate system.

It is true that such a terminology is necessary and useful for the physical interpretation of the line element. However, we can also deal with the notion of the space-time as a geometrical object in the sense of Riemannian geometry, independent of the coordinate system; and moreover it corresponds to the fact that general relativity, if viewed from a mathematical aspect, is nothing but a theory of analytical invariants.**** If we take such a point of view it is desirable to introduce the concept of the 'staticness' of the space-time (not of the line element!).

* The transformation $r = r(\bar{r}, \bar{t})$, $t = t(\bar{r}, \bar{t})$ which transforms the line element of the form (1.2) into the form (1.3), where \bar{r} and \bar{t} correspond to (1.3), is given by solving

$$-r^2 = \bar{r}^2 \{-A(\partial \bar{r} / \partial \bar{r})^2 + C(\partial \bar{t} / \partial \bar{r})^2\}, \quad 0 = -A \cdot \partial \bar{r} / \partial \bar{r} \cdot \partial \bar{r} / \partial \bar{t} + C \cdot \partial \bar{r} / \partial \bar{r} \cdot \partial \bar{t} / \partial \bar{t},$$

where A and C are known functions of r and t , with respect to two unknown functions \bar{r} and \bar{t} .

** To denote various kinds of the space-times we shall use the same notations [A], [B], ..., etc. as those used in the Reference 2).

*** Concerning the various forms of the line element of [A] and transformations connecting these forms, detailed research is given in the Reference 10).

**** The series of papers above cited have been worked out along this line of thought.

In this paper, we shall, as a first trial, introduce the concept of staticness as an intrinsic property of the space-time basing on the staticness of the line element of the form (1.2) in the usual sense by using the theory of the characteristic-system of the spherically symmetric space-times.* Then we shall obtain some properties of the static S_0 's thus defined, and lastly shall give some examples. As will be seen later it seems difficult to introduce the staticness of S_0 basing on the staticness in the usual sense of the line element of the form (1.3).

§ 2. Semi-static S and static S

Strictly speaking, the line element of an arbitrary spherically symmetric space-time is not necessarily reducible to the form (1.2) and those whose line elements are reducible to this form are called S_I . An S_I is an S_0 whose $B \neq \text{const.}$ in any spherically symmetric coordinate system.¹⁾ In general relativity, however, when we refer to a spherically symmetric space-time we usually mean this S_I implicitly. Hence in this paper we shall also confine ourselves to this S_I and shall denote it simply by S . S_{II} i.e. S_0 whose $B = \text{const.}$ in any spherically symmetric coordinate system is not dealt with throughout this paper.

For convenience' sake we shall call a coordinate system in which (1.2) holds a *canonical* coordinate system of the S . Evidently it is a kind of spherically symmetric coordinate systems. Next we shall call an S *semi-static* when its A in a c.c.s. (abbreviation of the canonical coordinate system) is given by a function of r alone. In other words a semi-static S is an S whose line element is reducible to the form

$$ds^2 = -A(r)dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + C(r, t)dt^2 \quad (2.1)$$

by taking the coordinate system suitably. Especially when C is of the form $\varphi(r)\psi(t)$ in (2.1), the ds^2 is transformable into the form

$$ds^2 = -A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + C(r)dt^2 \quad (2.2)$$

by the transformation

$$\bar{r} = r, \quad \bar{t} = f(t), \quad (2.3)$$

where $f(t) = \int \sqrt{\psi(t)} dt$. We shall call an S *static* when its line element is reducible to the form (2.2). Conversely we can easily prove that A in any c.c.s. of a static S must also be a function of r alone and C must be of the form $\varphi(r)\psi(t)$. These results come from the fact that the semi-staticness and the staticness above defined are intrinsic properties of the space-times as is shown in the Reference 1). But they can also be conjectured from the fact that a transformation of r and t which keeps the type (2.1) and changes the form of g_{ij} must be of the type (2.3)**. (r being assumed to be positive always.)

In the next section, using the properties of the characteristic systems introduced in

* Recently Møller defined the notion of the staticness of the coordinate system of S_0 by using (1.2) also, but he did not touch on the invariant character of his definition.¹¹⁾

** On the other hand, transformations of (r, θ, ϕ, t) which keep the form of g_{ij} invariant form a group called the group of motions of the space-time. Møller called this motion a generalized Lorentz transformation.¹¹⁾

the Reference 1), we shall obtain new invariant expressions of the semi-staticness and the staticness.

§ 3. Invariant expressions of semi-staticness and staticness

For any S a set of four intrinsic scalars $\overset{1}{\rho}$, $\overset{2}{\rho}$, $\overset{3}{\rho}$ and $\overset{4}{\rho}$ is determined uniquely.¹⁾ In a c.c.s. of the S ρ 's are given by

$$\overset{1}{\rho} = 4(\xi + \eta - a - \beta), \quad \overset{2}{\rho} = M + N, \quad \overset{3}{\rho} = M - N, \quad \overset{4}{\rho} = 2\eta, \quad (3.1)$$

where

$$M = 2(a + \beta - 2\eta), \quad N = 2(a - \beta) / \cosh 2\xi, \quad 2\xi = \tanh^{-1} \left\{ 2\sqrt{\frac{C}{A}} \frac{r}{a - \beta} \right\}, \quad (3.2)$$

$$a = K_{12}^{\cdot\cdot 12} = K_{13}^{\cdot\cdot 13} = -A' / 2A^2 r, \quad \beta = K_{24}^{\cdot\cdot 24} = K_{34}^{\cdot\cdot 34} = C' / 2ACr, \\ \gamma = K_{12}^{\cdot\cdot 24} = K_{13}^{\cdot\cdot 34} = -\dot{A} / 2ACr, \quad \eta = K_{23}^{\cdot\cdot 33} = (A^{-1} - 1)r^{-2}, \quad (3.3)$$

$$\xi = K_{14}^{\cdot\cdot 14} = -\{2(\ddot{A} - C'') + A' C' / A - \dot{A} \dot{C} / C - \dot{A}^2 / A + C'^2 / C\} / 4AC$$

are the non-vanishing components of the curvature tensor $K_{ij}^{\cdot\cdot lm}$, ($i, j, \dots = 1, \dots, 4$), primes and dots indicate the derivatives with respect to r and t respectively, and $(x^1, x^2, x^3, x^4) = (r, \theta, \phi, t)$. Furthermore for an S a non-constant scalar F is determined uniquely to within an m -transformation and it becomes $-\log r$ in a c.c.s.*. Therefore if we put

$$\lambda = \exp(-F), \quad (3.4)$$

then λ is a scalar intrinsic to the S and

$$\lambda = r \quad (3.5)$$

holds in a c.c.s. Hereafter we shall assume that the above relations hold for the characteristic system we are dealing with by performing an m -transformation suitably, or in other words, we shall deal only with the characteristic systems satisfying the above relations.

Now we shall call a scalar φ *static and spherically symmetric*** when φ is a function of λ i.e. $\varphi = \varphi(\lambda)$. Then, since this relation is of tensor form, this definition is independent of the coordinate system and it coincides with the usual definition in a c.c.s. Since $\overset{2}{\rho}$, ($a = 1, \dots, 4$), is spherically symmetric¹⁾, we have only to say that $\overset{2}{\rho}$ is static instead of saying that $\overset{2}{\rho}$ is static and spherically symmetric. Using such a terminology we can prove the following theorems:

Theorem 1. *A necessary and sufficient condition that an S be semi-static is that $\overset{4}{\rho}$ be static.*

* If we transform a characteristic system of an S by a motion, it is invariant in form in general. But in some special cases the resulting system is different in form from the original one and gives a new characteristic system for the same g_{ij} . This transformation is called an m -transformation of the characteristic system. In general F is also form-invariant under m -transformations and it changes its form only in some special S 's and only under some special m -transformations.

** As to the definition of a spherically symmetric scalar, see the Reference 1).

Proof. Since the relation $\overset{1}{\rho} = 2\gamma = 2(A^{-1} - 1)r^{-2}$ holds for any c.c.s., the theorem is obvious. We can also deduce this theorem from the identity $\nabla_i F = \alpha u_i - \bar{\alpha} \beta_i$ and the theorem [5.5] in the Reference 1), where ∇_i denotes covariant derivative.

Next, from the definition of the standard coordinate system for g_{ij} we have*

Corollary. Any c.c.s. of a semi-static S is standard for g_{ij} , and in this coordinate system (3.1) becomes

$$\overset{1}{\rho} = 4(\xi + \eta - \alpha - \beta), \quad \overset{2}{\rho} = 4(\alpha - \eta), \quad \overset{3}{\rho} = 4(\beta - \eta), \quad \overset{4}{\rho} = 2\gamma. \quad (3.6)$$

From this corollary we have

Theorem 2. A necessary and sufficient condition that an S be static is that both $\overset{4}{\rho}$ and $\overset{3}{\rho}$ be static.

Next we shall deal with $\overset{2}{\rho}$ and $\overset{1}{\rho}$. By this theorem we have a one to one correspondence between static S 's and the pairs of $\overset{4}{\rho}(\lambda)$ and $\overset{3}{\rho}(\lambda)$. In general, since ρ 's are spherically symmetric, $\overset{1}{\rho}$ and $\overset{2}{\rho}$ are not only functions of λ but also depend on one another variable. (Of course, in a c.c.s., we can take this variable as t .) If $\overset{4}{\rho}$ of these two scalars is a function of λ alone, then A in (1.2) becomes a function of r alone and the S becomes semi-static. Though various S 's are obtained by taking $\overset{3}{\rho}$ arbitrarily, as is seen from (3.3) and (3.6), $\overset{2}{\rho}$ is determined uniquely from $\overset{4}{\rho}$ by the following relation:

$$\text{for semi static } S: \quad \overset{2}{\rho} = \lambda \overset{4}{\rho}', \quad (\overset{a}{\rho}' = d\overset{a}{\rho}/d\lambda, \quad a = 1, \dots, 4). \quad (3.7)$$

Further when not only $\overset{4}{\rho}$ but also $\overset{3}{\rho}$ are functions of λ alone the S becomes static and in a c.c.s. A and C are determined to within a transformation of the type (2.3). Then $\overset{2}{\rho}$ and $\overset{1}{\rho}$ are given respectively by (3.7) and the relation

$$\text{for static } S: \quad \overset{1}{\rho} = \lambda^2/4 \cdot (\overset{3}{\rho} + 2\overset{4}{\rho}) (\overset{3}{\rho} - \lambda \overset{4}{\rho}') (1 + \lambda^2 \overset{4}{\rho}'/2)^{-1} + \lambda (\overset{3}{\rho}' + \overset{4}{\rho}'). \quad (3.8)$$

Here it must be noticed that $Z \equiv 1 + \lambda^2 \overset{4}{\rho}'/2 \neq 0$ because it is equal to A^{-1} in a c.c.s.

The scalar Z^{-1} is an intrinsic scalar of the static S and gives A of the canonical form of its line element. Now we shall give another intrinsic scalar of the S which gives C in the c.c.s. It is obtained by expressing C in tensor form using λ and ρ 's. From (3.3) and (3.6) we have

$$\overset{3}{\rho} = 2 \{ C' Z / \lambda C - \overset{4}{\rho}' \}, \quad (3.9)$$

which shows that the scalar defined by

$$\mu = \exp \{ \frac{1}{2} \lambda (\overset{3}{\rho} + 2\overset{4}{\rho}) Z^{-1} d\lambda \} \quad (3.10)$$

is an intrinsic scalar of the S and gives C in a c.c.s. Evidently μ is determined to within a constant multiplier. Especially when $Z = 1$ and $\mu = \text{const.}$ the S is the Minkowski space-time [B].

* A spherically symmetric coordinate system in which $\gamma = 0$ holds is called standard for g_{ij} .¹⁾

Remark. At first sight it seems also possible to define the semi-staticness of an S by using

$$ds^2 = -A(r, t)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + C(r)dt^2, \quad (3.11)$$

in place of (2.1). In this case, however, ρ 's are not functions of r alone in general and that it is difficult to express the condition in an invariant form by using the characteristic systems. Hence from the standpoint of the characteristic system, we can not say that such an S has a semi-static property of invariant meaning. For this reason we do not introduce such a semi-staticness. By similar reasons we also do not introduce the semi-staticness and the staticness using A and C in an isotropic polar coordinate system.

§ 4. S_p and S_q

In this section we shall deal with two kinds of special static S 's which are often treated in general relativity.

An S whose $\overset{2}{\rho} = \overset{3}{\rho}$ is denoted by S_{15} in the Reference 1). If we use the notation S_p in place of S_{15} , then from the theorem of the Reference 1) we have

Theorem 3. *The line element of an S_p is reducible to the form*

$$ds^2 = -A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + A^{-1}dt^2, \quad (4.1)$$

and *vice versa*.

From this theorem we have

Theorem 4. *An S_p is static.*

We shall call (4.1) the *reduced canonical form* of the line element of the S_p and the c.c.s. in which (4.1) holds *reduced*.

As is seen from the result of the last section we have a one to one correspondence between S_p 's and $\overset{4}{\rho}(\lambda)$. In the r.c.c.s. (abbreviation of the reduced c.c.s.), this correspondence becomes the one between $\overset{4}{\rho}(\lambda)$ and $A(r)$. When $\overset{4}{\rho} = 0$ the S_p is [B]. Of course $\overset{4}{\rho}$ and $\overset{2}{\rho}(=\overset{3}{\rho})$ are expressible in terms of $\overset{4}{\rho}$ as follows:

$$\text{for an } S_p: \quad \overset{2}{\rho} = \overset{3}{\rho} = \lambda \overset{4}{\rho}', \quad \overset{4}{\rho} = 2\lambda \overset{4}{\rho}' + \lambda^2 \overset{4}{\rho}''. \quad (4.2)$$

Next we shall consider line elements of the form

$$ds^2 = -A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + dt^2, \quad (4.3)$$

shall denote by S_q an S whose line element is reducible to this form, and shall call (4.3) the reduced canonical form and the c.c.s. reduced. The space-time of Einstein universe is an example of S_q . Evidently we have

Theorem 5. *An S_q is static.*

From (3.3) and (3.6) we easily obtain

Theorem 6. *A necessary and sufficient condition that an S be an S_q is given by*

$$\overset{4}{\rho} = \overset{4}{\rho}(\lambda), \quad \overset{3}{\rho} + 2\overset{4}{\rho} = 0, \quad (4.4)$$

Hence as in the case of S_p , we have a one to one correspondence between $\overset{4}{\rho}(\lambda)$ and S_q 's. S_q is [B] when $\overset{4}{\rho}=0$. Further we have

$$\text{for an } S_q: \quad \overset{2}{\rho} = -\overset{1}{\rho} = \lambda \overset{4}{\rho}, \quad \overset{3}{\rho} + 2\overset{4}{\rho} = 0. \quad (4.5)$$

The following theorem is easily proved:

Theorem 7. *The line element of an S which is both S_p and S_q at the same time is reducible to the form*

$$ds^2 = -(1+m)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + dt^2, \quad (4.6)$$

where m is a constant, and vice versa.

We shall denote such an S by $S_r(m)$ or S_r simply. That an $S_r(m)$ is an S_p is also seen from the fact that (4.6) is transformable into the form

$$ds^2 = -(1+m)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + (1+m) dt^2. \quad (4.7)$$

Hence an S_r has two reduced canonical forms of line element and in a c.c.s. the line element is given by (4.6) to within a transformation of the form (2.3).

$S_r(0)$ is [B]. When $m \neq 0$, however, $S_r(m)$ is not [B] since non-vanishing components of K_{ij}^{lm} are given by $K_{23}^{23} = mr^{-2}$ in a c.c.s. From (4.2) and (4.3) we have

$$\text{for an } S_r(m): \quad \overset{1}{\rho} = -\overset{2}{\rho} = -\overset{3}{\rho} = 2\overset{4}{\rho} = 4m\lambda^{-2}. \quad (4.8)$$

From the above results we also have

Corollary. *An S_p or an S_q whose $\overset{4}{\rho} = \text{const.}/\lambda^2$ is an S_r .*

The S_p , S_q and S_r play important rôles in the theory of the superposition of static S 's which will be published before long.

§ 5. Some examples

In this section we shall give some examples of the static S 's treated in the preceding sections selecting from papers on general relativity. We can easily verify that various formulae hitherto obtained hold for these S 's.

(i) de Sitter space-time [A] whose line element in the r.c.c.s. is given by (1.4). This space-time is an S_p and it holds that

$$\overset{1}{\rho} = \overset{2}{\rho} = \overset{3}{\rho} = 0, \quad \overset{4}{\rho} = -2k^2. \quad (5.1)$$

(ii) Space-time of Schwarzschild's exterior solution whose line element in the r.c.c.s. is given by

$$ds^2 = -(1-2m/r)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + (1-2m/r) dt^2, \quad (5.2)$$

where m is a constant. This space-time is also an S_p and it holds that

$$\overset{1}{\rho} = -2\overset{2}{\rho} = -2\overset{3}{\rho} = 6\overset{4}{\rho} = -24m\lambda^{-3}. \quad (5.3)$$

(iii) Space-time of Einstein universe [C] whose line element in the r.c.c.s. is given by

$$ds^2 = -(1 - r^2/R^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + dt^2, \quad (5.4)$$

where R is a constant. This space-time is an S_q and it holds that

$$\dot{\rho} = \ddot{\rho} = 0, \quad \ddot{\rho} = -2\dot{\rho}^4 = 4/R^2. \quad (5.5)$$

(iv) Space-time of Schwarzschild's interior solution whose line element in a c.c.s. is given by

$$ds^2 = -(1 - r^2/R^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) + (a - b\sqrt{1 - r^2/R^2})^2 dt^2, \quad (5.6)$$

where a , b and R are constants. This space-time is static though it is neither an S_p nor an S_q , and it holds that

$$\dot{\rho} = \ddot{\rho} = 0, \quad \ddot{\rho} = 4aR^{-2}(a - b\sqrt{1 - r^2/R^2})^{-1}, \quad \dot{\rho}^4 = -2/R^2. \quad (5.7)$$

(v) Lastly we shall give the space-time of relativistic cosmology as an example of S which is neither static nor semi-static. The line element of this space-time takes the following form in an isotropic polar coordinate system:*

$$ds^2 = -e^{2g(t)}(1 + r^2/4R^2)^{-2}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) + dt^2, \quad (5.8)$$

where R is a constant. If we denote this S by $S(L)$, we have

Theorem 8. *If an $S(L)$ is semi-static, then it is static. Moreover it must be one of $[A]$, $[B]$ and $[C]$. Hence $S(L)$'s other than $[A]$, $[B]$ and $[C]$ are neither static nor semi-static.*

Proof. As is easily seen (5.8) gives a standard coordinate system for g_{ij} and non-vanishing components of K_{ij}^{im} for this (5.8) are given by

$$u = \eta = -(\dot{g}^2 + e^{-2g}/R^2), \quad \xi = \beta = -(\ddot{g} + \dot{g}^3). \quad (5.9)$$

Hence by the formulae concerning ρ 's, we have

$$\dot{\rho} = \ddot{\rho} = 0, \quad \ddot{\rho} = 4(\beta - \eta) = 4(e^{-2g}/R^2 - \ddot{g}), \quad \dot{\rho}^4 = 2\eta = -2(\dot{g}^2 + e^{-2g}/R^2), \quad (5.10)$$

and by a suitable m -transformation, we have

$$\lambda = e^{-F} = r e^{g(t)} (1 + r^2/4R^2)^{-1}. \quad (5.11)$$

Therefore, in general, $\dot{\rho}$ is not a function of λ alone and the condition that $\dot{\rho}$ be a function of λ alone is $\dot{\rho} = \text{const.}$ From this we can consider the following two cases:

(a) When $\dot{g} = 0$ i.e. $g = \text{const.}$ In this case the $S(L)$ becomes $[C]$ or $[B]$ according as $1/R^2 \neq 0$ or $1/R^2 = 0$ respectively.

(b) when $\dot{g} \neq 0$. It holds that

$$\dot{g}^2 + e^{-2g}/R^2 = \text{const.}, \quad (5.12)$$

from which we have

$$\ddot{g} - e^{-2g}/R^2 = 0 \quad \text{i.e.} \quad \ddot{\rho} = 0. \quad (5.13)$$

* In an isotropic coordinate system of an S , the form of the line element is not necessarily determined contrary to the case of c.c.s., and in some special S 's it can take various forms. (See § 1.)

Solving this we obtain the following results:⁽¹⁰⁾

(1) When $e^g \dot{g} \neq \text{const.}$ It holds that

$$e^g = c_1 e^{2kt}, \quad \text{or} \quad e^g = c_2 e^{kt} + c_3 e^{-kt}, \quad (4c_2 c_3 = 1/R^2), \quad (5.14)$$

where c 's are arbitrary constants, according as $1/R^2 = 0$ or $1/R^2 \neq 0$, and the space-time is [A].

(2) When $e^g \dot{g} = \text{const.}$ It holds that

$$A = c_4, \quad \text{or} \quad e^g = c_5 t + c_6, \quad (c_5^2 = -1/R^2), \quad (5.15)$$

where again c 's are arbitrary constants, according as $1/R^2 = 0$ or $1/R^2 \neq 0$, and the space-time is [B]. Thus the theorem is proved.

This theorem can also be conjectured from the form of the transformation which transforms (5.8) into its canonical form.⁽¹⁰⁾

§ 6. Conclusion

In this paper we have introduced the concept of the 'staticness' of a spherically symmetric space-time which is independent of the coordinate systems, and then expressed it by using tensor equations. By virtue of this study, when a spherically symmetric space-time is given by designating its g_{ij} in any coordinate system we can easily determine whether it is static or non-static. The staticness introduced in this paper is an intrinsic property of the space-time we are considering and is not a concept depending on the apparent form of its line element. Thus, for example, it is to be concluded that de Sitter space-time is static in spite of the fact that its line element of the form (1.5) or (1.7) is non-static according to the usual terminology.

Lastly we shall add that the present research also gives the foundation of the theory of the superposition of static spherically symmetric space-times which will be published before long by the present writer.

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Upper Bound of the Pseudoscalar Coupling Constant in Beta-decay

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Recently the large value of pseudoscalar coupling constant in β -decay has come into discussion. In this paper the allowed shape β -spectra of He^6 and B^{12} are investigated with the mixture of tensor and pseudoscalar interactions, and it is concluded the ratio of the pseudoscalar coupling constant to that of tensor must lie in the region $-55 < G_P/G_T < 19$.

§ 1. Introduction

The β -spectrum of RaE was analysed by Petschek and Marshak¹⁾ last year and it was concluded that P must be included among the other interactions. (We abbreviate the five interaction types in the Fermi theory of β -decay as S , V , T , A and P hereafter.) Though it was pointed out recently by one of the authors that this conclusion is not necessarily correct, P still seems necessary if the spin of RaE is zero.²⁾ However, according to the relation of Ahrens, Feenberg and Primakoff³⁾ the nuclear matrix element $\langle \beta \gamma_5 \rangle$ in P is so small that we must take G_P , the coupling constant of P , very large ($|G_P/G_T| \approx 133$) to make P match to T . Recently it was shown by Ruderman⁴⁾ that, if nuclear force is mainly $\beta \gamma_5$ -type, $\langle \beta \gamma_5 \rangle$ is markedly large, and consequently G_P need not be so large as expected ($|G_P| \sim |G_T|$) to explain the β -decay of RaE . However, if nuclear force arises chiefly from pseudovector coupling between π -mesons and nucleons, G_P should still remain large. According to the latest π -meson theory in which the damping effect is taken into account,⁵⁾ it is very doubtful whether a large $\beta \gamma_5$ -type nuclear force exists, even if the coupling between π -mesons and nucleons is pseudoscalar one; though the conclusion is not to be taken too seriously, in view of many ambiguities in the present π -meson theory.

We feel the largeness of G_P not very agreeable aesthetically, and it seems probable that G_P is of the same order as G_T . However, we should discuss this problem on the firmer foundations. The β -spectrum of RaE will be analysed by Takebe⁶⁾ in the case of large G_P taking into account another terms pointed out by Ahrens, Feenberg and Primakoff.³⁾ In this paper, however, we shall find the upper bound of G_P from another viewpoint, i.e. from the fact that a very large G_P destroys the spectrum shapes of high energy allowed transitions. (He^6 and B^{12}).

§ 2. Theory

We assume for the moment that the nuclear force is the ordinary one and not of $\beta \gamma_5$ -type which is apt to make nucleon pairs. Then, $\langle \beta \gamma_5 \rangle$, the nuclear matrix element

in the second forbidden of P , becomes $i/(2M) \cdot \int \sigma^i$, where M is the nucleon mass. If we use the non-relativistic relation $\int \sigma \approx -\int \beta \sigma$, this nuclear matrix element is the same as the one in T except for the constant factor, and the ambiguity concerning nuclear matrix elements does not exist. Since the factor $1/(2M)$ is very small (we take the rest mass of an electron as unity), it has little effect on the ordinary energy β -spectrum even for $|G_P/G_T| \approx 133$. However, in the matrix element, the factor which comes from the lepton part has almost linear energy dependence and the term $\int \beta \gamma_5$ becomes significant in the high energy β -decay, compared with the ordinary allowed part which is almost independent of energy.

In order to make the situation clearer concerning P , it is convenient to introduce a transformation similar to the Dyson transformation in π -meson theory. Namely, instead of a field variable φ of π -meson we make use of $\zeta^{h*} \beta \gamma_5 \zeta^h$ of the lepton part. We shall use the following notation: capital letters for the nucleons, and small letters for the leptons. The isotopic spin components $T_3=1$ corresponds to neutron, $T_3=-1$ to proton, $\tau_3=1$ to electron and $\tau_3=-1$ to neutrino. All the Dirac matrices γ_μ are taken Hermitian and $\bar{\zeta} = \zeta^{h*} \gamma_4$. Ψ , ζ^h and A_μ are the field variables of nucleons, leptons and photons respectively, and Ω denotes the nuclear force and commutes with the operators in the lepton quantities. M , $M-\Delta M$ and m are the rest masses of a neutron, a proton and an electron respectively, and the rest mass of a neutrino is taken as zero. Then the total Hamiltonian density before the transformation is

$$\begin{aligned} H = & \sum_{i=1}^3 \bar{\Psi} \Gamma_i \frac{\partial}{\partial x_i} \Psi + \bar{\Psi} \left(M - \frac{1-T_3}{2} \Delta M \right) \Psi + \sum_{i=1}^3 \bar{\psi} \gamma_i \frac{\partial}{\partial x_i} \psi \\ & + \bar{\psi} \frac{1+\tau_3}{2} m \psi + \sum_{\mu=1}^4 i e \bar{\Psi} \Gamma_\mu \frac{1-T_3}{2} A_\mu \Psi - \sum_{\mu=1}^4 i e \bar{\psi} \gamma_\mu \frac{1+\tau_3}{2} A_\mu \psi \\ & + \bar{\Psi} \Omega \Psi + \sum_{i=1}^2 f_P \bar{\Psi} i \Gamma_5 T_i \Psi \bar{\psi} i \gamma_5 \tau_i \psi, \end{aligned} \quad (1)$$

where

$$f_P = -(1/2) G_P \quad (2)$$

and the terms having no relation to β -decay are neglected. We transform it by the following unitary transformation:

$$H' = U H U^{-1}, \quad U = \exp \left(\frac{i f_P}{2 M_0} \sum_{i=1}^2 \int d\mathbf{r} \bar{\Psi} \Gamma_4 \Gamma_5 T_i \Psi \bar{\psi} i \gamma_5 \tau_i \psi \right), \quad (3)$$

where M_0 is the mean rest mass of a neutron and a proton. The coupling constant f_P of β -decay is so small that the first order term with respect to f_P only will be retained. Then, the transformed Hamiltonian density for the β -decay interaction is

$$H'_\beta = \frac{f_P}{2 M_0} \sum_{i=1}^3 \sum_{l=1}^2 \bar{\Psi} i \Gamma_5 \Gamma_i T_l \Psi \frac{\partial}{\partial x_i} (\bar{\psi} i \gamma_5 \tau_l \psi)$$

$$\begin{aligned}
& + \frac{f_P}{2M_0} \sum_{i=1}^2 \bar{\Psi} \Gamma_4 \Gamma_5 T_i \Psi \left[\sum_{i=1}^3 2 \bar{\psi} \gamma_4 \gamma_5 \gamma_i \tau_i \frac{\partial \psi}{\partial x_i} \right. \\
& \quad \left. - \sum_{i=1}^3 \frac{\partial}{\partial x_i} (\bar{\psi} \gamma_4 \gamma_5 \gamma_i \tau_i \psi) + m \bar{\psi} \gamma_4 \gamma_5 \tau_i \psi \right] \\
& - \frac{f_{P\ell}}{2M_0} \sum_{i=1}^3 \sum_{l=1}^2 [\bar{\Psi} \Gamma_5 \Gamma_i A_l T_l T_3 \Psi \bar{\psi} i \gamma_5 \tau_i \psi + \bar{\Psi} \Gamma_4 \Gamma_5 T_l \Psi \bar{\psi} i \gamma_4 \gamma_5 \gamma_i A_l \tau_i \psi] \\
& - \frac{i f_P}{2M_0} \sum_{i=1}^2 \bar{\Psi} \{ \Gamma_5 T_i, \mathcal{Q} \} \Psi \bar{\psi} i \gamma_5 \tau_i \psi,
\end{aligned} \tag{4}$$

where $\{ \}$ means an anticommutator.

The first term of (4) in which we are interested now is the same as A (axialvector) in regard to the nucleon part, but the lepton part behaves as if it had a one-order higher forbiddenness because it is differentiated. (The larger Z becomes, the more complicated the situation is.) The last term of (4) is related to the nuclear force, and from this expression it is easily understood that when a large $\beta\gamma_5$ -type nuclear force exists this term becomes very large, but its largest term (unforbidden) has the selection rule, spin change zero, parity change yes. Spin change one, parity change no transition in which we are interested now appears only as the forbidden one-order higher. If we make this forbidden affect the ordinary allowed transition, the transition probability of the unforbidden, spin change zero, parity change yes transition becomes generally as large as that of the ordinary allowed transition*. However, the transitions which seem to belong to spin change zero, parity change yes transitions (there are more than ten examples) have distinctly larger ft -values than those of the allowed transitions.⁸⁾ Therefore, it seems very improbable that there exists a nuclear force which cancel the effect of the first term of (4) through the last term of (4), and so we may neglect the last term of (4) in the following calculation. It can be readily seen that the other terms can be neglected, too. (Though there may remain some ambiguities about the radiative correction.)

Therefore, we calculate the correction factor for the case of spin change one using the first term of (4) only. Since it has proved almost certain by the recent experiment⁹⁾ on the electron-neutrino angular correlation of He^6 that the G - T type interaction is T , we regard $|\beta\sigma|^2$ term of T as the main and assume those coming from P to be relatively small. (It is evident that extraordinary large value of G_P , [$|G_P| \sim 2M|G_T|$], is unfavorable.) Thus we neglect the square term in P and take into account only the interference term between P and T . We further assume $aZ \ll 1$ because we are only interested in He^6 and B^{12} . Arranging the main terms in a suitable way, the correction factor can be written in the notations of Konopinski and Uhlenbeck¹⁰⁾ as

* In section 3 we take the magnitude of the pseudoscalar nuclear matrix element (including $G_P/(2M_0)$) as about one-hundredth as large as $G_T|\beta\sigma|$ in T . On the other hand, a nuclear matrix element is about one hundred times as large as the one-order higher forbidden nuclear matrix element. (The latter includes one more r , and the nuclear radius is about 1/100 in the usual units of $\hbar=c=m=1$.) In §3 we examine the β -spectra of He^6 and B^{12} . The $\log_{10} ft$ -value for He^6 is 2.91 which is the smallest among the known ft -values, and the one for B^{12} is 4.18 which is a medium value for allowed transition.

$$C = G_T^2 \left| \int \beta \sigma |^2 L_0 + 2 \frac{G_T G_P}{2M_0} \left| \int \beta \sigma |^2 \left(\frac{1}{3} K L_0 + N_0 \right) \right. \right. \\ \left. \left. - 2 \frac{G_T G_P}{2M_0} \left(\frac{1}{2FP^2} \right) \left[\int \beta \sigma g_{-1}^* \int r \frac{(\sigma \cdot r)}{r} \left(\frac{f-1}{r} \right)' - \int \beta \sigma f_1^* \int r \frac{(\sigma \cdot r)}{r} \left(\frac{g_1}{r} \right)' \right] \right], \quad (5)$$

where the suffixes of f_π and g_π are quantum numbers $\chi's^{11)}$ and ' denotes the radial derivative. In deriving (5), we have used the non-relativistic relation $\int \beta \sigma \approx -\int \sigma$ which is allowable for He^6 and B^{12} and the relations between the phases of the nuclear matrix elements.¹²⁾ The derivative terms of g_{-1} and f_1 are neglected, because they are small when $aZ \ll 1$ and should be classified in the higher forbidden transition. In the first two terms of (5) the effects of the finite nuclear size¹³⁾¹⁴⁾¹⁵⁾ are not expressed explicitly. According to the recent investigation by one of the authors,¹⁵⁾ in the case of $aZ \ll 1$ these effects are expressed effectively by regarding the value of $aZ/(2\rho)$ as a variable parameter in the approximate formulae ($aZ \ll 1$) of Konopinski and Uhlenbeck¹⁰⁾ without any assumption about the charge distribution in the nucleus. $aZ/(2\rho)$ appears in the second term of (5), but it is energy independent and much smaller than the first term, so that the finite nuclear size effect is not significant. Also it is easily shown using the wave function in reference 15 that the last derivative term is energy independent and negligible too. The energy dependence of (5) is nearly linear with respect to the electron energy, and rather slight mixture of P can cause an appreciable energy dependence to the total correction factor.

§ 3. Determination of the upper bound of G_P from the experimental data on He^6 and B^{12}

The comparison with the experiments are carried out using the experimental data on He^6 and B^{12} . The β -ray of He^6 has the maximum energy $7.9mc^2$ and simple, and it is certain that the spin change is $0 \rightarrow 1$. Using the first two terms of (5) the correction factor and the Kurie plot are shown in Fig. 1 in the case of $G_P/(2M_0G_T) = -0.02$ and $G_P/(2M_0G_T) = 0.04$. We use the experimental data of Wu *et al.*¹⁶⁾ To our regret this experiment has not sufficient precision, but we can conclude

$$-0.02 < G_P/(2M_0G_T) < 0.04,$$

i.e.

$$-74 < G_P/G_T < 148.$$

The highest energy group of the β -ray of B^{12} has the spin change $1 \rightarrow 0$ almost certainly, and its maximum energy is very high ($27.3mc^2$). This β -decay is complex and the second highest energy group has the maximum energy $18.5mc^2$, but this group is about 4% of the total β -ray¹⁷⁾ and does not cause a detectable deviation of the Kurie plot from straight line above $W=16mc^2$. The next highest energy group has the maximum energy less than $13.6mc^2$, but it seems the entire decay scheme has not yet been established.

Therefore, we discard the lower energy part of the spectrum. Using the first two terms of (5) we show the correction factor and the Kurie plot in Fig. 2 in the case of $G_P/(2M_0G_T)=0.005$ and -0.015 .

We use the experimental data of Hornyak and Lauritsen.¹⁸⁾ From Fig. 2 we can see that

$$-0.015 < G_P/(2M_0G_T) < 0.005, \\ \text{i.e.}$$

$$-55 < G_P/G_T < 19.$$

If the decay scheme (including branching ratio) becomes much clearer by some other method than that of the Kurie plot (e.g. with the method of β - γ coincidence), the lower energy part of the spectrum will be useful, and will serve to narrow the allowable region for G_P/G_T ; especially it may reduce the upper bound for $-G_P/G_T$ for which no more than the rather large upper bound 55 can be assigned from the above analysis.

In the case of B^{12} the energy is so high that we must pay attention to the higher forbidden terms of T (including the finite de Broglie wavelength effect¹⁹⁾ which have the same selection rule as in the allowed transition and negligible in most cases compared with the ordinary allowed terms.

Since they are expected to be small, we have only to investigate the interference terms with the main term, $G_T\beta\sigma$. They are

$$2G_T^2[\int\beta\sigma^*\int\beta\sigma r^2\{-(K^2/6)L_0+(K/3)N_0\}+\int\beta\sigma^*\int(\beta\sigma\cdot r)r(2/3)KN_0 \\ -\int\beta\sigma^*\int\beta\sigma\times r\{(1/3)KL_0-N_0\}]. \quad (6)$$

The effect of finite de Broglie wavelength which comes from L_0 of the first term in (5) should be included in the first term in the brackets of (6). If we put $\int\beta\sigma r^2=\int(\beta\sigma\cdot r)r=\int\beta\sigma\cdot r^2$, the first term in the brackets of (6) plus the part due to the finite de Broglie wavelength effect of the first term in (5) as well as the second term in the brackets of

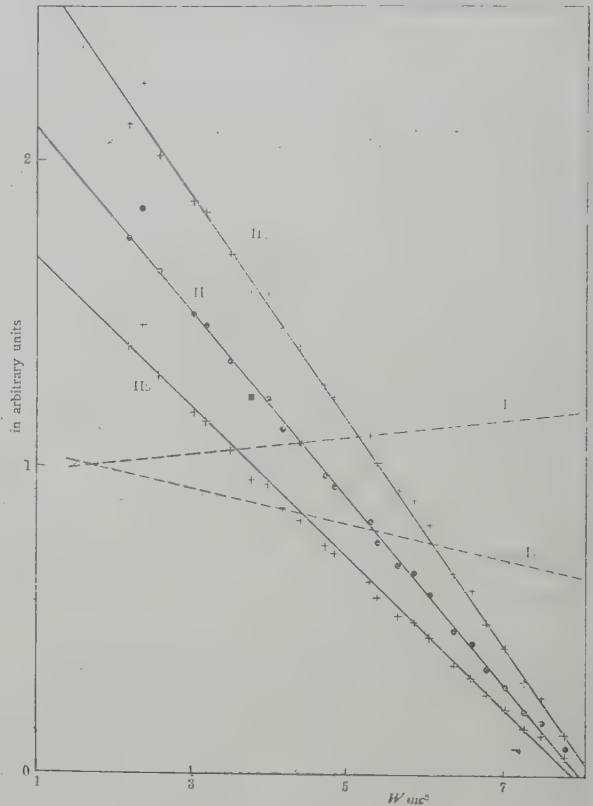


Fig. 1. Correction factor and Kurie plot for He^6 using the first two terms of (5).

Ia: Correction factor for $G_P/(2M_0G_T)=0.04$.

Ib: Correction factor for $G_P/(2M_0G_T)=-0.02$.

II: Kurie plot for $G_P/(2M_0G_T)=0$.

IIa: Kurie plot for $G_P/(2M_0G_T)=0.04$.

IIb: Kurie plot for $G_P/(2M_0G_T)=-0.02$.

(6) shows the energy dependence at most 1/10 times as large as that coming from the second term of (5) in which we put $|G_P/(2M_0G_T)|=0.005$. The last term of (6) is

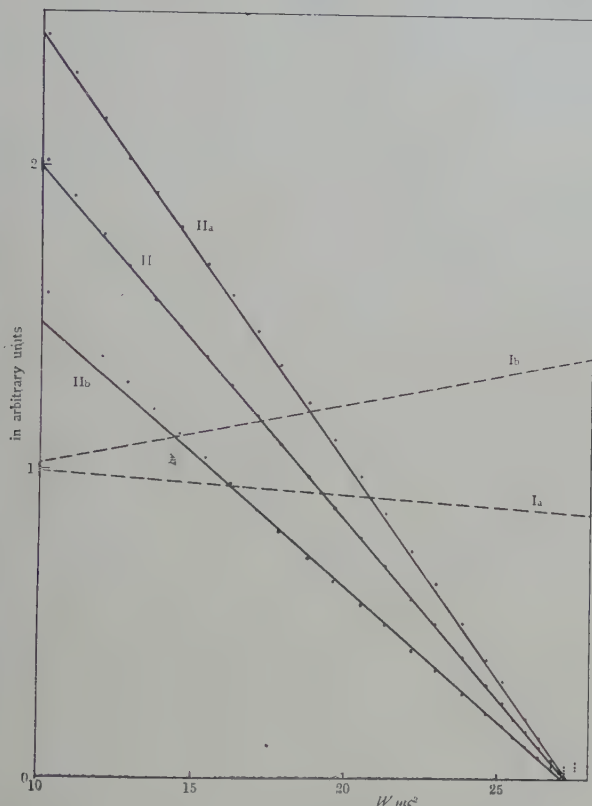


Fig. 2. Correction factor and Kurie plot for B^{12} using the first two terms of (5).

Ia: Correction factor for $G_P/(2M_0G_T)=0.005$.

Ib: Correction factor for $G_P/(2M_0G_T)=-0.015$.

II: Kurie plot for $G_P/(2M_0G_T)=0$.

IIa: Kurie plot for $G_P/(2M_0G_T)=0.005$.

IIb: Kurie plot for $G_P/(2M_0G_T)=-0.015$.

It would be almost evident to which curve belong the plotted points, so that the same marks are used common to the three Kurie plots. The deviation of the Kurie plot II from the straight line below $W=12mc^2$ is due to the mixture of other transitions.

almost energy independent. First, it is very improbable for $\int \beta \sigma r^2$ and $\int (\beta \sigma \cdot r) r$ to be much larger than the above value from the viewpoint of ft -value; second, even if we take them larger, the first two terms of (6) do not cancel the energy dependence of (5) perfectly, because they have nearly quadratic forms as to the electron energy in contrast with the linear energy dependence of (5). From the above two facts, the negligibility of the higher forbidden terms is almost certain. The higher forbidden terms in P can be treated in the same manner, but they are very small.

Consequently, the coupling constant of P can not be so large, and it must lie in the region $-55 < G_P/G_T < 19$. The upper bound for G_P/G_T , 19, is very stringent, but the one for $-G_P/G_T$, 55, is less stringent, although this value is considerably smaller than the value 133 which has been proposed by Ahrens, Feenberg and Primakoff⁽³⁾ in connection with the β -decay of RaE . However, we expect that more elaborate experiments will reduce this rather large upper bound.

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Čerenkov Radiation and Supersonic Flows

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The propagation character of light in moving medium is investigated by an application of the characteristic theory of hyperbolic partial differential equations with intimate connection with the theory of compressible fluid, and then, the Čerenkov radiation is treated theoretically as a special case when the velocity of medium exceeds the light velocity in the medium, analogously to shock waves—such as result from supersonic flight of bullets through air.

§ 1. Introduction

The Čerenkov radiation¹⁾ may be considered to be an electromagnetic shock wave and is analogous to the shock wave produced when a projectile travels through air at a speed greater than that of sound. Like the pressure shock wave, Čerenkov wave is conical with the apex of the cone coincident with the particle producing disturbance. The radiation is produced when any charged particle traverses a dielectric medium at a speed greater than that of light in the medium. The direction of the radiation is forward at an angle θ , from the direction of the particle, where θ is determined by the relation

$$\cos \theta = 1/\beta n. \quad (A)$$

Here n is the index of refraction of the dielectric medium. Mathematically, those shock phenomenon may be described as the solutions of the linear inhomogeneous wave equations approximately or exactly;²⁾ and it is easy to derive relation (A) from Huygence's principle by the superposition of spherical waves diverging from a point source moving at a speed greater than that of waves emerging from itself. (illustrated in Fig. 1). For the Čerenkov wave, the pattern of disturbance can be given by the solution of Maxwell equation in the medium, with the charge and current distribution caused by the point charge moving with constant velocity.

However, from the relativistic stand point of view, the different description may be also possible, which is carried through in the coordinate system : particle at rest, medium moving : in place of the laboratory system : medium at rest, particle moving.

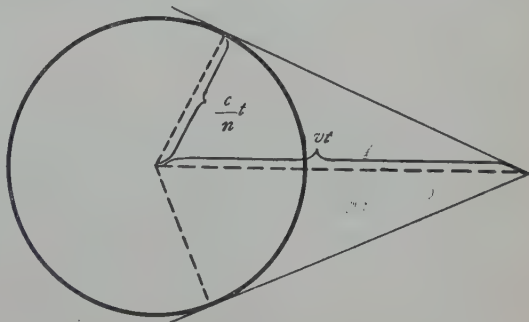


Fig. 1.

Although the relativistic invariance of this theory is guaranteed if we use the four dimensional tensor notation, it must be kept in mind that the uniform motion of the medium has to be taken into account and it makes the velocity of disturbance different from the so-called light velocity in that medium, c/n which is that in the coordinate system: medium at rest. The relativistic principle is thus violated in the above sense that it is possible in principle to detect an absolute motion by referring it to the motion of medium, and considerations were already made by Jauch and Watson³⁾ about these points.

The quite similar relation can be found in Hydrodynamics, if we consider the correspondencies of the velocity of medium to that of fluid and the light velocity in the medium to the sound velocity; and the propagation of disturbance in fluid may be delayed or accelerated by the motion of fluid and its velocity, which is equal to that of wave front, is usually different from the sound velocity and the more faster the motion of fluid is, the more larger the difference is.

According to the analogy with the propagation of light in medium, it may be said that Hydrodynamics is usually described in the coordinate system: fluid moving: though, of course, there can not be any transformation corresponding to Lorentz one, except when the velocity of fluid is constant and the non-linear character may be lost, and then, the Galilei transformation the transformation to the system "fluid at rest"—is permitted. In the above exceptional case, the correspondence may be fully substantiated, and investigated mathematically in our following discussions. This analogy made above between Hydrodynamics and Electrodynamics in medium teaches us also that Čerenkov radiation in the coordinate system "medium moving at a speed greater than c/n , particle at rest," corresponds to the discontinuous change of density appearing in fluid when the fluid velocity exceeds to that of sound, as in the cases of the flow through rocket nozzles, flight of bullets and detonation of explosives.

In the next section, we write down the Maxwell equations in the coordinate system "medium moving, particle at rest," according to Jauch and Watson⁴⁾ and the correspondence to fluid-dynamics will be expressed in the subsequent discussions mathematically by applications of the characteristic theory of the hyperbolic equation⁵⁾ which has been predominantly used in the investigations about propagations of discontinuity surfaces appearing in supersonic flows and fluid-dynamical shock waves.

§ 2. Field equations in a medium⁶⁾

As is well known, the Maxwell equations in a medium are, in the absence of charges and currents,

$$\text{rot } \mathbf{E} = -\dot{\mathbf{B}}, \quad \text{div } \mathbf{B} = 0, \quad (1)$$

$$\text{rot } \mathbf{H} = \dot{\mathbf{D}}, \quad \text{div } \mathbf{D} = 0, \quad (2)$$

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{B} = \mu \mathbf{H}, \quad (3)$$

with the electric and magnetic susceptibilities ϵ and μ respectively. The equations (1)

and (2) can be also written in the relativistically invariant form

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (1')$$

$$\partial_\mu G^{\lambda\mu} = 0 \quad (2')$$

by introducing two antisymmetrical tensors

$$F_{\lambda\mu} = \begin{vmatrix} 0 & B_3 & -B_2 & E_1 \\ -B_3 & 0 & B_1 & E_2 \\ B_2 & -B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{vmatrix},$$

$$G_{\lambda\mu} = \begin{vmatrix} 0 & B_3 & -B_2 & n^2 E_1 \\ -B_3 & 0 & B_1 & n^2 E_2 \\ B_2 & -B_1 & 0 & n^2 E_3 \\ -n^2 E_1 & -n^2 E_2 & -n^2 E_3 & 0 \end{vmatrix}.$$

The relations (3) hold only for the medium at rest and have to be generalized for any coordinate system "the medium moving with arbitrary constant velocity v ," and this can be really done by the following linking equation between two tensors $G_{\lambda\mu}$ and $F_{\lambda\mu}$,

$$G_{\lambda\mu} = F_{\lambda\mu} + \kappa (F_{\mu\sigma} v^\sigma v_\lambda - F_{\lambda\sigma} v^\sigma v_\mu)$$

where v^μ is the four vector of the medium velocity and equal to $dx^\mu/d\tau$. (τ is the proper time of the medium) and $\kappa = n^2 - 1$.

Introducing the four potentials ϕ_λ from (1)' defined by

$$F_{\lambda\mu} = \partial_\lambda \phi_\mu - \partial_\mu \phi_\lambda,$$

we obtain the field equations for them

$$(\partial^\mu \partial_\mu - \kappa \partial_\mu v^\mu \partial_\sigma v^\sigma) \phi_\lambda = 0 \quad (4)$$

with Lorentz condition

$$\chi \equiv \partial^\rho \phi_\rho - \kappa v^\sigma \partial_\sigma v^\rho \phi_\rho = 0.$$

The equation (4) differs from the ordinary wave equation considerably, if κ is not zero, and the second terms in the bracket represent the change of the propagation character of ϕ_λ waves due to the motion of the medium; though, of course, this effect comes out from the Lorentz transformation from the system "medium at rest," to the system "medium moving," and may be physically superficial. Our purpose is to investigate the hyperbolic character of the equation (4) analogously to that of Euler-Lagrange equation of fluid dynamics.

For this aim, it is convenient to rewrite the equation (4) in the following three dimensional form

$$\left[\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\kappa}{c^2(1-\beta^2)} \left\{ \sum_{i,k=1}^3 v_i v_k \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} - 2 \sum_i v_i \frac{\partial}{\partial x_i} \frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} \right\} \right] \phi_\lambda = 0, \quad (4')$$

where $v_i (i=1, 2, 3)$ are the components of the 3-dimensional velocity vector \mathbf{v} . It will be easily seen that the equations, thus written, are quite similar to the equation for density, ρ , in fluid dynamics, if the suitable linear approximations are made; or, in the following non-linear equations of the general fluid motion⁷⁾

$$\left. \begin{aligned} \partial \rho / \partial t + \sum_{i=1}^3 u_i \partial \rho / \partial x_i + \rho \sum_{i=1}^3 \partial u_i / \partial x_i &= 0, \\ \partial u_i / \partial t + \sum_{k=1}^3 u_k \partial u_i / \partial x_k + (c^2(\rho) / \rho) \partial \rho / \partial x_i &= 0, \quad i=1, 2, 3, \end{aligned} \right\} \quad (5)$$

approximating ρ , and u_i by $\rho_0 + \delta \rho$ and $v_i + \delta u_i$ respectively, (where ρ_0 and v_i are constants) and neglecting such that terms of order higher than first in $\delta \rho$, δu_i and their derivatives, we obtain the following linear equation for $\delta \rho$

$$\left[\Delta - \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} - (1/c_0^2) \left\{ 2 \sum_i v_i \frac{\partial^2}{\partial x_i \partial t} + \sum_i \sum_k v_i v_k \frac{\partial^2}{\partial x_i \partial x_k} \right\} \right] \delta \rho = 0, \quad (5)'$$

where $c_0 = c(\rho_0)$ and denotes the sound velocity. Then, applying the fluid dynamical considerations to the equations (4)', we investigate the propagation character of electromagnetic disturbance in moving medium in the following section.

§ 3. Characteristic theory of the field equations

The wave front may be defined, for example in one dimensional space, as a point, beyond which in one side, disturbance exists and in another, does not; and this point moves with time, tracing a line straight in some cases and curved in general, in two dimensional space-time, according to the natures of original field equations, which is called "characteristics" and usually denoted by the implicit form $\varphi(x, t) = 0$.

The extension of this conception to many dimensional cases is quite obvious and the characteristics may be written down $\varphi(x_i, t) = 0$ ($i=1, 2, 3$), in the 3-dimensional case. As is well known from the theory of hyperbolic differential equations, the equations which the characteristics have to satisfy - characteristic equations -, can be given for the equations (4)' as follows⁸⁾

$$c^2 \sum_{i=1}^3 \varphi_i^2 - \varphi_t^2 - g(\varphi_t - \sum_{i=1}^3 v_i \varphi_i)^2 = 0, \quad (6)$$

where $g = \kappa / (1 - \beta^2)$.

The integration of above first rank partial differential equation is equivalent to the integration of the following canonical equations representing the light ray⁹⁾

$$\frac{dx_i}{c^2 \varphi_i + g v_i (\varphi_t - \sum v_i \varphi_i)} = \frac{dt}{-\varphi_t - g(\varphi_t - \sum v_i \varphi_i)} = \frac{d\varphi_i}{0} = \frac{d\varphi_t}{0}. \quad (6)'$$

Especially, the solution satisfying the initial condition, $x_i = 0$ ($i=1, 2, 3$) at $t=0$,

represents the light ray emerging from the origin in the four dimensional world, the so-called "light cone", and can be given by

$$r^2 - c^2 t^2 + \{ \kappa / (1 + \kappa) (1 - \beta^2) \} (\mathbf{v} \cdot \mathbf{r} / c + ct)^2 = 0. \quad (7)$$

When the medium moves in the positive direction of x_3 axis, using the cylindrical co-ordinate $x_3 = z$, $x_1^2 + x_2^2 = R^2$, we can reduce the equation (7) to

$$z^2 - c^2 t^2 + R^2 + a(\beta z + ct)^2 = 0, \quad (7)'$$

where $a = \kappa / (1 + \kappa) \cdot (1 - \beta^2)$.

(i) Discussions in one dimension

Putting $R=0$ in eq. (7)', we obtain the two real branches of the light cone

$$\begin{aligned} z - a^+ ct &= 0, \\ z - a^- ct &= 0, \end{aligned} \quad (8)$$

with

$$\begin{aligned} a^+ &= (\beta n + 1) / (n + \beta), \\ a^- &= (\beta n - 1) / (n - \beta) \end{aligned} \quad (8)'$$

and these values of a^\pm give the velocities of the wave front measured in the unit of c and these shifts of them from c/n represent the influence of the motion of medium. It follows from (8)' immediately that always $a^+ > a^-$ for all possible values of β and β dependence of them can be given by

$$\begin{aligned} 1 \geq a^+ \geq 1/n & \quad \text{for all values of } \beta (\leq 1), \\ 1 \geq a^- \geq 0 & \quad \text{for } 1/(\beta n) \begin{matrix} < 1 \\ = 0 \\ > 1 \end{matrix} \quad \text{i.e. } v \begin{matrix} < \\ = \\ > \end{matrix} c/n, \text{ respectively.} \end{aligned} \quad (9)$$

The above relations show that for those values of v smaller than c/n , which is the so-called light velocity in the medium, strictly speaking, referred to the system "medium at rest," a^- is negative, opposite to a^+ and represents the propagation of waves in the negative direction of z axis, whereas a^+ represents that in the positive direction, but if the velocity of the medium exceeds the value c/n , all waves propagate itself to the positive direction of z axis and one can not find any wave transmitted to the negative direction, because those values of v make both a^+ and a^- , positive, or dz/dt increasing. This result comes from the fact that the velocity of waves spreading to the negative direction of z axis has to be compensated by the motion of the medium moving in the inverse direction which mediate the electromagnetic waves, and quite similar to that obtained in fluid dynamics. In fact, eq. (5) gives the following characteristic equation¹⁰⁾,

$$c^2 \sum_{i=1}^3 \varphi_i^2 - (\varphi_i + \sum_{i=1}^3 u_i \varphi_i)^2 = 0 \quad (6)''$$

with

$$c = \sqrt{p'(\rho)},$$

which is quite similar to (6) except the point that u_i and c are not constant but unknown independent variables, resulting from the non-linearity of the starting eq. (5).

The relation between the velocity of disturbance, i.e. the characteristic ray, and that of fluid motion may be given from (6)'' by

$$\sum_{i=1}^n (dx_i/dt - u_i)^2 = c^2$$

without any approximation.¹¹⁾ Especially, in one dimensional case where the fluid moves in the positive direction of z axis with velocity u , the above equation reduces to

$$\frac{dz}{dt} = u + c = a'_+$$

or

$$= u - c = a'_-$$

As is obviously seen from the above relations, the status is quite same in this case as in the former and corresponding to (9), we also obtain the similar dependence of the velocity of disturbance a'^{\pm} , to the fluid velocity u , i.e., a'^{+} is always positive for all u (u is surely positive) and a'^{+} is positive or negative, according to u is greater or smaller than the sound velocity c respectively, and equal to zero for the fluid velocity equal to c . These correspondencies indicate the intimate connection between supersonic flows and super light motion of medium and then, in the next subsection, we investigate the pattern of Cerenkov disturbance in two dimensional space, connected with the shock wave in fluid.

(ii) Shock front and Mach angle

In Hydrodynamics, the shapes of shock fronts or discontinuity surfaces caused by sudden changes of the fluid density, have been calculated in the two dimensional steady flow. In this case, the terms referred to time derivatives in equations (5) may be dropped and then, eq. (6)'' may be reduced to

$$c^2 \sum_{i=1}^2 \varphi_i^2 - \left(\sum_{i=1}^2 u_i \varphi_i \right)^2 = 0. \quad (10)$$

The same is also valid for our case and eq. (6) reduces to

$$c^2 \sum_{i=1}^2 \varphi_i^2 - g \left(\sum_{i=1}^2 v_i \varphi_i \right)^2 = 0$$

or

$$c^2 (\varphi_x^2 + \varphi_y^2) - g (v_x \varphi_x + v_y \varphi_y)^2 = 0 \quad (11)$$

under the similar condition.

The existence of the discontinuity lines or shock fronts in plane, across which there exists the discontinuous jump of the field strength, is guaranteed by the condition that eq. (10) permits the real value for $dy/dx (= -\varphi_x/\varphi_y)$, the tangent of those lines, determining the geometrical nature of the discontinuity. And then, it follows immediately

$$v^2 = v_x^2 + v_y^2 > c^2/n^2$$

or
$$v > c/n,$$

i.e. if and only if, the velocity of the medium exceeds the light velocity in that medium, the discontinuity surface, resulting from the sudden change of the strength of electromagnetic potentials, appears.

It is quite obvious that the similar relation between the velocity of fluid and the existence of discontinuity surface can be derived from (10). The shape of the discontinuity can be obtained by the integration of characteristic equations derived above and in fluid dynamics, it gives the so-called Mach-angle¹²⁾ though the exact solution of eq. (10) can not be obtained because of the non-linear character. However, for eq. (11) the velocity of medium v are constant and the integration can be performed exactly as follows. Assuming that the medium is moving in the positive direction of x -axis, without loss of any generality, we can get the following differential equation for the tangent of the discontinuity lines from (11).

$$dy/dx = \pm 1 \sqrt{\lambda^2 - 1}, \quad (12)$$

where, $\lambda^2 = g\beta^2$.

Denoting the angle between these discontinuity lines and x -axis, the direction of motion of medium, as u ; we get

$$\tan u = 1/(\lambda^2 - 1) \quad (13)$$

or
$$u = \arctan \{1/(\lambda^2 - 1)\}$$

$$= \arcsin (1/\lambda).$$

This corresponds to the Mach-angle in fluid dynamics, which gives the shape of discontinuity surface photographed instantaneously, appearing in violent disturbance—such as result from the flow through rocket nozzles, or from supersonic flight of projectiles; and then, we can expect analogously the shape of the shock front of Čerenkov radiation from above calculations. In fact, we can prove from (13) the experimental result (A) for the direction of radiation by introducing the angle θ between the normal to the discontinuity line and the direction of motion of the medium and transforming the value of $\cos \theta$ ($= \sin u$) into that referred to the laboratory system, according to the Lorentz transformation

$$y = y', \quad x = (x' + vt')/\sqrt{1 - \beta^2},$$

which gives the following equations for $t' = 0$.

$$\tan u' = y'/x' = 1/\sqrt{\beta^2 n^2 - 1}$$

or
$$\cos \theta' = 1/\beta n.$$

It has been recognized that the characteristic theory of the hyperbolic equation was often very useful for the study of shock waves, and it was proved that this was also

true for Čerenkov radiation, though in this case, the usual Fourier analysis makes us enable to obtain the exact solution, because of the linearity of the equation, and through analysis of the propagation of light in moving medium and compressible fluid dynamics, mathematically very similar in nature, the correspondences between them became much more clearer.

The writer expresses his cordial thanks to Prof. Y. Tanikawa for his kind interest and to Prof. K. Husimi for his valuable advices.

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Hamiltonian Formalism in Non-local Field Theories

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The Yang-Feldman's method is generalized in order to obtain a covariant Hamiltonian formulation of quantized fields with non-local interactions. The interaction Hamiltonian is constructed according to the perturbation theory in such a way that the equation of motion in the interaction representation is integrable. The calculations are carried out actually up to the fourth order approximation.

§ 1. Introduction and summary

It has been generally believed that for fields with non-local interactions the Hamiltonian formalism is impossible and only the S -matrix exists which connects free fields in the infinite past and in the infinite future^{1), 2)}. With the S -matrix alone, however, there remain questions how to treat unambiguously the problems involving bound states such as energy levels, line-breadths and mutual scatterings of composite particles. The aim of this paper is to find a generalization of the current covariant Hamiltonian formalism which will offer a field-theoretical basis for these problems and, at the same time, will give a S -matrix which is obviously unitary.

Recently, Umezawa-Takahashi³⁾ and Katayama⁴⁾ have shown that fields with interaction Lagrangians containing higher derivatives of field quantities can be quantized in a covariant way by generalizing the method of Yang-Feldman⁵⁾. Their method can be applied, but with modifications, to the fields with non-local interactions developed by Kristensen-Møller¹⁾ and Bloch²⁾. We shall confine ourselves to the solutions of field equations which are described by the same number of independent variables as the free fields and agree with those of the free fields in the limit of vanishing coupling constant. These solutions will be described well by the integral equations of Yang-Feldman's type which are solved by iteration. These equations, however, are not unique in that any homogeneous terms, which satisfy the free field equations and vanish as the coupling constant does, can be added to them. We can determine in a covariant way these additive terms together with the interaction Hamiltonians by successive approximations in the coupling constant such that these Hamiltonians satisfy the integrability condition and there exist unitary operators $U(\sigma, \sigma')$ which connect states on surfaces σ and σ' . Hamiltonians thus obtained are expressed as infinite power series of the coupling constant, each term of which is an integral of products of field operators over the whole space-time and has an explicit dependence on the entire form of surface σ on which their argument lies. These Hamiltonians are not unique, but the different systems are connected by unitary transformations in a way similar to the local theories,

The calculations have been carried out actually up to the fourth order approximation. The S -matrix constructed from these Hamiltonians is obviously in agreement up to the third order with those obtained according to the theories of Kristensen-Møller and Bloch since the in- and out-fields are defined by the same integral equations. In the fourth order, however, our equations differ by one special term, which will show that S -matrix defined by them is in general not unitary.

We shall not investigate the convergence of the Hamiltonians as infinite series, which seems to have the same character as that of the S -matrix. Then, in our formalism the integrability condition for the equation of motion is satisfied for any two points which lie not only on a space-like surface but also on a surface which can be time-like in the finite domain of space-time. It is a characteristic of the fields with non-local interactions that the Hamiltonians can not be defined without such an extension. Indeed, in the local theories we have interaction Hamiltonians which consist of a finite number of terms and satisfy the integrability condition if surfaces σ are restricted to the space-like ones, but if Hamiltonians are allowed to be infinite series σ will not always have to be space like.

If we choose flat surfaces we can obtain easily Hamiltonians in the Schrödinger representation. In the first approximation the results agree with those of recent Pauli's theory⁽⁶⁾. It seems to be possible that the degree of singularities of these Hamiltonians is low enough if we choose the form factor appropriately. This can be verified with the form factor of Kristensen-Møller for the first order Hamiltonian. In this way, it will be possible to treat the problems involving bound states without divergence difficulties, for instance, according to the Tamm-Dancoff's theory⁽⁷⁾. In another way, the Hamiltonian formalism will also offer a basis to extend the Bethe-Salpeter equation^(8,9) after Gell-Mann and Low⁽¹⁰⁾ such that its kernel is free from singularities due to the effect of the form factor.

§ 2. Method of quantization

For the sake of simplicity, we shall consider charged scalar fields ϕ , ϕ^* and a neutral scalar field u with non-local interactions of Kristensen-Møller's type.⁽¹⁾ The similar results are obtained for spinor fields ψ and ψ^* . In what follows the same notations as Yang-Feldman's⁽⁵⁾ and Kristensen-Møller's will be used if not otherwise stated. The Lagrangian is given by

$$\int L dx = \int L_0(x) dx + g \int \phi^*(1) \Phi(1, 2, 3) u(2) \phi(3) d1 d2 d3, \quad (1)$$

$$L_0(x) = -(\partial_\mu \phi^* \cdot \partial_\mu \phi + M^2 \phi^* \phi) - 1/2 \cdot (\partial_\mu u \cdot \partial_\mu u + m^2 u^2), \quad (2)$$

where $\Phi(1, 2, 3)$ is a form factor satisfying the Hermitian condition $\Phi(1, 2, 3) = \Phi^*(3, 2, 1)$. Numbers 1, 2, 3; 1', 2', 3'; ... which denote points in space-time will always be used as variables of integration, and to simplify the expressions we shall write

$$\Phi(1, 2, 3) d1 d2 d3 \equiv d(1, 2, 3). \quad (3)$$

Field equations are given by

$$\left. \begin{aligned} (\square - M^2)\phi(x) &= -g\{u(2)\phi(3)\delta(x-1)d(1\ 2\ 3), \\ (\square - m^2)u(x) &= -g\{\phi^*(1)\phi(3)\delta(x-2)d(1\ 2\ 3), \\ (\square - M^2)\phi^*(x) &= -g\{\phi^*(1)u(2)\delta(x-3)d(123)\}. \end{aligned} \right\} \quad (4)$$

The solutions of (4), which agree with those of the free fields in the limit $g \rightarrow 0$, will be described by the integral equations

$$\left. \begin{aligned} \phi(x) &= \phi(x, \sigma) - g \int \frac{\varepsilon(x-1) - \varepsilon(\sigma, 1)}{2} \Delta(x-1) u(2) \phi(3) d(123) + \sum_{n=1}^{\infty} g^n \varphi_n(x, \sigma), \\ u(x) &= u(x, \sigma) - g \int \phi^*(1) \frac{\varepsilon(x-2) - \varepsilon(\sigma, 2)}{2} D(x-2) \phi(3) d(123) + \sum_{n=1}^{\infty} g^n v_n(x, \sigma), \\ \phi^*(x) &= \phi^*(x, \sigma) - g \int \phi^*(1) u(2) \frac{\varepsilon(x-3) - \varepsilon(\sigma, 3)}{2} \Delta(x-3) d(123) + \sum_{n=1}^{\infty} g^n \varphi_n^*(x, \sigma). \end{aligned} \right\} \quad (5)$$

Here we assume that $\phi(x, \sigma)$ and $u(x, \sigma)$, being the independent variables which describe the motion of fields, satisfy the free field equations and commutation relations

$$(\square - M^2)\phi(x, \sigma) = 0, \quad (\square - m^2)u(x, \sigma) = 0, \quad (6)$$

$$\left. \begin{aligned} [\phi(x, \sigma), \phi^*(x', \sigma)] &= i\Delta(x-x'), \quad [u(x, \sigma), u(x', \sigma)] = iD(x-x'), \\ [\phi(x, \sigma), \phi(x', \sigma)] &= [\phi(x, \sigma), u(x', \sigma)] = 0, \end{aligned} \right\} \quad (7)$$

and are connected by a unitary operator $U(\sigma, \sigma')$ for any two surfaces σ and σ'

$$\phi(x, \sigma) = U^{-1}(\sigma, \sigma') \phi(x, \sigma') U(\sigma, \sigma'), \quad u(x, \sigma) = U^{-1}(\sigma, \sigma') u(x, \sigma') U(\sigma, \sigma'). \quad (8)$$

The additive terms $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ should satisfy the free field equations

$$(\square - M^2)\varphi_n(x, \sigma) = 0, \quad (\square - m^2)v_n(x, \sigma) = 0, \quad (n=1, 2, \dots), \quad (9)$$

and should be determined successively such that there exist the above unitary operator $U(\sigma, \sigma')$ and consequently the interaction Hamiltonian. It will be prescribed that they are expressed as integrals over space-time of products of Heisenberg operators $\phi^*(1), u(2), \phi(3), \phi^*(1'), \dots$ and depend on the form of σ explicitly. If they all vanish for both $\sigma = -\infty$ and $+\infty$

$$\varphi_n(x, \sigma = \pm \infty) = v_n(x, \sigma = \pm \infty) = 0, \quad (n=1, 2, \dots), \quad (10)$$

the equations (5) for $\sigma = \pm \infty$ agree with those of Kristensen-Møller¹⁾ and Bloch²⁾ which define the incoming and outgoing fields, with an obvious consequence that the same S -matrix as theirs is obtained from the Hamiltonian. As shown in § 3, the conditions (10) can be satisfied up to the third order, but a particular term appears in $\varphi_4(x, \sigma)$ which can not be made to vanish in general for both $\sigma = -\infty$ and $+\infty$.

If the difference between σ and σ' is infinitesimal, $U(\sigma, \sigma')$ can be written with some Hermitian operator $H(x'/\sigma)$ as

$$U(\sigma, \sigma') = 1 - i \int_{\sigma'}^{\sigma} H(x'/\sigma) dx'. \quad (11)$$

Then, we have from (8) and (5) for a point x' on σ

$$\left. \begin{aligned} \frac{\delta\psi(x, \sigma)}{\delta\sigma(x')} &= \frac{1}{i} [\psi(x, \sigma), H(x'/\sigma)] \\ &= -g \{ A(x-1)u(2)\psi(3)\delta(x'-1)d(123) - \frac{\partial}{\partial\sigma(x')} \sum_{n=1}^{\infty} g^n \varphi_n(x, \sigma), \\ \frac{\delta u(x, \sigma)}{\delta\sigma(x')} &= \frac{1}{i} [u(x, \sigma), H(x'/\sigma)] \\ &= -g \{ \psi^*(1)D(x-2)\psi(3)\delta(x'-2)d(123) - \frac{\partial}{\partial\sigma(x')} \sum_{n=1}^{\infty} g^n v_n(x, \sigma), \end{aligned} \right\} \quad (12)$$

where $\partial/\partial\sigma(x')$ means a functional derivative with respect to σ which appears explicitly.

Now, we write the solutions of (5) which are obtained by iteration as

$$\left. \begin{aligned} \psi(x) &= \psi(x, \sigma) + \sum_{n=1}^{\infty} g^n \psi_n(x, \sigma), \\ u(x) &= u(x, \sigma) + \sum_{n=1}^{\infty} g^n u_n(x, \sigma), \end{aligned} \right\} \quad (13)$$

and define $H(x/\sigma)$ as a power series

$$H(x/\sigma) = \sum_{n=1}^{\infty} g^n H_n(x/\sigma), \quad (14)$$

where $\psi_n(x, \sigma)$, $u_n(x, \sigma)$ and $H_n(x/\sigma)$ are all expressed as integrals of products of operators in the interaction representation $\psi^*(1, \sigma)$, $u(2, \sigma)$, $\psi(3, \sigma)$, $\psi^*(1', \sigma)$, ... and factors depending on σ explicitly. If the additive terms $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ are known up to $n=i$, we have $\psi_n(x, \sigma)$ and $u_n(x, \sigma)$ up to $n=i$ from (5) and we can write down the equations (12) in the $i+1$ -th approximation. If we obtain $H_{i+1}(x/\sigma)$, $\varphi_{i+1}(x, \sigma)$ and $v_{i+1}(x, \sigma)$ by solving these equations and this procedure proceeds to any higher orders, it can be shown as follows that the Hamiltonian thus obtained satisfies the integrability condition which is extended from that in the local theories in order to have a Hamiltonian formalism in the case of non-local interactions.

As seen from the form of equation (5), $\varphi_n(x, \sigma)$, $v_n(x, \sigma)$, $\psi_n(x, \sigma)$, $u_n(x, \sigma)$ and $H_n(x/\sigma)$ will all depend on σ explicitly through the sign-functions $\varepsilon(\sigma, 1)$, $\varepsilon(\sigma, 2)$, $\varepsilon(\sigma, 3)$, $\varepsilon(\sigma, 1')$, ... Only if surfaces σ divide the whole space-time into two simply connected regions and are space-like at an infinite distance, these sign-functions have a well-defined meaning for σ which is not always space-like. Accordingly, as made clear by considering equations (12) as a limit of coupled difference equations when the space-time is divided into small elements, the integrability conditions of (12) for $\psi(x, \sigma)$ and $u(x, \sigma)$, that they should be determined uniquely when their initial values $\psi(x, \sigma_0)$ and $u(x, \sigma_0)$ are given for a specified surface σ_0 , are for any two points x' and x'' lying on σ (we can not confine ourselves to the case where the distance between x' and x'' is infinitesimal even if σ is space-like)

$$\left\{ \begin{aligned} \frac{\partial^2}{\partial\sigma(x'')\partial\sigma(x')} - \frac{\partial^2}{\partial\sigma(x')\partial\sigma(x'')} \} \psi(x, \sigma) &= 0, \\ \frac{\partial^2}{\partial\sigma(x'')\partial\sigma(x')} - \frac{\partial^2}{\partial\sigma(x')\partial\sigma(x'')} \} u(x, \sigma) &= 0, \end{aligned} \right\} \quad (15)$$

or noting that $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ are written in Heisenberg operators

$$\left. \begin{aligned} \left\{ \frac{\partial^2}{\partial \sigma(x'') \partial \sigma(x')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} \varphi_n(x, \sigma) &= 0, \\ \left\{ \frac{\partial^2}{\partial \sigma(x'') \partial \sigma(x')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} v_n(x, \sigma) &= 0, \quad (n=1, 2, \dots) \end{aligned} \right\} \quad (16)$$

If $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ are obtained as one-valued functions of σ on which they depend explicitly, the condition (16) is satisfied. In other words, we should determine $H_n(x/\sigma)$ such that $\partial \varphi_n(x, \sigma)/\partial \sigma(x')$ and $\partial v_n(x, \sigma)/\partial \sigma(x')$ appearing in (12) are actually integrable.

$H_n(x/\sigma)$, $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ are not determined uniquely at each stage of the approximation, corresponding to the fact that the following unitary transformations are always possible. If the equation

$$\frac{\partial \psi(x, \sigma)}{\partial \sigma(x')} = \frac{1}{i} [\psi(x, \sigma), H(x'/\sigma)] \quad (17)$$

holds, $\psi'(x, \sigma)$ obtained by the unitary transformation with some Hermitian operator $G(\sigma)$ depending on σ ,

$$\psi'(x, \sigma) = e^{iG(\sigma)} \psi(x, \sigma) e^{-iG(\sigma)}, \quad (18)$$

satisfies the following equation in the lowest order approximation in the expansion of the parameter appearing in $G(\sigma)$

$$\frac{\partial \psi'(x, \sigma)}{\partial \sigma(x')} = \frac{1}{i} [\psi'(x, \sigma), H'(x'/\sigma) + \frac{\partial G(\sigma)}{\partial \sigma(x')}] \quad (19)$$

where $H'(x/\sigma)$ is the one transformed from $H(x/\sigma)$ according to (18).

The only restriction imposed on the form factor $\Phi(1, 2, 3)$ to perform the above program is that the integrals containing the form factor have a well-defined meaning, which will be realized if we introduce a suitable damping factor. The essential effect of the form factor will appear in the convergence of the series. This problem will not be considered here. It can be said safely that our method gives at least the asymptotic solutions correctly which agree with those of free fields in the limit $g \rightarrow 0$.

§ 3. Derivation of the interaction Hamiltonian

Following the above method we shall derive $H_n(x/\sigma)$, $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ successively up to the fourth order approximation. In this section we shall omit the notation (3) for simplicity, noting that 1, 2, 3; 1', 2', 3'; ... are integral variables multiplied always by form factors $\Phi(1, 2, 3)$, $\Phi(1', 2', 3')$, ...

(1) First order

The first order equations of (12) are

$$\left(\frac{\partial \varphi_1(x, \sigma)}{\partial \sigma(x')}\right)_l = - \int D(x-1) u(2, \sigma) \psi(3, \sigma) \delta(x'-1) + i[\psi(x, \sigma), H_1(x'/\sigma)], \quad (20)$$

$$\left(\frac{\partial v_1(x, \sigma)}{\partial \sigma(x')}\right)_l = - \int \psi^*(1, \sigma) D(x-2) \psi(3, \sigma) \delta(x'-2) + i[u(x, \sigma), H_1(x'/\sigma)],$$

where the suffix l means to take the lowest order approximation of the term in the bracket, that is, to replace the Heisenberg operators $\psi^*(1)$, $u(2)$, \dots by the operators in the interaction representation $\psi^*(1, \sigma)$, $u(2, \sigma)$, \dots . In order that the second terms in (20) may be of the same form as the first terms, $H_1(x'/\sigma)$ must be of a form

$$H_1(x'/\sigma) = - \{\psi^*(1, \sigma) u(2, \sigma) \psi(3, \sigma) \{a\delta(x'-1) + b\delta(x'-2) + a\delta(x'-3)\}\}, \quad (21)$$

where a and b are real constants. Inserting (21) into (20) we have for the first equation

$$\left(\frac{\partial \varphi_1(x, \sigma)}{\partial \sigma(x')}\right)_l = \int D(x-1) u(2, \sigma) \psi(3, \sigma) \{(a-1)\delta(x'-1) + b\delta(x'-2) + a\delta(x'-3)\}. \quad (22)$$

Returning $u(2, \sigma)$ and $\psi(3, \sigma)$ back to the Heisenberg operators $u(2)$ and $\psi(3)$, where their orders are prescribed such that ψ^* , u and ψ always stand from the left in this order, and using an integral

$$\int_0^\sigma \delta(x'-1) dx' = \frac{\varepsilon(\sigma, 1)}{2} + \text{const.}, \quad (23)$$

we can integrate (22) and obtain

$$\varphi_1(x, \sigma) = \int D(x-1) u(2) \psi(3) \left\{ (a-1) \frac{\varepsilon(\sigma, 1)}{2} + b \frac{\varepsilon(\sigma, 2)}{2} + a \frac{\varepsilon(\sigma, 3)}{2} \right\} + C, \quad (24a)$$

where C is a term independent of σ . The condition (10) can be satisfied if we choose $C=0$ and

$$2a + b = 1. \quad (25)$$

These restrictions are also necessary to have a correspondence to the local case where (24a) should vanish for any σ . In the same way, under the condition (10) we have (25) and

$$v_1(x, \sigma) = \int \psi^*(1) D(x-2) \psi(3) \left\{ a \frac{\varepsilon(\sigma, 1)}{2} + (b-1) \frac{\varepsilon(\sigma, 2)}{2} + a \frac{\varepsilon(\sigma, 3)}{2} \right\}. \quad (24b)$$

The arbitrariness of (21) within the limit of (25) is explained by the unitary transformation (18). If we choose

$$G(\sigma) = -g \int \psi^*(1, \sigma) u'(2, \sigma) \psi'(3, \sigma) \left\{ (a-a') \frac{\varepsilon(\sigma, 1)}{2} + (b-b') \frac{\varepsilon(\sigma, 2)}{2} + (a-a') \frac{\varepsilon(\sigma, 3)}{2} \right\} \quad (26)$$

with $2a' + b' = 1$, we have the interaction Hamiltonian (21) with new constants a' and b'

in place of a and b .

(ii) *Second order*

It will be convenient to add $\varphi_1(x, \sigma)$ and $v_1(x, \sigma)$ obtained above to the second terms in (5) and write

$$\left. \begin{aligned} \phi(x) &= \phi(x, \sigma) - g \int \frac{\varepsilon(x-1) - \varepsilon(\sigma, 123)}{2} A(x-1) u(2) \phi(3) + \sum_{n=2}^{\infty} g^n \varphi_n(x, \sigma), \\ u(x) &= u(x, \sigma) - g \int \phi^*(1) \frac{\varepsilon(x-2) - \varepsilon(\sigma, 123)}{2} D(x-2) \phi(3) + \sum_{n=2}^{\infty} g^n v_n(x, \sigma), \end{aligned} \right\} \quad (27)$$

where we have written

$$\varepsilon(\sigma, 123) \equiv a\varepsilon(\sigma, 1) + b\varepsilon(\sigma, 2) + c\varepsilon(\sigma, 3). \quad (28)$$

Then, the second order equations of (12) are

$$\left. \begin{aligned} \left(\frac{\partial \varphi_2(x, \sigma)}{\partial \sigma(x')} \right)_i &= - \int A(x-1) \{ u_1(2, \sigma) \phi(3, \sigma) + u(2, \sigma) \phi_1(3, \sigma) \} \delta(x', 123) \\ &\quad + i[\phi(x, \sigma), H_2(x'/\sigma)], \\ \left(\frac{\partial v_2(x, \sigma)}{\partial \sigma(x')} \right)_i &= - \int D(x-2) \{ \phi_1^*(1, \sigma) \phi(3, \sigma) + \phi^*(1, \sigma) \phi_1(3, \sigma) \} \delta(x', 123) \\ &\quad + i[u(x, \sigma), H_2(x'/\sigma)], \end{aligned} \right\} \quad (29)$$

with

$$\left. \begin{aligned} \phi_1(x, \sigma) &= - \int \frac{\varepsilon(x-1) - \varepsilon(\sigma, 123)}{2} A(x-1) u(2, \sigma) \phi(3, \sigma), \\ u_1(x, \sigma) &= - \int \phi^*(1, \sigma) \frac{\varepsilon(x-2) - \varepsilon(\sigma, 123)}{2} D(x-2) \phi(3, \sigma), \end{aligned} \right\} \quad (30)$$

$$\delta(x', 123) \equiv \frac{\partial}{\partial \sigma(x')} \frac{\varepsilon(\sigma, 123)}{2} = a\delta(x'-1) + b\delta(x'-2) + c\delta(x'-3). \quad (31)$$

In order that the second terms in (29) may be of the same form as the first terms, we must put

$$\begin{aligned} H_2(x'/\sigma) &= - \int \{ c\phi_1^*(1, \sigma) u(2, \sigma) \phi(3, \sigma) + d\phi^*(1, \sigma) u_1(2, \sigma) \phi(3, \sigma) \\ &\quad + c\phi^*(1, \sigma) u(2, \sigma) \phi_1(3, \sigma) \} \delta(x', 123), \end{aligned} \quad (32)$$

where c and d are real constants. Inserting (32) into (29) and using (30) we have for the first equation

$$\begin{aligned} \left(\frac{\partial \varphi_2(x, \sigma)}{\partial \sigma(x')} \right)_i &= \int A(x-1) \phi^*(1', \sigma) D(2-2') \phi(3', \sigma) \phi(3, \sigma) \left\{ (1-d) \right. \\ &\quad \times \frac{\varepsilon(2-2') - \varepsilon(\sigma, 1'2'3')}{2} \delta(x', 123) + d \frac{\varepsilon(2'-2) - \varepsilon(\sigma, 123)}{2} \delta(x', 1'2'3') \Big\} \\ &\quad + \int A(x-1) u(2, \sigma) A(3-1') u(2', \sigma) \phi(3', \sigma) \left\{ (1-c) \frac{\varepsilon(3-1') - \varepsilon(\sigma, 1'2'3')}{2} \right. \end{aligned}$$

$$\times \delta(x', 123) + c \frac{\varepsilon(1'-3) - \varepsilon(\sigma, 123)}{2} \delta(x', 1'2'3') \}. \quad (33)$$

In order that the expression in the bracket on the right of (33) may have a form $\partial/\partial\sigma(x')$, we must choose

$$c=d=1/2. \quad (34)$$

Returning $\phi^*(1, \sigma)$, $\phi^*(1', \sigma)$, ... back to the Heisenberg operators we have

$$\begin{aligned} \varphi_2(x, \sigma) = & -\frac{1}{2} \int \mathcal{A}(x-1) D(2-2') \phi^*(1') \phi(3') \phi(3) \frac{\varepsilon(2-2') - \varepsilon(\sigma, 1'2'3')}{2} \\ & \times \frac{\varepsilon(2'-2) - \varepsilon(\sigma, 123)}{2} \\ & -\frac{1}{2} \int \mathcal{A}(x-1) \mathcal{A}(3-1') u(2) u(2') \phi(3') \frac{\varepsilon(3-1') - \varepsilon(\sigma, 1'2'3')}{2} \\ & \times \frac{\varepsilon(1'-3) - \varepsilon(\sigma, 123)}{2}, \end{aligned} \quad (35a)$$

where the integration constant is chosen such that $\varphi_2(x, \sigma)$ vanishes for $\sigma = -\infty$ and $+\infty$ under the restriction (25). In deriving (35a) the order of $\phi(3')$ and $\phi(3)$ is not determined. If we choose the reverse order in place of (35a), $\varphi_4(x, \sigma)$ and all terms higher than the fourth are affected. This alteration will correspond to making some unitary transformation. In the same way we have with (34)

$$\begin{aligned} v_2(x, \sigma) = & -\frac{1}{2} \int \mathcal{A}(1-3') D(x-2) \phi^*(1') u(2') \phi(3) \frac{\varepsilon(1-3') - \varepsilon(\sigma, 1'2'3')}{2} \\ & \times \frac{\varepsilon(3'-1) - \varepsilon(\sigma, 123)}{2} + \text{Herm. conj.} \end{aligned} \quad (35b)$$

(iii) Third order

Using (35a) and (35b), we have the third order equation of (12) for $\varphi_3(x, \sigma)$

$$\begin{aligned} \left(\frac{\partial \varphi_3(x, \sigma)}{\partial \sigma(x')} \right) = & -\int \mathcal{A}(x-1) \{ u_2(2, \sigma) \phi(3, \sigma) + u_1(2, \sigma) \phi_1(3, \sigma) \\ & + u(2, \sigma) \phi_2(3, \sigma) \} \delta(x', 123) + \frac{1}{2} \int \mathcal{A}(x-1) D(2-2') \{ \phi_1^*(1', \sigma) \phi(3', \sigma) \phi(3, \sigma) \\ & + \phi^*(1', \sigma) \phi_1(3', \sigma) \phi(3, \sigma) + \phi^*(1', \sigma) \phi(3', \sigma) \phi_1(3, \sigma) \} \frac{\partial}{\partial \sigma(x')} \zeta(2, 2') \zeta(2', 2) \\ & + \frac{1}{2} \int \mathcal{A}(x-1) \mathcal{A}(3-1') \{ u_1(2, \sigma) u(2', \sigma) \phi(3', \sigma) + u(2, \sigma) u_1(2', \sigma) \phi(3', \sigma) \\ & + u(2, \sigma) u(2', \sigma) \phi_1(3', \sigma) \} \frac{\partial}{\partial \sigma(x')} \zeta(3, 1') \zeta(1', 3) + i[\phi(x, \sigma), H_3(x'/\sigma)], \end{aligned} \quad (36)$$

where we have introduced a new notation

$$\zeta(x, y') \equiv \frac{\varepsilon(x - y') - \varepsilon(\sigma, 1' 2' 3')}{2}, \quad \zeta(x''', y'') \equiv \frac{\varepsilon(x''' - y'') - \varepsilon(\sigma, 1'' 2'' 3'')}{2}, \dots, \\ (x, y = 1, 2, 3). \quad (37)$$

In the approximations higher than the third, it will be convenient to proceed as follows. Terms on the right of (36) except the last one are all developed and graphically equivalent terms are grouped together by the interchanges of the integration variables. Each group is expressed as an integral of the products of definite propagation functions such as $\Delta(3-1')$, $D(2'-2'')$, ..., definite operators such as $\psi^*(1, \sigma)$, $u(2, \sigma)$, ... and a factor consisting of sign-functions and δ -functions. This factor is divided into two parts as simply as possible, one part having the form $\partial/\partial\sigma(x')$ and the other having not. The Hamiltonian is chosen in such a way that the latter part is canceled by the last term in (36). For instance, we have as one of five graphically independent groups appearing in (36)

$$-1/2 \int \Delta(x-1) \Delta(3-1') \Delta(3'-1'') u(2, \sigma) u(2', \sigma) u(2'', \sigma) \psi(3'', \sigma) \\ \times \left\{ \frac{\partial}{\partial\sigma(x')} \zeta(1', 3) \zeta(3', 1'') \zeta(1'', 3') \right. \\ \left. + \zeta(3, 1') \zeta(3', 1'') \delta(x', 123) + \zeta(1', 3) \zeta(1'', 3') \delta(x', 1'' 2'' 3'') \right\}.$$

The second and the third terms in the bracket are canceled if we choose as one part of $H_3(x'/\sigma)$

$$-1/2 \cdot \int \Delta(3-1') \Delta(3'-1'') \psi^*(1, \sigma) u(2, \sigma) u(2', \sigma) u(2'', \sigma) \psi(3'', \sigma) \\ \times \{ \zeta(3, 1') \zeta(3', 1'') \delta(x', 123) + \zeta(1', 3) \zeta(1'', 3') \delta(x', 1'' 2'' 3'') \}.$$

We shall write the results alone since the actual calculations are lengthy and tedious. In what follows some simplifications of expressions will be made. $\delta(x', 123)$, $\delta(x', 1' 2' 3')$, $\delta(x', 1'' 2'' 3'')$, ... be written as δ , δ' , δ'' , ...; products of operators in the interaction representation such as $\psi^*(1'', \sigma) u(2, \sigma) \psi(3', \sigma) \psi(3, \sigma)$ and products of Heisenberg operators such as $\psi^*(1) u(2) u(2') \psi(3'')$ be written as $[1'' 2' 3' 3]$ and $\{1' 2' 2' 3''\}$, respectively, noting that 1, 2 and 3 always represent the arguments of ψ^* , u and ψ , respectively. The results are

$$H_3(x'/\sigma) = -\frac{1}{2} \int \Delta(3-1') \Delta(3'-1'') [1' 2' 2'' 3''] \{ \zeta(3, 1') \zeta(3', 1'') \delta \\ + \zeta(1', 3) \zeta(1'', 3') \delta'' \} \\ - \frac{1}{2} \int \Delta(1-3') D(2'-2'') [1' 1'' 2' 3' 3] \{ \zeta(1, 3') \zeta(2', 2'') \delta + \zeta(3', 1) \zeta(2'', 2') \delta'' \} \\ - \frac{1}{2} \int \Delta(3-1') D(2'-2'') [1' 1'' 2' 3' 3'] \{ \zeta(3, 1') \zeta(2', 2'') \delta + \zeta(1', 3) \zeta(2'', 2') \delta'' \}, \\ \varphi_3(x, \sigma) = -\frac{1}{2} \int \Delta(x-1) \Delta(3-1') \Delta(3'-1'') \{ 2' 2'' 2'' 3'' \} \zeta(1', 3) \zeta(3', 1'') \zeta(1'', 3')$$

$$\begin{aligned}
& -\frac{1}{2} \int \mathcal{A}(x-1) D(2-2') \mathcal{A}(1'-3'') \{1'' 2'' 3' 3\} \zeta(2', 2) \zeta(1', 3'') \zeta(3'', 1') \\
& -\frac{1}{2} \int \mathcal{A}(x-1) \mathcal{A}(3-1') D(2'-2'') \{1'' 2 3'' 3'\} \zeta(1', 3) \zeta(2', 2'') \zeta(2'', 2') \\
& \quad \quad \quad (39a)
\end{aligned}$$

$$-\frac{1}{2} \int \mathcal{A}(x-1) D(2-2') \mathcal{A}(3'-1'') \{1' 2'' 3'' 3\} \zeta(2', 2) \zeta(3', 1'') \zeta(1'', 3'),$$

$$\begin{aligned}
v_3(x, \sigma) = & -\frac{1}{2} \int D(x-2) \mathcal{A}(1-3') D(2'-2'') \{1' 1'' 3'' 3\} \zeta(3', 1) \zeta(2', 2'') \zeta(2'', 2') \\
& -\frac{1}{2} \int D(x-2) \mathcal{A}(1-3') \mathcal{A}(1'-3'') \{1'' 2'' 2' 3\} \zeta(3', 1) \zeta(1', 3'') \zeta(3'', 1') \\
& + \text{Herm. conj.}, \quad (39b)
\end{aligned}$$

where we choose the integration constants such that $\varphi_3(x, \sigma)$ and $v_3(x, \sigma)$ vanish for $\sigma = -\infty$ and $+\infty$ with (25), noting that $\zeta(x, y') \zeta(y', x)$ vanishes for $\sigma = -\infty$ and $+\infty$. Our choice of $H_3(x'/\sigma)$ as (38) has reduced the numbers of independent terms from five to four in (39a) and (39b).

(iv) *Fourth order*

We have fourteen and sixteen graphically independent groups in the fourth order equations of (12). In the equation for $\varphi_4(x, \sigma)$ some groups consist of two members in which the orders of operators are different, and as the result of contraction we have new terms, the last ones in (40) and (41a) below. Such a term does not appear in $v_4(x, \sigma)$ and in the lower orders. The results are, writing simply $D(x-2)$, $\mathcal{A}(1-3')$, $D(2''-2''')$, ..., as $(x-2)$, $(1-3')$, $(2''-2''')$, ..., respectively, $\zeta(x, y')$ as (x, y') and $\zeta(x, y') \zeta(y', x)$ as $((x, y'))$,

$$\begin{aligned}
H_4(x'/\sigma) = & \frac{1}{2} \int (3-1') (3'-1'') (3''-1''') [1 2 2' 2'' 2''' 3'''] (3'', 1''') \left\{ (3, 1') \right. \\
& \times (3', 1'') \delta - \frac{1}{4} (3, 1') (1''', 3'') \delta - \frac{1}{4} (1', 3) (1''', 3'') \delta \left. \right\} \\
& + \frac{1}{2} \int (2-2') (1'-3'') (2''-2''') [1 1'' 1''' 3''' 3' 3] (2'', 2''') \left\{ (2, 2') (1', 3'') \delta \right. \\
& - \frac{1}{4} (2, 2') (2''', 2'') \delta - \frac{1}{4} (2', 2) (2''', 2'') \delta \left. \right\} \\
& + \frac{1}{2} \int (3-1') (2'-2'') (3'-1''') [1 1'' 2 2''' 3'' 3'''] (3', 1''') \\
& \times \{ (3, 1') (2', 2'') - ((2', 2'')) \} \delta \\
& + \frac{1}{2} \int (3-1''') (2-2') (3'-1'') [1 1' 2'' 2''' 3'' 3'''] (1'', 3') \left\{ (2', 2) (3, 1''') \delta \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} (3, 1''') (3', 1'') \delta + \frac{1}{4} (1''', 3) (3', 1'') \delta'' \} \\
& + \frac{1}{2} \int (3-1') (3'-1'') (2-2''') [1 \, 1''' \, 2' \, 2'' \, 3'' \, 3'''] \left\{ (1', 3) (1'', 3') (2, 2''') \delta'' \right. \\
& - (2''', 2) (3, 1') (3', 1'') \delta''' + \frac{1}{2} (2, 2''') ((3', 1'')) \delta + \frac{1}{2} (2''', 2) ((3', 1'')) \delta''' \} \\
& + \frac{1}{2} \int (2-2') (1'-3'') (3-1''') [1 \, 1'' \, 2'' \, 2''' \, 3' \, 3'''] (3'', 1') \left\{ (2', 2) (3, 1''') \delta'' \right. \\
& \quad \left. + \frac{1}{4} (3, 1''') (1', 3'') \delta + \frac{1}{4} (1''', 3) (1', 3'') \delta''' \right\} \quad (40) \\
& + \frac{i}{4} \int (3-1') (2'-2'') (1''-3''') (2'''-2) [1 \, 1''' \, 3'' \, 3'] \left\{ \frac{1}{2} ((2', 2'')) (2, 2''') \right. \\
& \quad \left. - (2'', 2') (3''', 1'') (2, 2''') - (2', 2'') (3, 1') (1'', 3''') \right\} \delta
\end{aligned}$$

+ Herm. conj.,

$$\begin{aligned}
\varphi_4(x, \sigma) = & -\frac{1}{2} \int (x-1) (3-1') (3'-1'') (3''-1''') \{ 2 \, 2' \, 2'' \, 2''' \, 3'''' \} (1', 3) ((3'', 1''')) \\
& \times \left\{ (1'', 3') - \frac{1}{4} (3, 1') \right\} - \frac{1}{2} \int (x-1) (2-2') (1'-3'') (2''-2''') \{ 1'' \, 1''' \, 3''' \, 3' \, 3 \} \\
& (2', 2) ((2'', 2''')) \left\{ (3'', 1') - \frac{1}{4} (2, 2') \right\} - \frac{1}{2} \int (x-1) (2-2') (1'-3'') (3'-1''') \\
& \times \{ 1'' \, 2'' \, 2''' \, 3''' \, 3 \} ((2, 2')) (1', 3'') (3', 1''') - \dots \quad (41a) \\
& + \frac{i}{2} \int (x-1) (3-1') (2'-2'') (1''-3''') (2'''-2) \{ 1''' \, 3''' \, 3' \} \\
& \times \left\{ (2, 2''') (1', 3) ((2', 2'')) - (1'', 3''') (1', 3) ((2', 2'')) - \frac{1}{4} ((2, 2''')) ((2', 2'')) \right\} \\
& + \frac{1}{2} (3, 1') (2', 2'') (1'', 3''') (2''', 2) - \frac{1}{2} (2, 2''') (3''', 1'') (2'', 2') (1', 3) + C,
\end{aligned}$$

$$\begin{aligned}
v_4(x, \sigma) = & -\frac{1}{2} \int (x-2) (3-1') (3'-1'') (3''-1''') \{ 1 \, 2' \, 2'' \, 2''' \, 3'''' \} (1', 3) ((3'', 1''')) \\
& \times \left\{ (1'', 3') - \frac{1}{4} (3, 1') \right\} - \frac{1}{2} \int (x-2) (3-1') (2'-2'') (3'-1''') \{ 1 \, 1'' \, 2''' \, 3'' \, 3'''' \} \\
& \times (1', 3) (3', 1''') ((2', 2'')) - \dots \quad (41b)
\end{aligned}$$

+ Herm. conj.,

where (41a) except the last term and (41b) vanish for $\sigma = -\infty$ and $+\infty$, but the last

term in (41a) can not be made to vanish for both $\sigma = -\infty$ and $+\infty$ by choosing the integration constant C appropriately. If we choose C such that $\varphi_4(x, \sigma)$ vanishes for $\sigma = -\infty$, we have

$$\begin{aligned} \varphi_4(x, \sigma = +\infty) = & \frac{i}{2} \int \Delta(x-1) \Delta(3-1') D(2'-2'') \Delta(1''-3''') D(2'''-2) \{1'''3''3'\} \\ & \times \{\theta(2-2''') \theta(3'''-1'') \theta(2''-2') \theta(1'-3) \\ & - \theta(3-1') \theta(2'-2'') \theta(1''-3''') \theta(2'''-2)\}, \end{aligned} \quad (42)$$

where

$$\theta(x) = \frac{\varepsilon(x) + 1}{2}. \quad (43)$$

In the local limit

$$\Phi(1, 2, 3) \rightarrow \delta(1-2) \delta(2-3) \quad (44)$$

(42) vanishes, but in general it will not. From the examination of (42) in the form of Fourier integrals in momentum space it seems unlikely that there exist form factors other than (44) which are analytic and make (42) vanish. Bloch⁽⁹⁾ has proved on a basis of the equations containing no additive terms

$$\left. \begin{aligned} \phi(x) &= \phi(x, -\infty) - g \int \frac{\varepsilon(x-1) + 1}{2} \Delta(x-1) u(2) \phi(3), \\ u(x) &= u(x, -\infty) - g \int \phi^*(1) \frac{\varepsilon(x-2) + 1}{2} D(x-2) \phi(3), \end{aligned} \right\} \quad (45)$$

$$\left. \begin{aligned} \phi(x) &= \phi(x, +\infty) - g \int \frac{\varepsilon(x-1) - 1}{2} \Delta(x-1) u(2) \phi(3), \\ u(x) &= u(x, +\infty) - g \int \phi^*(1) \frac{\varepsilon(x-2) - 1}{2} D(x-2) \phi(3), \end{aligned} \right\} \quad (46)$$

that if the in-fields $\phi(x, -\infty)$ and $u(x, -\infty)$ satisfy the commutation relations (7) the out-fields $\phi(x, +\infty)$ and $u(x, +\infty)$ also satisfy the same relations. His proof seems to be doubtful. Indeed, if we calculate directly, for instance, $[\phi(x, +\infty), \phi(x', +\infty)]$ from (45) and (46) with the commutation relations of the in-fields, a term similar to (42) remains in the fourth order

$$\begin{aligned} [\phi(x, +\infty), \phi(x', +\infty)] = & g^4 \int \Delta(x-1) \Delta(x'-1''') \Delta(3-1') D(2'-2'') \\ & \times \Delta(1''-3''') D(2'''-2) [3'3''] \{\theta(2-2''') \theta(3'''-1'') \theta(2''-2') \theta(1'-3) \\ & - \theta(3-1') \theta(2'-2'') \theta(1''-3''') \theta(2'''-2)\}. \end{aligned} \quad (47)$$

That is to say, in-fields and out-fields defined by (45) and (46) will not be connected by a unitary S -matrix. If we retain, however, the term (42), (47) is canceled by it.

It will be apparent from the results in this section that $H_{n+1}(x, \sigma)$, $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ depend on σ explicitly through the products of n ξ -functions defined by (37).

In the local limit (44), $H_1(x/\sigma)$ agrees with the usual one and $H_n(x/\sigma)$ ($n=2, 3, \dots$) together with all additive terms $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ vanish for space-like σ . This can be seen easily from that $\varphi_1(x, \sigma)$, $v_1(x, \sigma)$, $\psi_1(x/\sigma)$, $u_1(x/\sigma)$ and terms of a form

$$\Delta(1-1') \frac{\epsilon(1-1') - \epsilon(\sigma, 1')}{2} \frac{\epsilon(1'-1) - \epsilon(\sigma, 1)}{2}$$

vanish. On the other hand, if σ contains some portions which are time-like, $H_n(x/\sigma)$, $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ ($n=2, 3, \dots$) all survive and guarantee the integrability condition which is not satisfied by $H_1(x/\sigma)$ alone.

§ 4. Hamiltonian in the Schrödinger representation

If we specify a surface σ_0 , which will have to be space-like in order that field operators may be defined initially on it, and divide the space-time region between σ_0 and any σ by a number of surfaces denoted by $\sigma_0, \sigma_1, \sigma_2, \dots, \sigma_{n-1}, \sigma_n = \sigma$, we have for the unitary operator defined by (8) and (11)

$$U(\sigma, \sigma_0) = U(\sigma_1, \sigma_0) U(\sigma_2, \sigma_1) \cdots U(\sigma, \sigma_{n-1}). \quad (48)$$

Then, noting that

$$U(\sigma, \sigma_0) \phi(x, \sigma) = \phi(x, \sigma_0) U(\sigma, \sigma_0), \quad U(\sigma, \sigma_0) u(x, \sigma) = u(x, \sigma_0) U(\sigma, \sigma_0),$$

we have

$$i \frac{\delta U(\sigma, \sigma_0)}{\delta \sigma(x)} = U(\sigma, \sigma_0) H(x/\sigma) = H_i(x/\sigma, \sigma_0) U(\sigma, \sigma_0), \quad (49)$$

where $H_i(x/\sigma, \sigma_0)$ is the one obtained from $H(x/\sigma)$ by replacing the operators $\psi^*(1, \sigma)$, $u(2, \sigma), \dots$ contained in it by $\psi^*(1, \sigma_0)$, $u(2, \sigma_0), \dots$, σ explicitly contained being unchanged. If we put, with any state vector Ψ_0 in the Hilbert space,

$$\Psi(\sigma) = U(\sigma, \sigma_0) \Psi_0, \quad (50)$$

we have a Tomonaga-Schwinger equation

$$i \frac{\delta \Psi(\sigma)}{\delta \sigma(x)} = H_i(x/\sigma, \sigma_0) \Psi(\sigma). \quad (51)$$

If the integrability condition (15) for the equations (12), which is expressed in terms of $H(x/\sigma)$ as³⁾

$$\frac{1}{i} [H(x/\sigma), H(x'/\sigma)] + \frac{\partial H(x/\sigma)}{\partial \sigma(x')} - \frac{\partial H(x'/\sigma)}{\partial \sigma(x)} = 0, \quad (52)$$

holds, the condition for (51) is also satisfied for σ which is not always space-like since this condition is nothing other than the unitary transform of (52) with $U(\sigma, \sigma_0)$.

The transformation to the Schrödinger representation will be performed most easily by making all σ flat. Writing $\sigma = t$ and $\sigma_0 = t_0$ we have from (51)

choice of t_0 if this is finite. (59) shows that derivative couplings appear in the first order. In the local limit, however, (60) all vanish except $F_1(x_1, x_2, x_3)$ which becomes

$$F_1(x_1, x_2, x_3) \rightarrow \delta(x_1 - x_2) \delta(x_2 - x_3). \quad (61)$$

We shall examine the degrees of singularity in the form factors (60). For illustration we shall choose $F_1(x_1, x_2, x_3)$ which has the highest singularity. Using the form factor given by Kristensen-Møller

$$\Phi(x_1, x_2, x_3) = (2\pi)^{-3} \int G(l_1, l_3) e^{i\{l_1 x_1 + l_3 x_3 - (l_1 + l_3)x_2\}} dl_1 dl_3, \quad (62)$$

where $G(l_1, l_3)$ is a function of Π^2 only

$$\Pi^2 = \left(\frac{l_1 - l_3}{2} \right)^2 - \frac{[(l_1 + l_3)(l_1 - l_3)]^2}{4(l_1 + l_3)^2}, \quad (63)$$

and expressing Δ -functions in Fourier integrals, we have in the case $a=0$ and $b=1$

$$\begin{aligned} F_1(x_1, x_2, x_3) = (2\pi)^{-6} \int 1/2 \cdot \{ G(2\Pi^2 = \sqrt{l_1^2 + M^2} \sqrt{l_3^2 + M^2} - M^2 - l_1 \cdot l_3) \\ + G(2\Pi^2 = -\sqrt{l_1^2 + M^2} \sqrt{l_3^2 + M^2} - M^2 - l_1 \cdot l_3) \} e^{i\{l_1(x_1 - x_2) + l_3(x_3 - x_2)\}} dl_1 dl_3. \end{aligned} \quad (64)$$

We can see by estimating the integral with respect to relative angle between l_1 and l_3 that (64) is less singular than (61). Especially, a value of (64) for $x_1 = x_2 = x_3 = 0$ turns out to be P^4/λ^2 as compared with P^6 of (61), where P is the upper limit of $|l_1|$ and $|l_3|$, and $1/\lambda^2$ is the cut-off value of Π^2 . These differences will be made more distinct if we consider space integrals of $F_1(x_1, x_2, x_3)$ over one of variables x_1 , x_2 or x_3 . We have

$$\left. \begin{aligned} \int F(x_1, x_2, x_3) dx_1 &= (2\pi)^{-3} \int 1/2 \cdot \{ G(2\Pi^2 = M\sqrt{l^2 + M^2} + \overline{M^2} - M^2) \\ &\quad + G(2\Pi^2 = -M\sqrt{l^2 + M^2} + \overline{M^2} - M^2) \} e^{i\mathbf{l} \cdot (\mathbf{x}_3 - \mathbf{x}_2)} d\mathbf{l}, \\ \int F(x_1, x_2, x_3) dx_2 &= (2\pi)^{-3} \int 1/2 \cdot \{ G(\Pi^2 = l^2) \\ &\quad + G(\Pi^2 = -M^2) \} e^{i\mathbf{l} \cdot (\mathbf{x}_1 - \mathbf{x}_3)} d\mathbf{l}. \end{aligned} \right\} \quad (65)$$

Then, if $G(\Pi^2)$ vanishes for $|\Pi^2| \geq M^2$, (65) are finite for any values of $x_3 - x_2$ and $x_1 - x_3$. In this way, we shall have a possibility to treat the problems involving bound states without divergence difficulties according to the Tamm-Dancoff's theory⁷⁾.

In conclusion, the author would like to express his appreciations to Professor H. Yukawa and Dr. Y. Katayama for many valuable discussions.

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Note added in proof: We can show as follows that, starting from the field equations (4), it is always possible to find the interaction Hamiltonian $H(x/\sigma)$ which satisfies (12). For a specified space-like surface σ^* , we have directly from (4)

$$\psi(x) = \psi(x, \sigma^*) - g \int \{ \varepsilon(x-1) - \varepsilon(\sigma^*, 1) \} / 2 \cdot \Delta(x-1) u(2) \psi(3) d(123), \quad (\text{N} \cdot 1)$$

$$\psi(x, \sigma^*) \equiv \int d\sigma_\mu' \{ \Delta(x-x') \partial_\mu' \psi(x') - \psi(x') \partial_\mu' \Delta(x-x') \}_{x'/\sigma^*}. \quad (\text{N} \cdot 2)$$

We assume that, for given operators $\psi(x, \sigma^*)$ and $u(x, \sigma^*)$ which satisfy (6) and (7), equation (N·1) and the corresponding one for $u(x)$ have a unique solution, that is, the form factor belongs to the Pauli's "normal class"²⁹⁾. If not unique, however, it will be possible to define by iteration an asymptotic solution which agrees with that of the free fields in the limit $g \rightarrow 0$. With these known operators $\psi(x)$ and $u(x)$, we write for general surface σ the equations (5) as

$$\psi(x, \sigma) = \psi_0(x, \sigma) - \varphi(x, \sigma), \quad u(x, \sigma) = u_0(x, \sigma) - v(x, \sigma), \quad (\text{N} \cdot 3)$$

$$\psi_0(x, \sigma) \equiv \psi(x) + g \int \{ \varepsilon(x-1) - \varepsilon(\sigma, 1) \} / 2 \cdot \Delta(x-1) u(2) \psi(3) d(123), \quad u_0(x, \sigma) \equiv \dots, \quad (\text{N} \cdot 4)$$

where additive terms $\varphi(x, \sigma)$ and $v(x, \sigma)$ should vanish for $\sigma = \sigma^*$. The known operators $\psi_0(x, \sigma)$ and $u_0(x, \sigma)$ defined by (N·4) are not in general connected by unitary transformations for fixed x and varying σ . By subtracting from them suitable operators $\varphi(x, \sigma)$ and $v(x, \sigma)$ which are functionals of known Heisenberg operators and dependent on σ *explicitly*, we can obtain operators $\psi(x, \sigma)$ and $u(x, \sigma)$ which satisfy (8). The way of subtraction will not be unique, but $\psi(x, \sigma)$ and another solution $\psi'(x, \sigma)$ are both connected by unitary transformations with $\psi(x, \sigma^*)$, and they are also connected with each other by some unitary transformation as (18). Thus, the existence of $\varphi(x, \sigma)$ and $v(x, \sigma)$ as one-valued functions of σ assures the possibility of finding $H(x/\sigma)$ which satisfies (12) and the integrability condition.

The results in § 3 are given for $\sigma^* = -\infty$. In this case we can define energy-momentum four-vectors $P_\mu(\sigma)$, which are constant and form the infinitesimal generators of the coordinate transformation for *finite* σ ,

$$P_\mu(\sigma) = U^{-1}(\sigma, \sigma_0) \{ P_\mu^{(0)}(\sigma_0) - \int_\sigma H_i(x'/\sigma, \sigma_0) d\sigma_\mu' \} U(\sigma, \sigma_0), \quad (\text{N} \cdot 5)$$

where $P_\mu^{(0)}(\sigma_0)$ are free energy-momentum operators written in $\psi(x, \sigma_0)$ and $u(x, \sigma_0)$, and $H_i(x/\sigma, \sigma_0)$ is defined by (49). It is shown from the results given in § 3 that, if the form factor contains a suitable damping factor such as $\exp[-\varepsilon|\ell_1 + \ell_2 + \ell_3|]$, the following equations hold except terms of order ε

$$\partial P_\mu(\sigma) / \partial \sigma(x) = 0, \quad \partial \psi(x) / \partial x_\mu = i[\psi(x), P_\mu], \quad \partial u(x) / \partial x_\mu = i[u(x), P_\mu]. \quad (\text{N} \cdot 6)$$

Many-body Problem in Quantum Field Theory

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The Feynman-Dyson theory is extended so as to include the scattering involving composite particles.

For this purpose we have clarified the relation of the Salpeter-Bethe wave functions to the probability amplitude by introducing co- and contra-variant components of state vectors keeping a close correspondence to the vector analysis in an oblique coordinate system. (§ 1).

The integral equations for various kinds of Feynman kernels are derived in a systematic way by making use of functional differentiations with respect to external source, and these equations are shown to have formal solutions expressed in terms of Fermion kernels and various interaction operators. (§ 2)

The procedure introduced by Gell-Mann and Low is fully utilized to derive integral equations for the covariant components with devices to suitably take account of the initial conditions. (§ 3)

Finally the method to construct the S -matrix is discussed. (§ 4) Not discussed are the problems of renormalization and of metastable states.

Introduction and summary

In recent years, the so-called Feynman-Dyson theory has widely been applied to scattering problems for open systems with great success, yet the theory in its original form can not immediately be adapted to such scattering problems that involve composite particles, since the hypothesis of adiabatic switching plays an essential rôle in the theory. For the scattering of particles by short range potentials, we can regard these particles at large distances as free from forces and the hypothesis has its validity. To our regret, however, it is not the case for the interaction between particles and fields, because the interaction of a particle with vacuum cannot be discarded even when the particles are separated far away from each other, hence the particle can no more appear to be free, but we must retain the interaction of the particle with vacuum by treating it as a dressed particle. This situation is especially worth noticing in problems involving composite particles.

In addition to such a basic requirement, we are motivated sometimes by the necessity of the relativistic theory of nuclear forces, in which the current means to deal with such problems in reference to the static potentials, a purely non-relativistic concept, is suspected to lose its applicability for high energy problems and for the $P_S(p_S)$ meson theory that requires a relativistic approach even at low energies.

For this purpose, we must start from a positron theoretic point of view. Indeed, Nambu¹⁾ proposed an equation to treat the two-body problem on the above basis

$$(\gamma\partial + \kappa)_1 (\gamma\partial + \kappa)_2 g(1, 2) = -e^2 (\gamma_\mu)_1 (\gamma_\mu)_2 \bar{D}_F(1-2) g(1, 2). \quad (1)$$

In this equation only the second order effect is taken into account, and later on this

equation was further extended by Salpeter and Bethe²⁾ and by Kita³⁾ upon the Feynman picture as to include the higher order effects

$$g(12) = g^0(12) + \int d\omega_3 d\omega_4 d\omega_5 d\omega_6 S'_F(1, 3) S'_F(2, 4) G(34; 56) g(56), \quad (2)$$

where g^0 and g represent the incident and scattered waves respectively. This equation enables us to treat the problems of bound states by dropping the incident wave g^0 as well as scattering problems.

The equation common to the above two cases is given in the form of an integro-differential equation by generalizing the equation (1).

An attempt to derive these equations from the conventional field theory was furnished by Gell-Mann and Low⁴⁾.

The essential point of their theory is the relation between Heisenberg operators and Feynman kernels

$$K(12; 34) = - (Vac | T[\phi(1)\phi(2)\bar{\phi}(3)\bar{\phi}(4)] | Vac). \quad (3)$$

Thick letters denote Heisenberg operators throughout this paper, and $Vac.$ is the abbreviation of the true vacuum.

The sole connection between the state vector Ψ_s in the Heisenberg representation and the Salpeter-Bethe wave function $g_s(12)$ given in their paper is

$$g_s(12) = (\Psi_s, T[\phi(1)\phi(2)] \Psi_s), \quad (\Psi_0 = Vac), \quad (4)$$

where Ψ_s is an eigenstate of the total energy-momentum operator P_μ , *i.e.*

$$(P_\mu - p_\mu^s) \Psi_s = 0, \quad (5)$$

p_μ^s being the eigenvalue.

It must be emphasized that only energy levels can be determined by these equations but that we need further investigations in obtaining such information as the expectation values of operators in bound states and the S matrix for the scattering involving bound states.

The purpose of the present paper is to clarify these points. In order to investigate the connection between Salpeter-Bethe wave functions and the state vectors, it is very instructive to note the close correspondence of the present problem to the vector analysis in an oblique coordinate system. The relation (4) suggests us to provide the Salpeter-Bethe wave functions with the name of "covariant components" of the state vector. In a similar way, "contravariant components" are also defined as the coefficients of the expansion of a state vector in terms of Heisenberg operators operated onto the true vacuum.

As a characteristic feature of the positron theory, both components do not coincide with each other in the presence of the interaction between fields. Only after the settlement of these components, expectation values of physical quantities are calculable as will be shown in § 1. (Also see Appendix.)

The temporal development of a system is then described by integral equations for Feynman kernels or covariant components which are studied in a systematic way with the

aid of functional differentiations with respect to external sources and of the limiting procedure of Gell-Mann and Low. (§ 2, § 3)

The method to construct the S matrix for scattering processes involving bound states is presented. It is stressed, for this purpose, that we must prepare a variety of asymptotic forms corresponding to the boundary conditions imposed upon the final states. (§ 4)

§ 1. Representation of state vectors

Our first task is to investigate the precise connection between the state vectors in the Heisenberg picture and the Salpeter-Bethe wave functions. The state vector Ψ_s , for instance, for one nucleon system, will be expressed as

$$\Psi_s = [\int f_s(x) \phi^*(x) d^3x + \dots] \Psi_0, \quad (1.1)$$

where x 's refer to four dimensional coordinates, while the integrations are extended over a three dimensional space.

In order to represent the state completely, we need in the integrand of (1.1) various combinations of operators such as

$$\phi^*(x), \phi^*(x)\phi(\xi), \phi^*(x)\phi(\xi), \dots,$$

corresponding to the meson cloud. Here ψ and ϕ mean the wave functions of nucleon and meson.

Hence we begin with examining what combinations of operators will be necessary and sufficient.

(a) Closed Set of Operators

Let the linear space expanded by the linear combinations of a set of operators O_1, O_2, \dots be \mathfrak{R} . If for a given Hamiltonian $\bar{H} = \int H(x) d^3x$ and for an arbitrary operator $O \in \mathfrak{R}$, results always

$$[\bar{H}, O] \in \mathfrak{R},$$

we call the set of operators $\{O_1, O_2, \dots\}$ as *closed*.

Theorem 1. In order to represent a stationary state Ψ_s completely by operating a set of operators to Ψ_0 as in (1.1), it is necessary and sufficient that the set is closed.

(Proof) Let Ψ_s be expressed as

$$\Psi_s = \sum_n a_s(n) O_n \Psi_0,$$

then the equation

$$(\bar{H} - E_s) \Psi_s = 0$$

may be read as

$$\begin{aligned} \bar{H} \Psi_s &= \sum_n \bar{H} a_s(n) O_n \Psi_0 \\ &= \sum_n a_s(n) ([\bar{H} O_n] + E_0 O_n) \Psi_0 \quad (\text{Note } (\bar{H} - E_0) \Psi_0 = 0) \\ &= E_s \sum_n a_s(n) O_n \Psi_0. \end{aligned}$$

Hence, if the set is closed, this equation can be transformed into a compact form of equations for a 's (sufficient), otherwise these equations for a 's are not closed (necessary).

This condition is equivalent to the invariance of the space \mathfrak{H} under time development.

From the above theorem, we see that if the motion of fields is described in terms of canonical variables q 's and p 's the set of operators of the form

$$q \dots q \ p \dots p \quad (1.2)$$

is closed, provided that H is a polynomial of these variables. The field variables corresponding to q and p are ψ and ψ^* for Fermions and ϕ and $\dot{\phi}$ for Bosons respectively, and we can readily construct a closed set of operators as (1.2).

For instance, for a one nucleon system, the state vector is expressed as

$$\begin{aligned} \Psi_s = & \left[\int f_s(x) \psi^*(x) d^3x + \int f_s(x, \xi) \psi^*(x) \dot{\phi}(\xi) d^3x d^3\xi \right. \\ & \left. + \int f'_s(x, \xi) \psi^*(x) \dot{\phi}(\xi) d^3x d^3\xi + \dots \right] T_0. \end{aligned} \quad (1.3)$$

In such relations, all arguments refer to the same time.

Rigorously speaking, however, we further need to fix the order of operators.

(b) Ordering Operators

For the ordering convention, the T -product introduced by Wick⁵⁾ is convenient for the present purpose. In connection with the T -product he has also introduced the S -product or the normal product, and we will attempt to apply the concept of the S -product to Heisenberg operators.

He defined the S -product in the interaction representation

$$:ABC\dots Z:$$

by dropping the contracted parts from the well ordered products of operators (i.e. creation operators to the left, destruction to the right) between colons. Hence we can reduce the S -product to

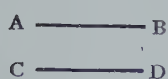
$$\begin{aligned} :ABC\dots Z: = & T(ABC\dots Z) - \sum A \cdot B \cdot T(C\dots Z) \\ & + \sum A \cdot B \cdot C \cdot D \cdot T(E\dots Z) - \dots, \end{aligned} \quad (1.4)$$

where thin letters refer to the interaction representation and the dot superscripts for a pair of operators are the contraction symbols defined by

$$A \cdot B = (\text{vac} | T(AB) | \text{vac}). \quad (\text{vac.} = \text{free vacuum}) \quad (1.5)$$

In the expression (1.4) all contracted terms are properly subtracted, since a term with n times contractions (Cf. Fig. 1) are subtracted in $\binom{n}{1}$ ways in the second term, added in $\binom{n}{2}$ ways in the third and so on, it is subtracted by only once as a whole as seen from

$$-\binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots = -1 + \sum_{s=0}^n (-1)^s \binom{n}{s} = -1. \quad (1.6)$$



n-line

Fig. 1.

The T -product for Heisenberg operators

$$T(ABC...Z) \quad (1.7)$$

is well defined. We shall, however, use the T -product in the sense of the T^* -product⁽⁶⁾ for later convenience. The direct decomposition of Heisenberg operators into creation and destruction operators is not

possible since they obey non-linear field equations contrary to the case of the interaction representation.

Since we cannot define the S -product for Heisenberg operators as Wick has done in the interaction representation, we start from the alternative definition (1.4), because the T -product and the contraction procedure may retain in the Heisenberg representation.

The S -product for Heisenberg operators

$$:ABC...Z: \quad (1.8)$$

is thus defined by referring to (1.4), and the contraction of Heisenberg operators by

$$A \cdot B = (Vac|T(AB)|Vac). \quad (1.9)$$

The physical meaning of the S -product here is not so clear as was the case in the interaction representation, but we shall find it useful as well as necessary in what follows.

Theorem 2. If the arguments of the operators A, B, \dots, Z are "space-likely" situated from each other, then holds the following identity

$$(:AB...Z:)^+ = :Z^+...B^+A^+;, \quad (+ : \text{Hermitian conjugate}).$$

In the interaction representation, this theorem holds trivially without the condition of space-like situation. We shall simply state the meaning of this theorem. We see

$$T(ABC...Z)^+ = T^{-1}(Z^+...C^+B^+A^+).$$

Even if the arguments are space-likely situated we cannot replace T^{-1} by T in general, since there may be non-commutative quantities involved in the operand of T^{-1} . It must be remembered, however, that their commutators should be c -numbers provided that they are space-likely situated, and therefore the extra terms arising from their non-commutativity become

$$\begin{aligned} T(AB)^+ - T(B^+A^+) &= (Vac|T(AB)^+ - T(B^+A^+)|Vac) \\ &= (A \cdot B)^* - (B^+ \cdot A^+), \end{aligned}$$

which are just cancelled out by contraction terms in (1.4).

(c) Co- and Contra-variant Components

We have seen that the state vector for a one nucleon system can be expressed as in (1.3), and from now on we shall employ the S -product for the combination of operators to express the state vector. Namely a state vector can be expressed by a linear combination of such S -products as

$$\phi^*, : \phi^* \phi :, : \phi^* \dot{\phi} :, : \phi \phi \phi :, \dots$$

operated onto the true vacuum. *

We call the coefficients of these operators

$$f_s(x), f_s(x, \xi), f'_s(x, \xi), \dots \quad (1.10)$$

the *contravariant components* of the state vector Ψ_s .

On the other hand, functions like

$$g_s(x) = (\Psi_0, \phi(x) \Psi_s), \quad g_s(x, \xi) = (\Psi_0, : \phi(x) \dot{\phi}(\xi) : \Psi_s), \dots \quad (1.11)$$

are called the *covariant components*, where operators should be understood as being *S-products*.

These nomenclatures are suggested by the close analogy of the present formalism with the vector analysis, in which the components x^i and x_i , defined by

$$x = x^i e_i, \quad x_i = (e_i, x)$$

are called *contra-* and *co-variant components* respectively, e 's are basic vectors in an oblique coordinate system.

Relations corresponding to

$$x_i = g_{ij} x^j, \quad g_{ij} = (e_i, e_j)$$

are also furnished in the present formalism, and as we shall see later Feynman kernels correspond to the metric tensor g_{ij} .

In addition to the above two kinds of components, we shall introduce another auxiliary kind of components defined by

$$X_s(x) = (\Psi_0, \phi(x) \Psi_s), \quad X_s(x, \xi) = (\Psi_0, T[\phi(x) \dot{\phi}(\xi)] \Psi_s), \dots, \quad (1.12)$$

for which we provide with the name of *S-B components*.

The utility of understanding the *T-product* in the sense of the T^* -product consists in the following identity

$$\left(\Psi_0, T \left[\dots \frac{\partial \phi(x)}{\partial x_\mu} \dots \right] \Psi_s \right) = \frac{\partial}{\partial x_\mu} (\Psi_0, T[\dots \phi(x) \dots] \Psi_s). \quad (1.13)$$

Theorem 3. Covariant and *S-B* components of a state can be expressed in terms of alternative components.

This theorem is an immediate consequence of the definition (1.4) which connects the *S-* and *T-products* with each other.

In what follows we denote such quantities as ϕ , ϕ^* , $\dot{\phi}$ and $\dot{\phi}$ indiscriminately by e , then the co- and contra-variant components g 's and f 's are given respectively by

$$g_{s,N} = (\Psi_0, : e_1 \dots e_n : \Psi_s), \quad N = (1, 2, \dots, n) \\ \Psi_s = \sum_N \int f_{s,N} : e_n^+ \dots e_1^+ : \Psi_0, \quad (+ : \text{Hermitian conjugate}) \quad (1.14)$$

where \int corresponds to the spatial integrations in (1.3).

If we choose $t_1 = \dots = t_n$, we see from the theorem 2

$$\begin{aligned} (\Psi_r, \Psi_s) &= \left(\sum_N \int f_{r,N} : e_n^+ \dots e_1^+ : \Psi_0, \Psi_s \right) \\ &= \sum_N \int f_{r,N}^* (\Psi_0, : e_1 \dots e_n : \Psi_s) \\ &= \sum_N \int f_{r,N}^* g_{s,N} (= \sum_N \int f_{s,N} g_{r,N}^*). \end{aligned} \quad (1.15)$$

Hence the ortho-normalization for state vectors is expressed by

$$\sum_N \int f_{r,N}^* g_{s,N} = \delta(r, s). \quad (1.16)$$

The relation (1.15) precisely corresponds to the formula to give the inner product of two vectors x and y

$$(x, y) = x_i y^i = x^i y_i.$$

The connection of the contravariant components with the covariant ones are given by

$$\begin{aligned} g_{s,N} &= (\Psi_0, : e_1 \dots e_n : \Psi_s) \\ &= (\Psi_0, : e_1 \dots e_n : \sum_{N'} \int f_{s,N'} : e_{n'}^+ \dots e_{1'}^+ : \Psi_0) \\ &= \sum_{N'} \int \mathfrak{R}(N|N') f_{s,N'}, \end{aligned} \quad (1.17)$$

where

$$\mathfrak{R}(N|N') = (\Psi_0, : e_1 \dots e_n : : e_{n'}^+ \dots e_{1'}^+ : \Psi_0). \quad (1.18)$$

If we choose $t_1 = \dots = t_n > t_1' = \dots = t_{n'}'$, \mathfrak{R} may be written as

$$\mathfrak{R}(N|N') = (\Psi_0, T[: e_1 \dots e_n : : e_{n'}^+ \dots e_{1'}^+ :] \Psi_0). \quad (1.18')$$

Since this is a mixed T -product, it can be reduced into a linear combination of T -product kernels like

$$K(N, N') = (\Psi_0, T[e_1 \dots e_n e_{n'}^+ \dots e_{1'}^+] \Psi_0) \quad (1.19)$$

by means of the relation (1.4).

The functions K 's in (1.19) are just the Feynman kernels. Hence, if K 's are known we can in principle determine the contravariant components from the covariant ones with reference to the equation (1.17).

The means to get information on Feynman kernels and covariant components will be discussed in the following sections.

(d) Expectation Values

We shall briefly describe how to evaluate the expectation value of a given physical

quantity Q in a given state \mathcal{V}_s .

The expectation value is written as

$$\langle Q \rangle_s = (\mathcal{V}_s, Q \mathcal{V}_s). \quad (1.20)$$

The state vector \mathcal{V}_s is given by

$$\mathcal{V}_s = \sum_N \int f_{s,N} : e_n^+ \dots e_1^+ : \mathcal{V}_0,$$

and hence

$$Q \mathcal{V}_s = \sum_N \int f_{s,N} Q : e_n^+ \dots e_1^+ : \mathcal{V}_0$$

If we choose the time coordinates in e 's to be earlier than that of Q , the above quantity may be rewritten as

$$\sum_N \int f_{s,N} T[Q : e_n^+ \dots e_1^+ :] \mathcal{V}_0.$$

The mixed T -product in the above expression can be decomposed into a linear combination of S -products, *i.e.*

$$T[Q : e_n^+ \dots e_1^+ :] = \sum_{N'} Q_{N',N} : e_{n'}^+ \dots e_{1'}^+ :.$$

Thus $\langle Q \rangle_s$ in (1.20) turns out to be

$$\begin{aligned} & (\mathcal{V}_s, \sum_{N,N'} \int f_{s,N} Q_{N',N} : e_{n'}^+ \dots e_{1'}^+ : \mathcal{V}_0) \\ &= \sum_{N,N'} \int f_{s,N} Q_{N',N} (\mathcal{V}_s, : e_{n'}^+ \dots e_{1'}^+ : \mathcal{V}_0) \\ &= \sum_{N,N'} \int g_{s,N'}^* Q_{N',N} f_{s,N}. \end{aligned} \quad (1.21)^*$$

Finally we must add an important comment.

The S — B or covariant components are defined, as we shall see in the following sections, for all configurations of arguments including the case when arguments are time-likely placed. Such a situation seems at first sight to be in apparent contradiction to our physical intuition compared to the cases of conventional many-time theories of Dirac⁽⁷⁾ and of Tomonaga⁽⁸⁾ and Schwinger⁽⁹⁾.

This difficulty is interpreted as follows:

The theorem 2, equations (1.15), (1.18') and (1.21) indicate that contravariant components have their physical meaning only when all arguments are space-likely situated, and in the calculations of physical quantities we need contravariant components as well as covariant ones. Indeed, to these expressions contribute only the values of covariant components for space-like region in conformity with our physical intuition.

* There is another simpler means to express the expectation values of physical quantities in terms of covariant components alone. See Appendix.

§ 2. Integral equations for the kernels

We have seen in the previous section that the Feynman kernels play the most fundamental rôle in the present theory. Furthermore, the dynamics of the system of interacting fields are governed by the integral equations for the Feynman kernels, so that in this section our efforts are devoted to the investigation of these equations.

After Dyson¹⁰⁾, the integral equation for the one-body kernel is given by

$$S'_F = S_F + S_F \Sigma^* S'_F = S_F + S'_F \Sigma^* S_F, \quad (2.1)$$

where Σ^* is the proper self energy part for a nucleon, and functions such as S_F and S'_F are defined in the absence of external fields by

$$\langle T[\psi(1)\bar{\psi}(2)] \rangle_{vac} = S_F(1-2), \quad \langle T[\phi(1)\bar{\phi}(2)] \rangle_{vac} = S'_F(1-2),$$

similarly

$$\langle T[\phi_a(1)\phi_b(2)] \rangle_{vac} = \delta_{ab} \Delta_F(1-2), \quad \langle T[\phi_a(1)\phi_b(2)] \rangle_{vac} = \delta_{ab} \Delta'_F(1-2).$$

The function Δ'_F satisfies an integral equation similar to (2.1)

$$\Delta'_F = \Delta_F + \Delta_F \Pi^* \Delta'_F = \Delta_F + \Delta'_F \Pi^* \Delta_F, \quad (2.1')$$

where Π^* is the proper self energy part for a meson.

The two-body kernel satisfies as discussed by several authors²⁾³⁾¹⁾ an integral equation

$$K(12; 34) = S'_F(12; 34) + \int_{-\infty}^{\infty} d\omega_5 d\omega_6 d\omega_7 d\omega_8 S'_F(1, 5) S'_F(2, 6) \\ \times G(56; 78) K(78; 34) \quad (2.2)$$

$$= S_F(12; 34) + \int_{-\infty}^{\infty} d\omega_5 d\omega_6 d\omega_7 d\omega_8 K(12; 56) \\ \times G(56; 78) S'_F(7, 3) S'_F(8, 4), \quad (2.2')$$

where

$$S'_F(12; 34) = S'_F(1, 3) S'_F(2, 4) - S'_F(1, 4) S'_F(2, 3).$$

In order to solve these equations we need to know the boundary condition.

For instance, for a two-body kernel

$$K(12; 34) = \Re(12|34) = - (Vac|T[\phi(1)\phi(2)\bar{\psi}(3)\bar{\psi}(4)]|Vac),$$

we may decompose it as

$$= - \sum_s (Vac|T[\phi(1)\phi(2)]|\Psi_s)(\Psi_s|T[\bar{\psi}(3)\bar{\psi}(4)]|Vac), \quad (2.3)$$

if $t_1=t_2=t > t_3=t_4=t'$. Here $T[\phi(1)\phi(2)]$ must operate onto Ψ_s as a destruction operator and $T[\bar{\psi}(3)\bar{\psi}(4)]$ as a creation operator, and hence the time dependence of the term in the summand should be like $\exp[-iE_s(t-t')]$ for $t > t'$ and similarly like $\exp[-iE_s(t'-t)]$ for $t' > t$. This is precisely the Feynman's positive frequency condition¹¹⁾ being equivalent to the outgoing wave condition. In this example, we divided

the operators tentatively between ϕ 's and ψ 's by the time label $t_1=t_2>t_3=t_4$, but it is obvious that positive frequency condition should hold for an *arbitrary* division of operators between two groups.

The iteration method to solve these integral equations will fail to satisfy the positive frequency condition when there are isobar states or bound states, since the solutions of the integral equations without the inhomogeneous terms corresponding to such states cannot be included by such a method.

In order to discuss the problem in more detail we choose an interaction Hamiltonian

$$H_{int} = i\eta : \bar{\psi} O_a \psi : \phi_a + Q_a \phi_a - H_{self}, \quad (2.4)$$

where H_{self} is the counter term to cancel divergences like self energies about which, however, we shall not discuss in this paper. The operators in (2.4) are to be understood as

$$\eta = \begin{cases} -e \\ f \end{cases}, \quad O_a = \begin{cases} \gamma_\mu \\ \gamma_5 \tau_a \end{cases}, \quad \phi_a = \begin{cases} A_\mu, & \text{Q.E.D.,} \\ \phi_a, & \text{symmetrical } Ps(ps) \end{cases} \quad (2.5)$$

in quantum electrodynamics and in symmetrical $Ps(ps)$ meson theory respectively. The c -number quantity Q_a represents the external source introduced formally for the later convenience.

In charged meson theories the existence of such a c -number source violates the charge conservation, so that we must regard Q_a to vanish in equations derived after the differentiation with respect to Q_a . Namely we must understand Q_a as a purely mathematical tool rather than a physical quantity.

From the Hamiltonian (2.5), we define the tranformation function U by

$$i \frac{\partial}{\partial \sigma(x)} U(\sigma, \sigma_A) = H_{int}(x) U(\sigma, \sigma_A) \quad \text{with} \quad U(\sigma_A, \sigma_A) = 1. \quad (2.6)$$

According to Gell-Mann and Low⁽¹⁾, there is an identity which connects Heisenberg operators to operators in the interaction representation

$$\begin{aligned} (Vac | T[A(1)B(2)...Z(n)] | Vac) \\ = \frac{(vac | T[U(\infty, -\infty)A(1)B(2)...Z(n)] | vac)}{(vac | U(\infty, -\infty) | vac)}, \end{aligned} \quad (2.7)$$

provided that there is no external field.

The expression (2.7) will simply be denoted by

$$\langle A(1)B(2)...Z(n) \rangle,$$

and also the following abbreviations are used;

$$U = U(\infty, -\infty), \quad \mathcal{U} = (vac | U(\infty, -\infty) | vac).$$

Then we have

$$i \frac{\partial U}{\partial Q_a(x)} = i \frac{\partial}{\partial Q_a(x)} T[\exp(-i \int H_{int}(x)(dx))] = T[\mathcal{U} \phi_a(x)]. \quad (2.8)$$

In a similar way, it follows

$$i \frac{\partial}{\partial Q_a(0)} (\text{vac} | T[U A(1) B(2) \dots Z(n)] | \text{vac}) \\ = (\text{vac} | T[U \phi_a(0) A(1) B(2) \dots Z(n)] | \text{vac}), \quad (2.9)$$

or

$$i \frac{\partial}{\partial Q_a(0)} [C \langle A(1) B(2) \dots Z(n) \rangle] \\ = C \langle \phi_a(0) A(1) B(2) \dots Z(n) \rangle \quad (2.9')$$

in virtue of the relation (2.7).

In these equations, we must always put $Q_a=0$ after differentiation as has been noticed before. Namely in what follows, we utilize differentiations with respect to the external source Q to derive integral equations connecting various kernels. Since, however, the identity (2.7) holds only when there is no external source, we define kernels by

$$\langle A(1) B(2) \dots Z(n) \rangle \\ = (\text{vac} | T[U A(1) B(2) \dots Z(n)] | \text{vac}) / C \quad (A)$$

rather than by

$$(V_{ac} | T[A(1) B(2) \dots Z(n)] | V_{ac}) \quad (B)$$

when the external source is present, since the true vacuum cannot definitely be defined in this case.

On differentiating equations for a kernel of the type (A) with respect to Q , we arrive at integral equations connecting various kernels of the type (A).

Then we may put (B) equal to (A) by letting Q vanish, giving desired integral equations connecting kernels of the type (B) when the external field is absent.

Then we can derive

$$- \left[\frac{\partial^2 C}{\partial Q_a(1) \partial Q_b(2)} \right]_{Q=0} = \delta_{ab} A'_F(1-2) \cdot C_{Q=0}. \quad (2.10)$$

If we put after Feynman¹¹⁾

$$C = e^{-L}, \quad (2.11)$$

we do not have to be concerned with the contributions from processes involving isolated loops. (Fig. 2)

The quantity L is given in the second order by

$$L_2 = \frac{1}{2} \eta^2 \iint d\omega_1 d\omega_2 S p [O_a S_F(1-2) O_a S_F(2-1)] A_F(1-2). \quad (2.12)$$

If we take account of the existence of a symbolical external field Q , we must supplement the following term;

$$L_2^{ext} = \frac{1}{2} \iint d\omega_1 d\omega_2 Q_a(1) Q_a(2) \mathcal{A}_F(1-2). \quad (2.12')$$

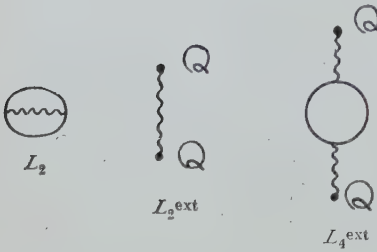


Fig. 2.

From (2.10) and (2.11), there results

$$\left[\frac{\delta^2 L}{\delta Q_a(1) \delta Q_b(2)} \right]_{Q=0} = \delta_{ab} \mathcal{A}'_F(1-2).$$

The usefulness of such an operational differentiation is exhibited by the following examples.

Differentiating the equation

$$CS'_F(1, 2) = \langle T[U\psi(1)\bar{\psi}(2)] \rangle_{vac},$$

we have from (2.9)

$$i \frac{\partial \mathcal{C}}{\partial Q_a(3)} \cdot S'_F(1, 2) + iC \frac{\partial}{\partial Q_a(3)} S'_F(1, 2) = \langle \psi(1) \bar{\psi}(2) \phi_a(3) \rangle.$$

Since we know

$$\begin{aligned} T[\psi(1) \bar{\psi}(2) \phi_a(3)] &= : \psi(1) \bar{\psi}(2) \phi_a(3) : + \langle \psi(1) \bar{\psi}(2) \rangle \phi_a(3) \\ &= : \psi(1) \bar{\psi}(2) \phi_a(3) : + \phi_a(3) \cdot S'_F(1, 2), \end{aligned}$$

and

$$-i \frac{\delta L}{\delta Q_a(3)} = \langle \phi_a(3) \rangle \quad (2.13)$$

we get immediately

$$i \frac{\partial}{\partial Q_a(3)} S'_F(1, 2) = \langle : \psi(1) \bar{\psi}(2) \phi_a(3) : \rangle. \quad (2.14)$$

The terms $\langle \phi_a(3) \rangle S'_F(1, 2)$ and $\langle : \psi(1) \bar{\psi}(2) \phi_a(3) : \rangle$ correspond to diagrams Fig. 3 and Fig. 4 respectively.

The right hand side of (2.14) is nothing but the definition of the vertex part, so that we may write

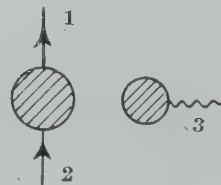


Fig. 3.

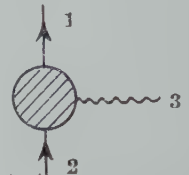


Fig. 4.

$$i \frac{\partial}{\partial Q_a(3)} S'_F(1, 2) = \gamma \int d\omega_4 d\omega_5 d\omega_6 S'_F(1, 4) \mathcal{O}_a(45; 6) S'_F(5, 2) \mathcal{A}'_F(6, 3), \quad (2.15)$$

where

$$\mathcal{O}_a(45; 6) = \mathcal{O}_a \delta(4-5) \delta(5-6) + \dots$$

is the vertex operator.

In the above example, the situation is rather special and is much simplified since there was a ready made quantity, the vertex operator. Hence we shall exhibit a general means to utilize the symbolical differentiation.

The problem to be dealt with is to find the relation between functions

$$\langle \phi(1) \bar{\phi}(2) \rangle = S'_F(1, 2),$$

and

$$\langle \phi(1) \bar{\phi}(2) \phi_a(0) \rangle = K_a(12; 0).$$

In the presence of an external field (though it is illusory), S'_F satisfies the following equation¹²⁾ but not (2.1)

$$\begin{aligned} S'_F(1, 2) = & S_F(1-2) + \eta \int d\omega_3 S_F(1-3) O_a \langle \phi_a(3) \rangle S'_F(3, 2) \\ & + \int d\omega_3 d\omega_4 S_F(1-3) \Sigma^*(3, 4) S'_F(4, 2). \end{aligned} \quad (2.16)$$

From (2.9) and the definitions of S'_F and K_a , it follows

$$CK_a(12; 0) = i \frac{\partial}{\partial Q_a(0)} [CS'_F(1, 2)]. \quad (2.17)$$

Multiplying (2.16) by C and differentiating it with respect to Q , we arrive at an integral equation for K_a

$$\begin{aligned} K_a(12; 0) = & \langle \phi_a(0) \rangle S_F(1-2) + \eta \int d\omega_3 S_F(1-3) O_b [\langle \phi_a(0) \phi_b(3) \rangle S'_F(3, 2) \\ & + \langle \phi_b(3) \rangle K_a(32; 0)] + \int d\omega_3 d\omega_4 S_F(1-3) \\ & \times \left[i \frac{\partial \Sigma^*(3, 4)}{\partial Q_a(0)} \cdot S'_F(4, 2) + \Sigma^*(3, 4) K_a(42; 0) \right]. \end{aligned} \quad (2.18)$$

It is convenient to write the equation (2.16) formally as

$$S'_F(1, 2) = S_F(1-2) + \mathcal{J}(1) S'_F(1, 2), \quad (2.16')$$

where \mathcal{J} is the integral operator in (2.16) and the argument 1 in parentheses refers to the argument to be operated.

With this notation, the equation (2.18) can be written as

$$\begin{aligned} K_a(12; 0) = & \langle \phi_a(0) \rangle S_F(1-2) + \mathcal{J}(1) K_a(12; 0) \\ & + \eta \int d\omega_3 S_F(1-3) O_b \langle \phi_a(0) \phi_b(3) \rangle S'_F(3, 2) \\ & + \int d\omega_3 d\omega_4 S_F(1-3) \left[i \frac{\partial \Sigma^*(3, 4)}{\partial Q_a(0)} \right] S'_F(4, 2). \end{aligned} \quad (2.18')$$

It is to be noticed that (2.16') may be transformed into the form

$$(1 - \mathcal{J}(1))^{-1} S_F(1-2) = S'_F(1, 2). \quad (2.16'')$$

The operator $(1 - \mathcal{J}(1))$ has not always its inverse when there are isobar states, but it is well-defined by the above equation if we can find $S_F(1-...)$ in the operand. Then

we are fortunate to find that the equation (2.18') can formally be solved by operating $(1 - \mathcal{J}(1))^{-1}$ on both sides of (2.18') after bringing the term $\mathcal{J}(1) K_a(12; 0)$ to the left.

$$K_a(12; 0) = \langle \phi_a(0) \rangle S'_F(1, 2) + \eta \int d\omega_3 S'_F(1, 3) O_b S'_F(3, 2) \langle \phi_a(0) \phi_b(3) \rangle \\ + \int d\omega_3 d\omega_4 S'_F(1, 3) \left[i \frac{\delta \Sigma^*(3, 4)}{\delta Q_a(0)} \right] S'_F(4, 2). \quad (2.19)$$

On putting $Q \rightarrow 0$ we reach

$$K_a(12; 0) = \eta \int d\omega_3 S'_F(1-3) O_a S'_F(3-2) A'_F(3-0) \\ + \int d\omega_3 d\omega_4 S'_F(1-3) \left[i \frac{\delta \Sigma^*(3, 4)}{\delta Q_a(0)} \right]_{Q=0} S'_F(4-2), \quad (2.19')$$

since $\langle \phi_a(0) \rangle$ vanishes due to Furry's theorem and $\langle \phi_a(0) \phi_b(3) \rangle$ reduces to $\delta_{ab} A'_F(3-0)$.

The above procedure is quite general and can be continued to obtain Feynman kernels with any number of meson wave functions starting from Fermion kernels. For instance, the kernel for the meson-nucleon scattering can be derived by differentiating (2.19). It must be noticed, however, that the differentiation of (2.19') does not yield a correct result.

The functions

$$(\Sigma^*)_{Q=0}, (\delta \Sigma^* / \delta Q)_{Q=0}, \dots$$

are calculated by graphical inspections.*)

In meson-nucleon scattering, the standard Salpeter-Bethe method deals with a pure two-body problem¹³⁾, while the present method to utilize (1.16'') deals with a one-body problem. This difference is interpreted that the former is essentially concerned with the integral equations for $(\delta^2 \Sigma^* / \delta Q \delta Q)$, on the other hand this quantity is treated as if already known to higher orders in the present method.**)

Comparing (2.19) with (2.13) and (2.14), we see

$$i \frac{\delta \Sigma^*(1, 2)}{\delta Q_a(0)} = \eta \int d\omega_3 O_a(12; 3) A'_F(3, 0) - \eta O_a \delta(1-2) A'_F(1, 0). \quad (2.20)$$

The first and second terms in the right hand side of this equation correspond to diagrams in Fig. 5 and in Fig. 6 respectively. Inserting (2.20) into (2.19), we find

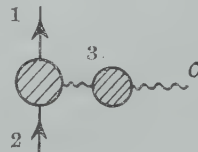


Fig. 5.



Fig. 6.

* The subscript $Q=0$ will be omitted sometimes unless it might lead us to confusions.

** We can derive the equation for $(\delta^2 \Sigma^* / \delta Q \delta Q)$ starting from the closed integral equations for the mass operator, polarization operator, vertex operator and Green functions given in the Schwinger theory of Green functions¹²⁾.

$$K_a(12; 0) = \langle \phi_a(0) \rangle S'_F(1, 2) + \eta \int d\omega_3 d\omega_4 d\omega_5 S'_F(1, 3) \mathcal{O}_a(34; 5) S'_F(4, 2) \mathcal{A}'_F(5, 0), \quad (2.21)$$

being equivalent to (2.14) and (2.15).

The illustration of the present method will be accomplished by applying it to the two-body problem.

We start from the equation (2.2), in which the interaction operator G is given by

$$G(56; 78) = \eta^2 O_a(5) O_a(6) \mathcal{A}_F(5-6) \delta(5-7) \delta(6-8) + \dots \quad (2.22)$$

The arguments 5 and 6 in the parentheses indicate where the matrices O 's are to be inserted. (Fig. 7)

On differentiating (2.2), we must take account of graphs for G in which the symbolical external source Q is involved. (Fig. 8)

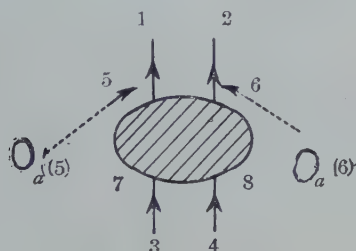
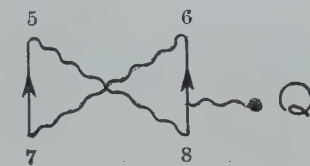


Fig. 7.



An example of the graph for G involving the source Q .

Fig. 8.

We search for the relation between

$$-\langle \phi(1) \phi(2) \bar{\phi}(3) \bar{\phi}(4) \rangle = K(12; 34),$$

and

$$-\langle \phi(1) \phi(2) \bar{\phi}(3) \bar{\phi}(4) \phi_a(0) \rangle = K_a(12; 34; 0).$$

They are related to each other by

$$CK_a(12; 34; 0) = i \frac{\partial}{\partial Q_a(0)} [CK(12; 34)].$$

Multiplying (2.2) by C and differentiating it with respect to $Q_a(0)$, we find

$$CK_a(12; 34; 0) = i \frac{\partial}{\partial Q_a(0)} [CS'_F(12; 34)] + C\mathcal{J}(12) K_a(12; 34; 0) + C\mathcal{J}_a(12; 0) K(12; 34), \quad (2.23)$$

where $\mathcal{J}(12)$ is defined similarly to (2.16') by writing (2.2) as

$$\begin{aligned} K(12; 34) &= S'_F(12; 34) + \mathcal{J}(12) K(12; 34) \\ &= S'_F(12; 34) + K(12; 34) \mathcal{J}^{\leftarrow}(34), \end{aligned} \quad (2.42)$$

and $\mathcal{J}_a(12; 0)$ by

$$\begin{aligned} & \mathcal{K}_a(12; 0)K(12; 34) \\ &= \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 i \frac{\partial}{\partial Q_a(0)} [S'_F(1, 5) S'_F(2, 6) G(56; 78)] K(78; 34). \end{aligned} \quad (2.25)$$

With reference to (2.2), (2.13), (2.17) and (2.21), the first term in (2.23) is evaluated as

$$\begin{aligned} & i \frac{\partial}{\partial Q_a(0)} [C S'_F(12; 34)] \\ &= C \langle \phi_a(0) \rangle S'_F(12; 34) + \eta C \int d\omega_5 d\omega_6 d\omega_7 \\ & \times [S'_F(12; 35) \mathcal{O}_a(56; 7) S'_F(6, 4) - S'_F(12; 45) \mathcal{O}_a(56; 7) S'_F(6, 3)] \mathcal{A}'_F(7, 0). \end{aligned} \quad (2.26)$$

The factor in the integrand of (2.25) is given by

$$\begin{aligned} & i \frac{\partial}{\partial Q_a(0)} [S'_F(12; 56) G(56; 78)] \\ &= S'_F(12; 56) [G_a(56; 78; 0) - \langle \phi_a(0) \rangle G(56; 78)] \\ &+ C^{-1} i \frac{\partial}{\partial Q_a(0)} [C S'_F(12; 56)] \cdot G(56; 78), \end{aligned} \quad (2.27)$$

where $G_a(56; 78; 0)$ is defined by

$$G_a(56; 78; 0) = i \partial G(56; 78) / \partial Q_a(0), \quad (2.28)$$

and we must insert (2.26) into the second term of (2.27). A simple graphical consideration leads to the result that G_a can always be expressed as

$$\begin{aligned} G_a(12; 34; 0) &= \int d\omega_5 \mathcal{O}_b(12; 34; 5) \langle \phi_a(0) \phi_b(5) \rangle, \quad (\text{generally}), \\ &= \int d\omega_5 \mathcal{O}_a(12; 34; 5) \mathcal{A}'_F(5-0), \quad (\text{for } Q=0). \end{aligned} \quad (2.28')$$

We shall see later that this form is more useful to construct the S matrix than (2.28). The operator \mathcal{O}_a corresponding, for instance, to Fig. 8 is given to the lowest order by

$$\eta^5 \mathcal{O}_b S_F(1-3) \mathcal{O}_c \cdot \mathcal{O}_c S_F(2-5) \mathcal{O}_a S_F(5-4) \mathcal{O}_b \cdot \mathcal{A}'_F(1-4) \mathcal{A}'_F(2-3).$$

The integral equation (2.23) can formally be solved as before

$$\begin{aligned} K_a(12; 34; 0) &= \langle \phi_a(0) \rangle K(12; 34) \\ &+ \eta \int d\omega_5 d\omega_6 d\omega_7 [K(12; 35) \mathcal{O}_a(56; 7) S'_F(6, 4) - K(12; 45) \mathcal{O}_a(56; 7) S'_F(6, 3)] \\ & \times \mathcal{A}'_F(7, 0) \\ &+ \frac{1}{2} \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 K(12; 56) [G_a(56; 78; 0) - \langle \phi_a(0) \rangle G(56; 78)] K(78; 34) \end{aligned}$$

$$+ \frac{1}{2} \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 (1 - \mathcal{F}(12))^{-1} C^{-1} \cdot i \frac{\delta}{\delta Q_a(0)} [CS'_F(12; 56)] \cdot G(56; 78) \\ \times K(78; 34). \quad (2.29)$$

The integrand of the last term is written by (2.26) as

$$(1 - \mathcal{F}(12))^{-1} C^{-1} \cdot i \frac{\delta}{\delta Q_a(0)} [CS'_F(12; 56)] \\ = \langle \phi_a(0) \rangle K(12; 56) + \eta \int d\omega_7 d\omega_8 d\omega_9 \times \\ \times [K(12; 57) \mathcal{O}_a(78; 9) S'_F(8, 6) - K(12; 67) \mathcal{O}_a(78; 9) S'_F(8, 5)] \mathcal{A}'_F(9, 0). \quad (2.30)$$

That all S'_F functions are replaced by K in the above expression corresponds to the method of distorted waves.

We have so far investigated the procedure to find the Feynman kernels involving meson wave functions from kernels of Fermions alone, and found that all kinds of kernels are expressed by the combination of kernels of Fermions alone

$$K(12 \dots; 1'2' \dots) = \langle \dots \phi(2) \phi(1) \bar{\phi}(1') \bar{\phi}(2') \dots \rangle$$

and the interaction operators like

$$(\Sigma^*)_{Q=0}, (G)_{Q=0}, (\partial \Sigma^* / \partial Q)_{Q=0}, (\partial G / \partial Q)_{Q=0}, \dots$$

It will be better to add a remark here, that we have only to refer to a Fermion kernel with one more particle when we intend to deal with a problem of pair creation. The meaning of this remark will become clearer in the next section.

Finally we give a relation between Fermion kernels with different number of particles.

$$i \int_{\sigma} d\sigma_{\mu} (1) (1' | \gamma_{\mu} | 1) K(\bar{1}, 2, \dots; \bar{1}', 2', \dots)_{\mu=1} \\ = -(m-n) K(2, \dots; 2', \dots), \quad (2.31)$$

where

$$K(\bar{1}, 2, \dots; \bar{1}', 2', \dots) = \langle \dots \phi(2) : \phi(1) \bar{\phi}(1') : \bar{\phi}(2') \dots \rangle \\ = K(1, 2, \dots; 1', 2', \dots) - S'_F(1, 1') K(2, \dots; 2', \dots),$$

and m and n stand for the numbers of $\bar{\phi}$'s and ϕ 's earlier than the surface σ .

This is readily shown if one remembers

$$i \int_{\sigma} d\sigma_{\mu} : \bar{\psi} \gamma_{\mu} \psi = \text{No. of particles} - \text{No. of anti-particles}. \quad (2.32)$$

§ 3. Integral equations for the covariant components

From the integral equations for the kernels investigated in the previous section, we can readily find those for the covariant components by adopting the limiting procedure of

Gell-Mann and Low⁴⁾, which we shall simply call the $G-L$ procedure. For instance, from (2.2) we obtain the equation

$$g_s(12) = g_s^0(12) + \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 S_F'(1, 5) S_F'(2, 6) G(56; 78) g_s(78), \quad (3.1)$$

where $g_s^0(12)$ is the covariant component for the incident wave consisting of a product of plane wave functions of the two incident particles. Each plane wave is normalized in reference to the procedure developed in § 1. A simpler means of normalization will be given in the next section.

Rigorously speaking, the $G-L$ procedure when applied to inhomogeneous equations, *i.e.* for unbound states, yields equations like

$$g_s(12) = c g_s^0(12) + \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 S_F'(1, 5) S_F'(2, 6) G(56; 78) g(78),$$

where c is an indefinite constant. Thus the covariant component $g_s(12)$ depends on the constant c . Since, however, we know how to normalize in reference to (1.16), we can in principle fix the constant c . We have put c equal to unity for simplicity since this choice seems to be plausible in that it leads to the same results with those for the perturbation expansion.* We must call our attention, however, when we intend to apply the $G-L$ procedure to meta-stable states for which the degeneracy of energy levels is essentially important, since the procedure is applicable only upon the assumption of the non-degeneracy of energy levels.

On dealing with the problem of bound states, we must drop the inhomogeneous term $g_s^0(12)$. Hence, the problem to derive integral equations for the covariant components seems rather trivial after those for kernels are settled.

There are such important subjects, however, that give rise to discussions only in the equations for the covariant components. The equation (3.1) can fix the first member of the covariant components of the state vector \mathcal{V}_s for a two-nucleon system.

The second member $g_s(12; 0)$ can be expressed in terms of the first member $g_s(12)$ by adopting the $G-L$ procedure to the equation (2.29). For simplicity we shall illustrate the procedure in the case of one-nucleon system. The $G-L$ procedure produces, when applied to (2.21), the equation

$$g_s(1; 0) = \langle \phi_a(0) \rangle g_s(1) + \eta \int d\omega_3 d\omega_4 d\omega_5 S_F'(1, 3) \mathcal{O}_a(34; 5) J_F'(5, 0) g_s(4),$$

(for $Q \rightarrow 0$) (3.2)

where

$$g_s(1; 0) = (\mathcal{V}_0, : \phi(1) \phi_a(0) : \mathcal{V}_s) = (\mathcal{V}_0, T[\phi(1) \phi_a(0)] \mathcal{V}_s).$$

Furry's theorem is of help to reduce the equation (3.2) into

* We can put c equal to unity if we only assume that the interaction between dressed particles are adiabatically switched on, but we need not assume for bare ones.

$$g_s(1; 0) = \eta \int d\omega_3 d\omega_4 d\omega_5 S'_F(1-3) \mathcal{O}_a(34, 5) \mathcal{A}'_F(5-0) g_s(4). \quad (\text{for } Q=0) \quad (3 \cdot 2')$$

The second member of the covariant components is thus determined when the first member is known, and the later members are also derived in completely the same way. Hence all the covariant components for a state can be fixed in principle.

It must be noticed that for components involving two or more mesons we must distinguish between covariant and S - B components. (Cf. eqs. (1.11) and (1.12).)

Next we investigate the component with a virtual nucleon-pair produced. For simplicity, we shall again consider a one-nucleon system, and for this purpose we must start from a two-body kernel as has been remarked in the previous section.

$$\begin{aligned} K(12; 34) &= -\langle \phi(1) \phi(2) \bar{\psi}(3) \bar{\psi}(4) \rangle \\ &= -\sum_s (\Psi_0, T[\phi(1) \phi(2) \bar{\psi}(3)] \Psi_s) (\Psi_s, \bar{\psi}(4) \Psi_0), \\ &\quad \text{for } t_1, t_2, t_3 > t_4. \end{aligned} \quad (3.3)$$

Application of the G - I procedure to the equation (2.2) with the above division yields the equation

$$\begin{aligned} X_s(12; 3) &= S'_F(2, 3) g_s(1) - S'_F(1, 3) g_s(2) \\ &\quad + \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 S'_F(1, 5) S'_F(2, 6) G(56; 78) X_s(78; 3), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} X_s(12; 3) &= (\Psi_0, T[\phi(1) \phi(2) \bar{\psi}(3)] \Psi_s), \\ g_s(12; 3) &= (\Psi_0, \phi(1) \phi(2) \bar{\psi}(3) : \Psi_s), \quad \text{and} \quad X_s(1) = g_s(1). \end{aligned}$$

From (3.4), we have the covariant component

$$g_s(12; 3) = \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 S'_F(1, 5) S'_F(2, 6) G(56; 78) X_s(78; 3). \quad (3.5)$$

(a) Method of Green Functions

We must solve the integral equation (3.4) for $X_s(12; 3)$, and for this purpose we introduce the method of Green functions. The equation (3.4) can be written with the notation defined in the previous sections as

$$(1 - \mathcal{J}(12)) X_s(12; 3) = S'_F(2, 3) g_s(1) - S'_F(1, 3) g_s(2). \quad (3.4')$$

We need some device on applying $(1 - \mathcal{J}(12))^{-1}$ onto the equation (3.4') in order to solve the function X , since the right hand member does not in this form involve the function $S'_F(12; \dots)$ only for which the operator $(1 - \mathcal{J}(12))^{-1}$ is well defined.

The equation (2.16) is changed into

$$(\gamma \partial_1 + x - i\eta O_a \langle \psi_a(1) \rangle) S'_F(1, 2) - i \int d\omega_3 \Sigma^*(1, 3) S'_F(3, 2) = -i\partial(1-2), \quad (3.6)$$

by utilizing the relation

$$(\gamma \partial + x) S_F(x) = -i\partial(x).$$

The equation (3.6) may formally be written as

$$\int d\omega_2 \mathcal{D}(1, 2) S'_F(2, 3) = -i\delta(1-3)$$

or

$$\int d\omega_2 S'_F(1, 2) \mathcal{D}(2, 3) = -i\delta(1-3) \quad (3 \cdot 6'')$$

with the help of the integral representation. \mathcal{D} is given by

$$\mathcal{D}(1, 2) = (\gamma\partial_1 + \alpha - i\eta U_a \langle \phi_a(1) \rangle) \delta(1-2) - i\Sigma^*(1, 2).$$

Hence the right hand side of the equation (3·4') becomes

$$\begin{aligned} & S'_F(2, 3)g_s(1) - S'_F(1, 3)g_s(2) \\ &= \int d\omega_5 [S'_F(2, 3)\delta(1-5) - S'_F(1, 3)\delta(2-5)]g_s(5) \\ &= -i \int d\omega_4 d\omega_1 S'_F(12; 34) \mathcal{D}(4, 5)g_s(5) \end{aligned}$$

for which the operator $(1 - \mathcal{J}(12))^{-1}$ is well defined, and the equation (3·4'') is solved as

$$X_s(12; 3) = -i \int d\omega_4 d\omega_5 K(12; 34) \mathcal{D}(4, 5)g_s(5). \quad (3 \cdot 7)^*$$

(b) Incident Waves Involving Composite Particles

When the incident wave involves composite particles, we must call our attention in applying the $G-L$ procedure to obtain the integral equations for the covariant components. As a typical example, we consider the problem of neutron-deuteron collisions.

The integral equation for the three-body kernel is given by

$$\begin{aligned} K(123; 456) &= S'_F(123; 456) + [\mathcal{J}(12) + \mathcal{J}(23) + \mathcal{J}(31)] K(123; 456) \\ &\quad + \mathcal{J}(123) K(123; 456), \end{aligned} \quad (3 \cdot 8)$$

where

$$\begin{aligned} & \mathcal{J}(123) K(123; 456) \\ &= \int d\omega_1' \dots d\omega_6' S'_F(1, 1') S'_F(2, 2') S'_F(3, 3') G_3(1'2'3'; 4'5'6') K(4'5'6'; 456). \end{aligned} \quad (3 \cdot 9)$$

The second term in the right hand side of the equation (3·8) refers to the scattering by the two-body interaction (Fig. 9), and the last one by the three-body interaction (Fig. 10). If we immediately apply the $G-L$ procedure to the equation (3·8), we cannot express the initial condition that the incident system consists of a neutron and a deuteron.

In order to avoid this difficulty, we operate $(1 - \mathcal{J}(12))^{-1}$ on both sides of the equation (3·8) from the left. Then the S'_F function turns out to be

* In order to reduce this expression by eliminating \mathcal{D} , the equation (2·2') is useful. We have

$$X_s(12; 3) = S_{F'}(2, 3)g_s(1) - S_{F'}(1, 3)g_s(2) - \int d\omega_5 d\omega_6 d\omega_7 d\omega_8 K(12; 56) G(56; 78) S_{F'}(7, 3)g_s(8).$$

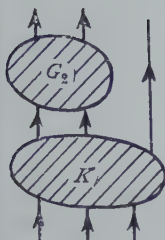


Fig. 9.

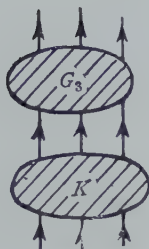


Fig. 10.

$$K(12; 45) S_F'(3, 6) + K(12; 56) S_F'(3, 4) \\ + K(12; 64) S_F'(3, 5), \quad (3 \cdot 10)$$

while the remaining members to be

$$[1 - (1 - \mathcal{J}(12))^{-1} (\mathcal{J}(23) + \mathcal{J}(31)) \\ - (1 - \mathcal{J}(12))^{-1} \mathcal{J}(123)] K(123; 456).$$

The last term can readily be obtained by replacing as

$$S_F'(1, 1') S_F'(2, 2') S_F'(3, 3') \rightarrow \frac{1}{2} K(12; 1'2') S_F'(3, 3')$$

in the operator $\mathcal{J}(123)$, since one can write (3.9) as

$$\mathcal{J}(123) K(123; 456) = \frac{1}{2} \int d\omega_1' \dots d\omega_6' S_F'(12; 1'2') S_F'(3, 3') \\ \times G_3(1'2'3'; 4'5'6') K(4'5'6'; 456).$$

While the second term, as the operand of $(1 - \mathcal{J}(12))^{-1}$ lacks the function $S_F'(12; \dots)$ or $S_F'(1, \dots) S_F'(2, \dots)$, but involves only a single S_F' . Hence we must again resort to the method of Green functions.

After such a device, the $G-L$ procedure provides us with the equation for the neutron-deuteron scattering. We have described the procedure very simply, however, the present method is considerably general and can be applied even to nucleus-nucleus collisions in principle. A more detailed discussion of this method will be given in another paper¹⁴⁾.

§ 4. Construction of the S matrix

In this section the method to construct the S matrix is investigated. This will perhaps be the final goal of the present theory.

The Feynman-Dyson theory developed so far provides us with a means to calculate the matrix $U(\infty, -\infty)$ as the final answer and the matrix U has tentatively been identified with the S -matrix. However, it is evidently not the case when there are composite particles involved before or after the collision. Hence we begin with the discussion of the differences between these two matrices.

Hypothesis of Adiabatic Switching on and off

The so-called hypothesis of adiabatic switching on and off is understood in this paper as

$$\lim_{\substack{t \rightarrow +\infty \\ t' \rightarrow -\infty}} (\Psi_0, T[A(t) \dots A'(t') \dots] \Psi_0) \\ = (\Psi_0, A(\infty) \dots U(\infty, -\infty) A'(-\infty) \dots \Phi_0) / (\Phi_0, U(\infty, -\infty) \Phi_0)$$

or more rigorously speaking, it must be expressed as

$$\begin{aligned}
& \lim_{\substack{\ell \rightarrow +\infty \\ \ell' \rightarrow -\infty}} \lim_{\substack{\tau \rightarrow +\infty \\ \tau' \rightarrow -\infty}} T[U(\tau, \tau') A(t) \dots A'(t') \dots] \\
&= \lim_{\substack{\tau \rightarrow +\infty \\ \tau' \rightarrow -\infty}} \lim_{\substack{\ell \rightarrow +\infty \\ \ell' \rightarrow -\infty}} T[U(\tau, \tau') A(t) \dots A'(t') \dots], \quad (4.1)
\end{aligned}$$

i.e. the hypothesis requires the commutativity of the two limiting procedures.

The first expression of (4.1) is interpreted to express the matrix element for the scattering of dressed particles rigorously, while the second one corresponds to a process in which bare particles are adiabatically dressed before scattering and again adiabatically undressed after. The Feynman-Dyson theory stands for the latter point of view, on which the connection of the Feynman kernel with the S matrix was investigated by Kita³⁹ and by Utiyama et al.¹²⁾

The adiabatic hypothesis will perhaps be valid if we confine ourselves to such a restricted class of collision problems in which no composite particle appears.

Since we are interested in a more general class of problems, however, we shall develop our discussions upon the former viewpoint.

As an example, the problem of nucleon-nucleon collisions is treated. The Salpeter-Bethe equation for this case is given on account of (2.24) by

$$(1 - \mathcal{J}(12))g_a(12) = g_a^0(12) \quad (4.2)$$

where $g_a^0(12)$ refers to the incident wave and

$$g_a^0(12) = g_{a'}(1)g_{a''}(2) - g_{a''}(1)g_{a'}(2), \quad (4.3)$$

each g stands for a plane wave defined by

$$g_{a'}(1) = (\Psi_0, \phi(1)\Psi_{a'}), \quad (4.4)$$

where $\Psi_{a'}$ is the state vector for a one nucleon state a' .

With (4.3), the equation (4.2) is solved in the form

$$g_a(12) = g_a^0(12) + \int d\omega_3 d\omega_4 d\omega_5 d\omega_6 K(12; 34)G(34; 56)g_{a'}(5)g_{a''}(6), \quad (4.5)$$

in virtue of the relation (2.2).

The first term corresponds to the incident wave and the second one to the scattered wave.

Once the first covariant component $g_a(12)$ is known, we can successively construct later components with reference to the procedure described in the previous section. Then we can define the S matrix as follows.

Consider, for instance, the multiple production of mesons by nucleon-nucleon collisions. The covariant component $g_a(12; 1'2' \dots n')$ referring to two nucleons and n mesons may asymptotically be expressed as

$$g_a(12; 1'2' \dots n') \sim \sum_b S_{ba} g_b^0(12; 1'2' \dots n'), \quad (4.6)$$

by which the S matrix is given. The function g_b^0 is defined in a similar way to (4.3) as the symmetrized (for Bosons) or antisymmetrized (for Fermions) direct product of the

covariant components of individual particles.

The S matrix defined by (4.6) is directly related to the transition probability amplitude, since (4.6) is derived from the relation for state vectors.

$$\mathcal{F}_a \sim S \Phi_a = \sum_b \Phi_b (\Phi_b, S \Phi_a) = \sum_b S_{ba} \Phi_b,$$

\mathcal{F}_a and Φ_a refer to the scattered and incident states.

On multiplying the above equation by $(\Psi_0, : \psi(1) \psi(2) \phi(1') \dots \phi(n') :$ from the left we have (4.6).

Corresponding to various kinds of boundary conditions, we must suitably rearrange the Feynman kernels which appear in equations like (4.5) to express covariant components in terms of incident waves. In order to construct the asymptotic forms of covariant components, we must replace δ_+ functions of energy, which are involved in Feynman kernels because of the positive frequency condition or equivalently the outgoing wave condition, by δ functions.

For the elastic scattering, we may better rewrite (4.5) as

$$\begin{aligned} g_a(12) = & g_a^0(12) + \int d\omega_3 d\omega_4 d\omega_5 d\omega_6 S'_F(12; 34) G(34; 56) g_{a'}(5) g_{a''}(6) \\ & + \frac{1}{2} \int d\omega_3 \dots d\omega_{10} S'_F(12; 34) G(34; 56) K(56; 78) \\ & \times G(78; 9\ 10) g_{a'}(9) g_{a''}(10), \end{aligned} \quad (4.7)$$

by means of equation (2.2).

We must replace each δ_+ in S'_F by δ in order to obtain the asymptotic form. This can readily be achieved if we only remember¹⁵⁾

$$\lim_{t, \infty} e^{-i\omega t} \delta_+(\omega) = \delta(\omega). \quad (4.8)$$

By letting t_1 and t_2 tend to $+\infty$, we have the asymptotic form

$$\begin{aligned} g_a(12) \sim & g_a^0(12) + \sum_b g_b^0(12) \int d\omega_3 d\omega_4 d\omega_5 d\omega_6 \bar{g}_b^0(34) G(34; 56) g_{a'}(5) g_{a''}(6) \\ & + \sum_b g_b^0(12) \cdot \frac{1}{2} \int d\omega_3 \dots d\omega_{10} \bar{g}_b^0(34) G(34; 56) \\ & \times K(56; 78) G(78; 9\ 10) g_{a'}(9) g_{a''}(10), \end{aligned} \quad (4.9)$$

since we can write for such a temporal division

$$\lim_{t_1, t_2 \rightarrow +\infty} S'_F(12; 34) = \sum_b g_b^0(12) \bar{g}_b^0(34). \quad (t_1, t_2 \rightarrow +\infty) \quad (4.10)$$

The relation (4.10) may be utilized to determine the normalization of the function g_b^0 for non-interacting particles without recourse to the contravariant components.*) Similarly

* As a result of the characteristic features of the renormalization procedure, the normalization of the function g^0 coincides with that in the interaction representation.

the normalization of the function g_b for interacting particles is determined subject to

$$K(12; 34) = \sum_b g_b(12) g_b(34), \quad \text{for } t_1, t_2 > t_3, t_4. \quad (4.10')$$

Comparing (4.6) for the case $n=0$ to (4.9), we have the desired S matrix for the elastic scattering

$$\begin{aligned} T_{ba} &= \int d\omega_3 d\omega_4 d\omega_5 d\omega_6 \bar{g}_b^0(34) G(34; 56) g_{a'}(5) g_{a''}(6) \\ &\quad + \frac{1}{2} \int d\omega_3 \dots d\omega_{10} \bar{g}_b^0(34) G(34; 56) K(56; 78) G(78; 9, 10) g_{a'}(9) g_{a''}(10) \\ &= \frac{1}{2} \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \bar{g}_b^0(12) G(12; 34) g_a(34), \end{aligned} \quad (4.11)$$

where $T = S - 1$.

In a similar way, we can treat inelastic collisions such as

$$p + p \rightarrow n + p + \pi^+, \quad (4.12)$$

and

$$p + p \rightarrow d + \pi^+ \quad (4.13)$$

by means of (2.29).

In the latter example, however, we slightly modify the above method to express the boundary condition that the final nucleons are bound to form a deuteron. For this purpose, we retain $K(12; 34)$ which appears in the equation expressing $g(12; 0)$ in terms of kernels and incident waves like (4.5), without rearranging it into S'_F as done in (4.7), and we let t_1 and t_2 tend to $+\infty$ to make use of the following relation instead of (4.10)

$$\lim_{t_1, t_2 \rightarrow +\infty} K(12; 34) = \sum_b g_b(12) \bar{g}_b(34), \quad (t_1, t_2 \rightarrow +\infty). \quad (4.14)$$

in which bound deuteron states are involved. In this way, we can derive a concrete expression of the S matrix for the process (4.13)

$$\begin{aligned} T_{ba} &= \frac{1}{2} \int d\omega_0 \dots d\omega_4 \bar{g}_a(12) g_a^*(0) \mathcal{O}_i(12; 34; 0) g_a(34) \\ &\quad - \frac{1}{2} i\eta \int d\omega_0 \dots d\omega_8 \bar{g}_a(12) g_a^*(0) \mathcal{O}_i(23; 0) \mathcal{D}(1, 4) \\ &\quad \times S'_F(34; 56) G(56; 78) g_a(78), \end{aligned} \quad (4.15)$$

where

$$g_a(0) = (\Psi_0, \phi_i(0) \Psi_f),$$

Ψ_f is the state vector for a one π^+ state. Note that

$$\int d\omega_4 \mathcal{D}(1, 4) S'_F(34; 56) = -i [\partial(1, 6) S'_F(3, 5) - \partial(1, 5) S'_F(3, 6)].$$

Finally it will be worth while noticing that the present treatment of the S matrix gives such a normalization as

$$(Vac|S|Vac)=1 \quad (4.16)$$

contrary to the conventional relation

$$(\text{vac}|U(\infty, -\infty)|\text{vac})=e^{-L} \neq 1. \quad (4.17)$$

Or more generally, for a stable one particle state " α " either elementary or composite we have

$$S_{\alpha\alpha}=1.$$

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Appendix

Evaluation of Expectation Values

In this appendix, we study the method to express the expectation values of physical quantities in terms of *covariant components alone*.

Let $g_{s,Q}$ be defined by

$$g_{s,Q}(12\dots n;0)=(\Psi_0, T[:e_1\dots e_n: Q(0)]\Psi_s).$$

Then for $t_1, t_2, \dots > t_0$, it is separated as

$$\begin{aligned} g_{s,Q}(12\dots n;0) &= \sum_r (\Psi_0, :e_1\dots e_n: \Psi_r) (\Psi_r, Q(0)\Psi_s) \\ &= \sum g_r(12\dots n) Q(0)_{rs}, \end{aligned} \quad (A.1)$$

where

$$g_r(12\dots n) = (\Psi_0, :e_1\dots e_n: \Psi_r), \quad Q(0)_{rs} = (\Psi_r, Q(0)\Psi_s),$$

and Q_{rs} is the expectation value of a physical quantity Q in question.

The relation (A.1) is regarded as the equation for Q_{rs} provided that covariant components $g_{s,Q}$ and g_r are known. It is interesting to see that this equation can explicitly be solved in general. We shall illustrate the procedure by a simple example.

Expectation value of the operator $\bar{\psi}_\alpha \psi_\beta$ in one nucleon states.

For brevity we shall omit the spinor indices α and β . From (3.5) we have

$$\begin{aligned} g_{s,Q} &= (\Psi_0, T[\psi(1): \bar{\psi}(2)\psi(2):]\Psi_s) = S'_F(1-2)g_s(2) + g_s(12;2) \\ &= S'_F(1-2)g_s(2) + \int d\omega_3 d\omega_6 d\omega_7 d\omega_8 S'_F(1-5)S'_F(2-6) \\ &\quad \times G(56;78)X_s(78;2), \end{aligned}$$

where X_s is given by (3.7).

On letting t_1 tend to $+\infty$, we have

$$\begin{aligned} g_{s,Q} &= \sum_r g_r(1)[\bar{g}_r(2)g_s(2) + \int d\omega_3 d\omega_6 d\omega_7 d\omega_8 \bar{g}_r(5) \\ &\quad \times S'_F(2-6)G(56;78)X_s(78;2)], \quad (t_1 \rightarrow +\infty). \end{aligned} \quad (A.2)$$

By comparing this relation with (A.1), we have

$$Q(2)_{rs} = (\Psi_r : \bar{\phi}_\alpha(2) \phi_\beta(2) : \Psi_s) \\ = \bar{g}_r(2_\alpha) g_s(2_\beta) + \int d\omega_1 d\omega_2 d\omega_3 d\omega_4 \bar{g}_r(5) S'_F(2_\beta - 6) G(56; 78) N_s(78; 2_\alpha). \quad (A \cdot 3)$$

The technique utilized here is essentially equivalent to that employed in the derivation of the S matrix. There is no essential objection in applying the above technique to a more complicated class of problems such as the calculation of the magnetic moment of a deuteron.

It is worth while noticing for practical purposes that we can evaluate the expectation values of physical quantities as well as the S matrix without recourse to the contravariant components. The contravariant components do not appear in practical calculations, but they will be of much help to understand the physical meaning of the covariant components.

It is interesting to see that the present method is developed upon the over all space-time point of view contrary to the case of the equal time formalism described in § 2, and that we can speak of the expectation values of the physically observable quantities alone, but not of those of the unobservable ones such as the probability of a nucleon to be bare.

It should further be stated that the normalization of the covariant components for a state involving n nucleons can be determined in reference to the relation

$$i \int d\sigma_\mu (\Psi_r : \bar{\phi}_\mu \phi : \Psi_s) = n(\Psi_r, \Psi_s), \quad (A \cdot 4)$$

the left hand member of which is calculable by means of the method given in this appendix. The eqs. (4·10') and (A·4) refer to the normalization for the renormalized amplitude, while the eq. (1·16) refers to that for the unrenormalized amplitude.

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Note added in proof. Recently a similar treatment appeared by Matthews and Salam. They have employed completely the same definition of the S product for Heisenberg operators with the present paper. The "Feynman amplitude" in their paper corresponds to the covariant component in the present paper. They have derived coupled equations for the covariant components. The equations given in the present paper are the separated forms of their equations.

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Letters to the Editor

On the Inverse Power Expansion of the $S_{\text{vac}}[\eta^*, \eta, J]$ and the Green-functions*

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By way of illustration we will choose the charged scalar field ψ (mass x) interacting with the neutral scalar ϕ (mass λ). If we introduce suitable source functions, η^*, η, J in the Lagrangian, as was done by J. Schwinger,¹⁾ we can obtain all relevant Green-functions by differentiating functionally $S_{\text{vac}}[\eta^*, \eta, J]$ by the source functions.

"Diffusion equations" for the $S_{\text{vac}}[\eta^*, \eta, J]$ can be derived with the help of the Feynman's path integral²⁾ or the Schwinger's variational principle³⁾;

$$\begin{aligned} \partial S_{\text{vac}}/\partial(a^{-1}) = & -i \int \int d^4x d^4y (\partial/\partial\eta(x)) (\square - x^2) \\ & \times \delta(x-y) (\partial/\partial\eta^*(y)) \cdot S_{\text{vac}}, \end{aligned}$$

$$\begin{aligned} \partial S_{\text{vac}}/\partial(b^{-1}) = & -\frac{i}{2} \int \int d^4x d^4y (\partial/\partial J(x)) \\ & (\square - \lambda^2) \delta(x-y) (\partial/\partial J(y)) \cdot S_{\text{vac}}, \end{aligned}$$

where the similar scale-change has been made as was done in our previous paper,⁴⁾

$$\begin{aligned} g & \rightarrow ab^{1/2}g, \quad a^{1/2}\phi \rightarrow \phi, \quad b^{1/2}\phi \rightarrow \phi; \\ a^{-1/2}\eta & \rightarrow \eta, \quad b^{-1/2}J \rightarrow J. \end{aligned}$$

These equations can be solved to lead to the inverse power expansion of the S_{vac} , if we know the "initial condition", $S_{\text{vac}}(a, b \rightarrow \infty)$. The conventional choice $L_{\text{int}} = g\phi^*\phi\phi$, however, gives the following result,

$$\begin{aligned} S_{\text{vac}}(a, b \rightarrow \infty) \\ \sim \int \int_{x>0} \exp[4i \int (g^{-1}\eta^*(x)\eta(x)J(x))^{1/2} \\ \times \cosh f(x) \cdot d^4x] D(f(x)), \end{aligned}$$

(which we may call "Functional Hankel Function") and leads to a diverging result. To obtain a converging result, therefore, we must modify L_{int} or $S_{\text{vac}}(a, b \rightarrow \infty)$ accordingly.

An equation which leads to the conventional perturbation expansion has also been obtained,

$$\partial S_{\text{vac}}/\partial g = - \int d^4x (\partial^2/\partial\eta^*(x)\partial\eta(x)\partial J(x)) \cdot S_{\text{vac}},$$

with the initial condition,

$$\begin{aligned} S_{\text{vac}}(g=0) \\ = \exp \left[-\frac{a}{2} \int \int d^4x d^4y \eta^*(x) \Delta_F(x-y; x) \eta(y) \right. \\ \left. - \frac{b}{4} \int \int d^4x d^4y J(x) \Delta_F(x-y; \lambda) J(y) \right]. \end{aligned}$$

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The Second Virial Coefficient at Very Low Temperatures

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By the solution of Bloch's equation for the density matrix, Green¹⁾ has obtained an expression suitable for the evaluation of the second virial coefficient at very low temperatures. Recently, Ichimura²⁾ has also obtained a similar expression by the method of the number representation of the grand partition function. Now it is known³⁾ that the phase shifts method can also be used to evaluate the second virial coefficient at very low temperatures. Starting from the expression of the second virial coefficient in terms of the phase shifts, we shall, in this note, obtain the same development as Green's except for the terms coming from the bound states of two molecules.

For simplicity, we shall restrict ourselves in the case of Boltzmann statistics. The second virial coefficient is given by the following expression:

$$B = -\frac{1}{2} (2\lambda^2)^{3/2} \sum_{l=0}^{\infty} (2l+1) \sum_n e^{-E_{nl}/kT} + F_G,$$

* Read at the Kyoto meeting of the Physical Society of Japan, May 3, 1953.

$$R_G = -\frac{1}{2} (2\lambda^2)^{3/2} \sum_{l=0}^{\infty} (2l+1) \int_0^{\infty} e^{-(\lambda^2 k^2/2\pi)} \frac{1}{\pi} \\ \times \frac{d\delta_l(k)}{dk} dk, \quad \lambda^2 = \frac{\hbar^2}{2\pi m k T}, \quad (1)$$

where E_{nl} 's ($E_{nl} < 0$) are the bound energy levels and $\delta_l(k)$'s the phase shifts of the l th partial wave. The subscript G of R_G means that it will turn out to be identical with Green's expression for the second virial coefficient, at least at very low temperatures.

Let $\phi(r)$ be the potential between two molecules and put

$$U(r) = \frac{m}{\hbar} \phi(r). \quad (2)$$

Solving the integral equation for the phase shifts by a simple interaction method, we obtain

$$\frac{1}{k^{2l+1}} \tan \delta_l = \sum_{n=1}^{\infty} (-1)^n \int_0^{\infty} dr_1 \dots dr_n \frac{j_l(r_1)}{k^{l+1}} U(r_1) K_l(r_1, r_2) U(r_2) \\ \dots K_l(r_{n-1}, r_n) U(r_n) \frac{j_l(r_n)}{k^{l+1}}, \quad (3)$$

where

$$K_l(r, r') = 1/k \cdot j_l(r) n_l(r'), \quad r < r' \\ = 1/k \cdot j_l(r') n_l(r), \quad r > r' \\ j_l(r) = \sqrt{\frac{\pi k r}{2}} J_{l+(1/2)}(kr), \\ n_l(r) = (-1)^l \sqrt{\frac{\pi k r}{2}} J_{-l-(1/2)}(kr). \quad (4)$$

By a formal expansion of the right-hand side of eq. (3) in powers of k we can obtain the expression for the phase shift in power series of k . Substituting this power series for $\delta_l(k)$ in eq. (1) and integrating over k term by term, we obtain finally

$$2B_G / \frac{\lambda^2}{2\pi} = R_0 + R_1 \frac{2\pi}{\lambda^2} + O(\lambda^{-4}),$$

$$R_0 = \int_0^{\infty} 4\pi r^2 U(r) \tau(r) dr,$$

$$R_1 = -2\pi \left[\int_0^{\infty} r^2 U(r) \tau(r) dr \right]^2 \\ + 4\pi \int_0^{\infty} \int_0^{\infty} dr dr' r r' U(r) U(r') \tau(r) \tau(r') K_0^{(2)}(r, r') \\ + 2\pi \int_0^{\infty} dr r^4 U(r) [1 - \tau(r)] \\ - 2\pi \sum_{n=2}^{\infty} (-1)^n \int_0^{\infty} \dots \int_0^{\infty} dr_1$$

$$\dots dr_n r_1^2 U(r_1) K_1^{(0)}(r_1, r_2) U(r_2)$$

$$\dots K_1^{(n)}(r_{n-1}, r_n) U(r_n) r_n^2,$$

$$\tau(r) = 1 - \frac{1}{r} \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} \dots \int_0^{\infty} dr_1$$

$$\dots dr_n K_0^{(0)}(r, r_1) U(r_1)$$

$$\dots K_0^{(n)}(r_{n-1}, r_n) U(r_n) r_n,$$

$$K_0^{(0)} = r, \quad r < r', \quad K_0^{(2)} = r/4 \cdot [r^2 + 3r'^2], \quad r < r',$$

$$= r', \quad r > r', \quad = r'/4 \cdot [r'^2 + 3r^2], \quad r > r',$$

$$K_1^{(0)} = r^2/3r', \quad r < r',$$

$$= r'^2/3r, \quad r > r'.$$

(5)

R_0 contains only the term coming from $\delta_0(k)$, and R_1 contains the terms coming from $\delta_0(k)$ and $\delta_1(k)$. It is easy to verify that R_0 and R_1 are no more than Green's corresponding terms, and so B_G is identical with Green's expression, at least at very low temperatures. This tells us that Green's expression does not contain the terms coming from the bound states of two molecules, that is, the first terms of B in eq. (1). Since both B in eq. (1) and the expression by Green are derived from the same definition of the second virial coefficient, it is clear that the two expressions should lead to the same result. The origin of this discrepancy is not clear, though it may be due to the mathematical assumption that all the eigenfunctions of the Schrödinger equation form a complete set of functions, which is necessary for the validity of the initial condition used to solve Bloch's equation.

In order to evaluate $\tau(r)$ in eq. (5), Green has found an integral equation which $\tau(r)$ must satisfy and transformed it into a differential equation with suitable boundary conditions. However, making use of the well-known expansion⁴ of $k^{2l+1} \cot \delta_l$ in powers of k^2 , we can obtain the final results without using the infinite series such as $\tau(r)$ in eq. (5). In fact by using the following expansions:

$$k \cot \delta_0 = 1/a_0 + W_{00} k^2 + W_{01} k^4 + \dots, \\ k^{2l+1} \cot \delta_l = 1/a_l + (W_{l,0} - D_{l,0}) k^2 \\ + (W_{l,1} - D_{l,1}) k^4 + \dots, \quad l \geq 2, \quad (6)$$

we can obtain the expression for B_G as follows:

$$\frac{2B_G}{\lambda^2/2\pi} = -4\pi a_0 \left[1 - \frac{\pi a_0^2 (1 + 3 \cdot W_{00}/a_0)}{\lambda^2} \right. \\ + \frac{3\pi^2 a_0^4 (1 + 5 \cdot W_{00}/a_0 + 5 \cdot W_{00}^2/a_0^2 - 5 \cdot W_{01}/a_0^3)}{\lambda^4} \\ + \frac{9a_1\pi}{a_0\lambda^2} - \frac{45a_1^2(W_{10} - D_{10})\pi^2}{a_0\lambda^4} + \dots \\ \left. + \frac{75a_2\pi^2}{a_0\lambda^4} + \dots \right]. \quad (7)$$

The coefficients of the expansions, i.e., a_i , W_{lm} and D_{lm} , can be obtained by solving the differential equations, which are equivalent to Green's. In the right-hand side of eq. (7), the first line comes from $\delta_0(k)$, the second from $\delta_1(k)$ and the third from $\delta_2(k)$.

An expansion such as eq. (7), though it may not converge, may be sufficient to determine B_G at very low temperatures, i.e., when λ^2 is very large. However, the temperature range in which such an expansion is valid will depend strongly upon the form of the potential function $\phi(r)$, especially upon a behaviour of $\phi(r)$ near $r=0$. Therefore, it seems to be rather meaningless, from the point of view of applying the theory to actual gases, to evaluate the second virial coefficient from the expansion eq. (7), unless we know the precise form of $\phi(r)$ and so the range of validity of the expansion is determined.

We wish to thank Professor A. Harasima.

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The Photon Self-energy Problem and Microscopic Space-time Structure

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In spite of many successes of the Tomonaga-Schwinger theory, which is based on the renormalization procedure, we are often encountered with cases in which results of computation contradict with formal requirement of gauge invariance. The most serious one is the non-vanishment of the photon self-energy term

$$\delta f_{\mu}(x) = -\frac{i\epsilon^2}{4} \int (dx') S_{\mu} \{ S_{\nu}(x'-x) \gamma_{\nu} \} A_{\nu}(x') \\ = K A_{\mu}(x), \quad (1)$$

where K is a constant. This term is neither gauge

invariant nor renormalizable in photon mass. Recently, Hara et al.¹⁾ have attempted to remove the difficulty of the photon self-energy by using the idea of renormalizing the light velocity which depends on the momentum of photon in their development. But in my opinion, this procedure is intrinsically equal to change the metric of space-time. In the general relativity to represent the gravitational field in microscopic world, ordinary Euclidian space is replaced by the Riemannian one. (In this case the velocity of light is no longer constant.) Moreover we are able to consider that the gravitational field, besides its influence on the space-time, acts like a medium with the dielectric and magnetic constants.²⁾ In the microscopic world the vacuum is not empty space but physical and real space. For instance, in the vacuum polarization problem charge renormalization can be interpreted as vacuum is polarizable matter, that is, the vacuum has material structure. Therefore, it seems natural to treat the electron field $\psi(x)$ and electromagnetic field $A_{\mu}(x)$ not in Euclidian empty space but in Riemannian space whose departure from Euclidian one is due to the interaction between the photon and vacuum electron. As a first step, we generalize the $g_{\mu\nu}(x)$ so that it becomes $\delta_{\mu\nu}$ of Minkowski space when the interaction between photon and vacuum electron is neglected. Using such a metric, we attempt to remove the photon self-energy.

In the Riemannian space which has metric $g_{\mu\nu}(x)$, Maxwell equation is

$$F_{\mu,\nu}^{\nu} = -j_{\mu}, \quad (2)$$

where $_{,\nu}$ indicates the covariant derivative. By the generalized Lorentz condition $A_{\mu,\nu}^{\nu} = 0$ (2) becomes

$$\square A_{\mu} + R_{\mu\nu} A^{\nu} = -j_{\mu}, \quad (2')$$

where $R_{\mu\nu}$ is contracted Riemannian-Christoffel tensor and $\square A_{\mu}$ is $g^{\nu\sigma}(A_{\mu})_{,\nu,\sigma}$. Considering the vacuum effect, to the right hand side of (2') we must add the induced current δj_{μ} . Therefore (2) becomes

$$\square A_{\mu} + R_{\mu\nu} A^{\nu} = -j_{\mu} - \delta j_{\mu}. \quad (3)$$

Here, $\square A_{\mu} = 0$ represents the law of propagation of electromagnetic potential without interaction of matter in the curved space-time. Now we postulate that $\square A_{\mu} = 0$ is not the propagation law of bare photon, but dressed photon with virtual electron-positron field. As the observed propagation law of photon is the one of dressed photon, our procedure to remove the photon self-energy term is that by renormalizing the $g_{\mu\nu}$ which appears in $\square A_{\mu}$ to usual metric of Minkowski space, therefore we treat $\square A_{\mu}$ as ordinary $\square A_{\mu} = \delta_{\rho\sigma} \partial^2 A_{\mu} / \partial x_{\rho} \partial x_{\sigma}$ and remove the

photon self-energy term δJ_μ by the excess term $R_{\mu\nu}A^\nu$. For this purpose, we must take $g_{\mu\nu}(x)$ so that following equation holds

$$R_{\mu\nu} = -K^* g_{\mu\nu}, \quad (4)$$

where K^* is defined in (1). If $-K^*$ is replaced by the universal constant, above equation is nothing but the Einstein equation of gravitational field in empty space.

In the ϵ^2 -approximation, we shall show that the eq. (4) holds by taking $g_{\mu\nu}$ suitably. We put $g_{\mu\nu}$ in the following form

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \epsilon_{\mu\nu}(x). \quad (5)$$

Here, the departure from the Minkowski space is represented by $\epsilon_{\mu\nu}(x)$ which is $O(\epsilon^2)$. In the ϵ^2 -approximation $R_{\mu\nu}$ is

$$R_{\mu\nu} = 1/2 \cdot \square \epsilon_{\mu\nu} + 1/2 \times (\partial^2 \epsilon / \partial x_\mu \partial x_\nu - \partial^2 \epsilon_{\mu\sigma} / \partial x_\nu \partial x_\sigma - \partial^2 \epsilon_{\nu\sigma} / \partial x_\mu \partial x_\sigma), \quad (6)$$

where

$$\epsilon_{\mu}{}^{\sigma} = \delta^{\sigma\rho} \epsilon_{\mu\rho}, \quad \epsilon_{\nu}{}^{\rho} = \delta^{\rho\sigma} \epsilon_{\rho\sigma}.$$

In order to express $\epsilon_{\mu\nu}(x)$ explicitly we separate (6) into two equations

$$1/2 \cdot \square \epsilon_{\mu\nu} = R_{\mu\nu}, \quad (7)$$

$$\partial^2 \epsilon / \partial x_\mu \partial x_\nu - \partial^2 \epsilon_{\mu\sigma} / \partial x_\nu \partial x_\sigma - \partial^2 \epsilon_{\nu\sigma} / \partial x_\mu \partial x_\sigma = 0. \quad (8)$$

(8) is satisfied if

$$\partial(\epsilon_{\mu}{}^{\sigma} - 1/2 \cdot \delta_{\mu}{}^{\sigma} \epsilon) / \partial x_\sigma = 0. \quad (8')$$

(7) is written by

$$\epsilon_{\mu\nu}(x) = -i \int (dx') D_F(x-x') R_{\mu\nu}(x').$$

Then, the left hand side of (8') becomes in our approximation

$$i \int (dx') D_F(x-x') \frac{\partial}{\partial x_\sigma} \times \left[R_{\mu}{}^{\sigma}(x') - \frac{1}{2} \delta_{\mu}{}^{\sigma} R(x') \right] \\ = i \int (dx') D_F(x-x') \left(R_{\mu}{}^{\sigma}(x') - \frac{1}{2} g_{\mu}{}^{\sigma} R(x') \right),_{\sigma}$$

Owing to the property of $R_{\mu\nu}$ (i.e. contracted Bianchi identity) it becomes 0, the compatibility of (7) and (8) is guaranteed.

In the theory of general relativity ($R_{\mu}{}^{\sigma} - 1/2 \times g_{\mu}{}^{\sigma} R$) is equated to the energy-momentum tensor $T_{\mu}{}^{\sigma}$ of the total system. In our formalism we shall also take $T_{\mu}{}^{\sigma}$ as the energy-momentum tensor which causes the non-flatness of the space-time. In the following, we take $T_{\mu\nu}$ as ϵ^2 -correction for the electro-magnetic energy-momentum tensor,

$$T_{\mu\nu} = \chi \delta \langle T_{\mu\nu}(F, A) \rangle \\ = \chi \delta \langle 1/2 \cdot \partial A_\lambda / \partial x_\nu (\mu \cdot \partial A^\lambda / \partial x_\nu) - \delta_{\mu\nu} (\partial A_\lambda / \partial x_\sigma) \rangle \\ = -\frac{i\chi}{4} \int (dx') \left[\frac{\partial A_\lambda(x)}{\partial x_\mu} \frac{\partial D_F(x-x')}{\partial x_\nu} \right. \\ \left. - \delta_{\mu\nu} \frac{\partial A_\lambda(x)}{\partial x_\sigma} \frac{\partial D_F(x-x')}{\partial x_\sigma} \right] \delta J^\lambda(x'), \quad (9)$$

where χ is normalization constant and $\delta \langle T_{\mu\nu} \rangle$ means ϵ^2 -correction for energy-momentum tensor of photon. $\epsilon_{\mu\nu}(x)$ is then

$$\epsilon_{\mu\nu}(x) = -i \int (dx') D_F(x-x') \\ \times \left[T_{\mu\nu}(x') - \frac{1}{2} \delta_{\mu\nu} T(x') \right] \\ = -i \int (dx') D_F(x-x') \chi \theta_{\mu\nu}(x'), \quad (10)$$

where

$$\theta_{\mu\nu}(x) = -\frac{i}{4} \int (dx') \left[\frac{\partial A_\lambda(x)}{\partial x_\mu} \frac{\partial D_F(x-x')}{\partial x_\nu} \right. \\ \left. + \delta_{\mu\nu} \frac{\partial A_\lambda(x)}{\partial x_\sigma} \frac{\partial D_F(x-x')}{\partial x_\sigma} \right] \delta J^\lambda(x'). \quad (11)$$

Above $\epsilon_{\mu\nu}(x)$ does not satisfy (8') but by taking the one-particle part (8) is satisfied. In virtue of (7) and (10) $R_{\mu\nu}$ can be written as

$$R_{\mu\nu} = \chi \langle \theta_{\mu\nu} \rangle_{1 \text{ ph.} - 0 \text{ ph.}}, \quad (12)$$

where

$$\langle \theta_{\mu\nu} \rangle_{1 \text{ ph.} - 0 \text{ ph.}} = -3/4 \cdot \delta_{\mu\nu} \chi \langle A_\lambda A^\lambda \rangle_{1 \text{ ph.} - 0 \text{ ph.}}. \quad (13)$$

If we take

$$\chi = 1 / \frac{3}{4} \langle A_\lambda A^\lambda \rangle_{1 \text{ ph.} - 0 \text{ ph.}}, \quad (14)$$

we obtain

$$R_{\mu\nu} = -K^* \delta_{\mu\nu}. \quad (15)$$

We have thus shown that eq. (4) holds in ϵ^2 -approximation.

Similar method will be applied to the charge renormalization procedure.

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Analytical Representation of General Spin

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The spin operators and their eigenstates are usually represented by matrices. But it is also possible to represent them analytically in several ways.¹⁾ An interesting example is shown as follows. The spin operators are assumed to be the following differential operators of the two real variables u and v

$$\begin{aligned} J_x &= 1/2(uv - \partial^2/\partial u \partial v), \\ J_y &= i/2(i \cdot \partial/\partial u - u \cdot \partial/\partial v), \\ J_z &= 1/4(u^2 - \partial^2/\partial u^2 - v^2 + \partial^2/\partial v^2), \end{aligned} \quad (1)$$

which operate on the state function $\psi(u, v)$. These expressions result from the substitutions

$$\begin{aligned} a_1^+ &= i(u - \partial/\partial u)/\sqrt{2}, \\ a_1 &= -i(u + \partial/\partial u)/\sqrt{2}, \\ a_2^+ &= i(v - \partial/\partial v)/\sqrt{2}, \\ a_2 &= -i(v + \partial/\partial v)/\sqrt{2} \end{aligned} \quad (2)$$

in the Schwinger's general theory of the angular momentum²⁾

$$J = 1/2 \cdot \sum_{\zeta=1}^2 \sum_{\zeta'=1}^2 a_{\zeta}^+ (\zeta | \sigma | \zeta') a_{\zeta'} \quad (3)$$

and the commutation relations $J_x J_y - J_y J_x = i J_z$ etc. are easily verified.

Now solving the eigenvalue problem

$$J_z \psi = m \psi, \quad (4)$$

$$(J_x^2 + J_y^2 + J_z^2) \psi = j(j+1) \psi, \quad (5)$$

we get as the spin eigenfunctions

$$\psi_{j, m} = H_{j+m}(u) \cdot H_{j-m}(v) \cdot \exp[-(u^2 + v^2)/2] \quad (6)$$

with $j=0, 1/2, 1, 3/2, \dots$

and $m=-j, -j+1, \dots, j$,

where $H_n(x)$ is a normalized Hermite polynomial, whence there follows the orthonormal condition

$$\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \psi_{j, m} \psi_{j', m'} = \delta_{jj'} \delta_{mm'}. \quad (7)$$

For the special case of $j=1/2$, we have the two eigenstates

$$\begin{aligned} \psi_{1/2, 1/2} &= (2/\pi)^{1/2} \cdot u \exp[-(u^2 + v^2)/2] \\ \psi_{1/2, -1/2} &= (2/\pi)^{1/2} \cdot v \exp[-(u^2 + v^2)/2] \end{aligned} \quad (8)$$

which satisfy the following relation

$$2(J_x J_y + J_y J_x) \psi_{1/2, m} = \delta_{m, \pm 1/2} \psi_{1/2, m}. \quad (9)$$

Therefore $2J$ corresponds to the Pauli spin exactly.

If we can adequately adapt the above formulation to the inner structure of the elementary particle, the spin wave functions (6) or (8) may be effective as the convergence factors.

This work was performed during the author's stay at Kyoto University. He wishes to express his gratitude to Profs. H. Yukawa and T. Inoue and the colleagues in Kyoto University for their hospitality and discussions.

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On the Magnetic Structure of Manganese Antimoide

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Manganese antimonide Mn_2Sb is known to crystallize in a tetragonal form, the Mn atoms being distributed on two kinds of sublattices. The saturation magnetic moment of this substance has been measured by Guillard¹⁾ to $0.963 \mu_B$ per Mn atom, which was interpreted by him with an assumption that the Mn atoms on one sublattice have a magnetic moment of $3 \mu_B$ and those on the other sublattice a magnetic moment of $5 \mu_B$, these two magnetic moments being antiparallel and thus resulting in $1 \mu_B$ per Mn atom. In order to interpret the observed anomalous thermal behavior or the saturation magnetization above room temperature, he supposed furthermore that the liberation of electrons from Mn atoms with a magnetic moment of $3 \mu_B$ might take place with increasing temperature. Néel,²⁾ on the other hand, has shown that both the temperature dependency of the saturation magnetic moment below room temperature and that of the paramagnetic susceptibility

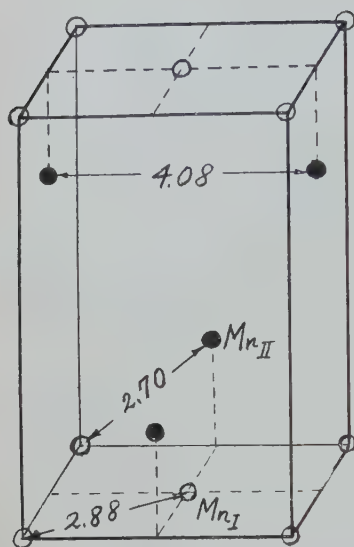


Fig. 1

above the Curie point (277°C) measured by Serres,⁵⁾ could be explained consistently with the assumption of two kinds of atoms mentioned above. Here it will be shown that, in spite of the apparent success of Néel's computation, the susceptibility data well above the Curie point can equally be explained with an assumption that all the Mn atoms possess the same magnetic moment $3\mu_B$.

First, we shall derive an empirical formula for the atomic susceptibility from the values observed by Serres, following Néel's procedure. In our case the Curie constant is $C_{3/2}=1.875$. With this constant the best fit can be obtained by the formula,

$$1/\chi_m = T/1.875 + 77.5 - 1560/(T - 630). \quad (1)$$

The calculated values from this formula are plotted in Fig. 2 along with the Néel's curve and the experimental one. Above 400°C the three curves may be considered to coincide within experimental errors, while below this temperature our curve lies rather far apart from the experimental one, yielding $\theta_p - \theta_f = 84^{\circ}$. One might regard this large difference between θ_p and θ_f as a failure of our model. However, Gingrich, Shull and Wilkinson⁴⁾ have recently observed that at high temperatures above the Curie point the observed paramagnetic scattering is only about half that calculated with Guillaud's model, though the room temperature observations agreed with this model. As the cross section of the paramagnetic scattering is proportional to $S(S+1)$, $\dots S$

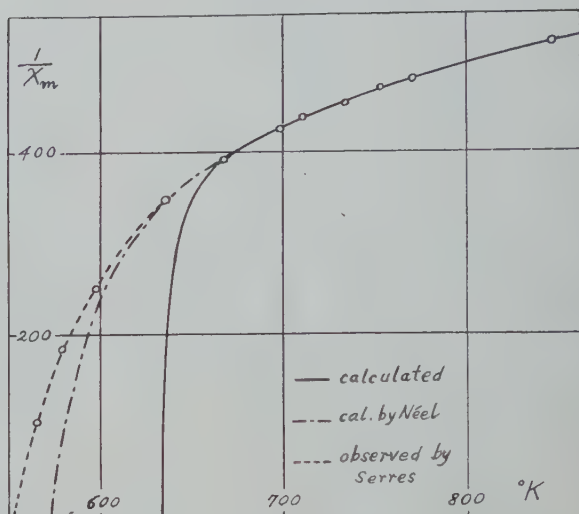


Fig. 2

being the spin angular momentum, the expected paramagnetic scattering for our model is six tenths of that for Guillaud's model. Accordingly, we may infer that our model is preferable to Guillaud's at high temperatures, presumably above about 400°C . Below this temperature, however, our empirical formula would have nothing to do with the real phenomena, as the paramagnetic scattering observations have proved that the magnetic structure at high temperatures is no longer identical with that at lower temperatures.

Further, we obtain the molecular field constants from the numerical constants in (1). They are, using Néel's notation,

$$n=414, \quad a=0.76, \quad \beta=0.48. \quad (2)$$

Here the positive signs of a and β show positive interactions in each own sublattice, while they were negative in Néel's results. The positive interactions seem to be preferable, as the atomic distances 2.88 \AA 4.08 \AA in each of the two sublattices, respectively, are larger than 2.81 \AA , which is supposed to be the lower limit of the distance which gives rise to positive interaction from the consideration of the magnetic properties of other manganese alloys, such as MnBi, MnSb and MnAs.⁵⁾

To understand the magnetic behavior more clearly, it is desirable to have detailed data of neutron diffraction, magnetic measurements, specific heat, etc. in the whole temperature range.

In conclusion, the writer wishes to express his

cordial thanks to Professor T. Nagamiya for his kind discussion on this problem.

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Charge Independence for V -particles*

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Assuming the charge independence for V -particles, the qualitative features of these unstable heavy particles are investigated.

In view of the present experimental material, there seems to be three charge states for V_1 :

- (1) V_1^0 : This particle has been most thoroughly investigated by many workers, and known to decay as $V_1^0 \rightarrow \rho + \pi^- + Q$, $Q \sim 37$ Mev.
- (2) V_1^+ : This particle was discovered by the Pasadena group.¹⁾ $V_1^+ \rightarrow \rho + \pi^0 + Q$, $Q \sim 40$ Mev.
- (3) V_1^- : One case was found in the cosmotron experiments²⁾ that seems to require the existence of V_1^- , although not conclusive. It is as yet not clear whether the isotopic spin of V_1 is 1/2 or 1 or higher. We shall, however, tentatively assign it as equal to 1, since this case is of special interest. Then from the cosmotron experiments,³⁾ V_3^0 or V_2^0 which is tentatively denoted as Π^0 should have a half integral isotopic spin⁴⁾ in reference to the process

$$\pi^- + \rho \rightarrow V_1^0 + \Pi^0, (\Pi^0 \rightarrow \pi^+ + \pi^-). \quad (1)$$

If we assume that there is no doubly charged counter particle to Π^0 , the isotopic spin of Π should be 1/2. In such a case Π^+ and Π^0 are treated just as proton

and neutron so long as we are concerned with their transformation properties in isotopic space. Hence the Π^0 -particle should be described by a complex wave function as well as the charged Π -particle, and we must distinguish between the Π^0 -particle and its anti-particle $\tilde{\Pi}^0$. This distinction leads to many interesting results as we shall see later.

From the above isotopic spin assignment we have the following results.

(1) The "even-odd" rule⁵⁾ is an inevitable consequence of the charge independence. If both the spin and isotopic spin of a hot particle* are integer or half-integer we call it an even particle, whereas if only one of them is integer and the other is half-integer we call it an odd particle. The even-odd rule holds for such an even-odd assignment of hot particles. Hence the large abundance and the striking stability of the V -particles against π^- or γ -decay are automatically guaranteed. Recently Pais derived this rule from his own theory of the " ω "-space⁶⁾ by imposing the conservation of the ω -parity, while in the present work it is derived with less new elements.

(2) In production processes, we have the following conservation law valid for the charge independent and electromagnetic interactions

$$n(V_1) - n(\Pi) = \text{const.}, \quad (2)$$

where $n(V_1)$ is the no. of V_1 -particles minus the no. of anti- V_1 -particles and $n(\Pi)$ the no. of Π^+ and Π^0 minus the no. of $\tilde{\Pi}^-$ and $\tilde{\Pi}^0$. This law is proved as follows.

From the above isotopic spin assignment for V_1^- and Π -particles, we have

$$q = I_3 + 1/2(n(N) + n(\Pi)), \quad (3)$$

where q and I_3 are the total charge and the third component of the isotopic spin of the system of hot particles.

There is another conservation law, the conservation of baryons**

$$b = n(V_1) + n(N) = \text{const.} \quad (4)$$

Since q , b and I_3 are conserved for the charge independent and electro-magnetic interactions, we have from (3) and (4)

$$n(V_1) - n(\Pi) = b - 2(q - I_3) = \text{const.}$$

* By a "hot particle", we mean a particle with strong nuclear interaction.

** The "baryon" is the collective name for the members of the nucleon family. This name is due to Pais. See ref. (6).

* After the completion of this work, the authors knew in a private letter from Prof. Nambu to Prof. Hayakawa that Dr. Gell-Mann has also developed a similar theory.

Especially in pion-nucleus or nucleon-nucleus impacts, (2) can be written as

$$n(V_1) = n(\Pi) = n(\Pi^+, \Pi^0) - n(\tilde{\Pi}^-, \tilde{\Pi}^0). \quad (5)$$

It must be noticed that in cases of heavy nuclei the Coulomb effects cannot be discarded and hence the validity of the conservation law for the electromagnetic interaction is necessary.

The conservation law (5) can forbid many processes, e.g.,

$$\pi^- + p \rightarrow V_1^+ + \tilde{\Pi}^-, \quad (6)$$

$$N + N \rightarrow V_1 + V_1, \text{ etc. } (N: \text{nucleon}). \quad (7)$$

Hence the production of V -particles in nucleon-nucleon collisions will be due to the processes such as

$$N + N \rightarrow N + V_1 + \Pi, \quad (8)$$

or
$$N + N \rightarrow N + N + \Pi + \tilde{\Pi}. \quad (9)$$

(3) Since the process (7) is forbidden, we may conjecture that the production of V -particles will result mainly in pion-nucleon (or nucleus) collisions rather than in nucleon-nucleon (or nucleus) collisions at energies where the cosmic ray experiments are being performed, i.e. at about 10 Mev in the laboratory system and about 2 Mev in the centre of mass system, in conformity with the experimental viewpoint.⁷⁾

(4) From the rarity of the σ -stars produced by negative heavy mesons the Bristol group⁸⁾ conjectured that the positive K -particles might be much more abundant than the negative ones.

Since the V_1 -particles are supposed to be more easily produced than the Π -mesons because of the lower excitation energy to transform a nucleon into a V_1 -particle than the energy required to create a heavy Π -meson, the production of V_1 -particles in high energy nuclear events will occur in such a manner that $n(V_1)$ assumes as large a value as possible for a fixed value of $n(\Pi^+, \Pi^0) + n(\tilde{\Pi}^-, \tilde{\Pi}^0) + n(V_1)$. Then from eq. (5), we may expect

$$n(V_1) \sim n(\Pi^+, \Pi^0) \gg n(\tilde{\Pi}^-, \tilde{\Pi}^0). \quad (10)$$

If we identify the K -particles with the Π -particles, then the conjecture of the Bristol group can be interpreted in terms of the relation (10). However, in the cloud chamber experiments both the positive and negative V -particles lighter than nucleon are observed comparably.⁹⁾ Thus it is an important problem to settle how many kinds of charged V -particles are present in nature.*

(5) The selection rules imposed by the charge conjugation and charge symmetry¹⁰⁾ cannot be applied to Π -meson decays. Since the real and imaginary

parts of the complex wave function of Π^0 have opposite parities under charge conjugation, we cannot apply the selection rule

$$n(v) + n(t) = \text{odd is forbidden}, \quad (11)$$

for transition among neutral Bosons, to Π^0 .

For charged Π -particles the CT transformation¹⁰⁾ when applied to Π^\pm alters its charge state, so that in this case, too, we cannot apply the second selection rule

$$n(v) + n(t) + n(\tau_3) = \text{odd is forbidden}, \quad (12)$$

for non-radiative transitions, to Π^\pm .

The authors thank Prof. S. Hayakawa for his kind criticism.

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The Distribution Function of Degenerating Ensemble (An Addition to "Generalization of Statistics")¹⁾

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November, 11, 1953

Conventionally, $2 \times$ Fermi's distribution function has been used for electron ensemble neglecting its spin.

The reason why it is applicable is as the followings.

Let x be a physical quantity now in question, and w Fermi distribution, then the mean value of the quantity is given by

$$\bar{x} = \sum_{\sigma} \int x \{ \epsilon(\sigma) \} w \{ \epsilon(\sigma) \} d\epsilon, \quad (1)$$

where ϵ and σ denotes its energy and spin respectively. Now $\epsilon(+1) = \epsilon(-1)$ for two directions of spin.

Taking first summation about σ , there appears automatically factor 2.

However, since it should be taken into account that Fermi distribution function is calculated under the condition that a energy state should be occupied by one and only one particle, it is dangerous to put $\epsilon(+1) = \epsilon(-1)$ without further proof.

And then we should start from the first, taking the fact into account that a state can be occupied by two particles.

For simplicity, we treat an ensemble of Fermi particles with spin 1/2. However we can extend to general case of capacity n as it is.

In our case $n=2$. We shall split the problem into two cases: (A) ideal gases so entirely free that there is no interaction among them, and (B) real gases that there are interaction among them.

(A) Ideal gases. There is no confusion in this case to get the distribution function straight forward.

But in order to discuss the problem parallel to the second case (B), we shall calculate the number of independent proper functions of the ensemble.

The statistical circumstances depend upon not only occupation number but also how many pairs of antiparallel spins are, or the total spin momentum of the ensemble.

To that purpose, let us calculate the number of independent proper functions when m_2 pairs of spins are antiparallel and m_1 spins parallel:

$$N = 2m_2 + m_1, \quad (2)$$

where N denotes the total number of Fermion.

The total spin momentum S is $m_1/2$ and it has $2S+1 = m_1+1$ Z -components, and further, each component is accompanied with d_{m_1, m_2} independent proper function, where d_{m_1, m_2} denotes the dimension of the representation $\left\{ 2, 2, \dots, 2, 1, \dots, 1 \right\}$ of symmetry group of the $N!$ th order.

(a) In the case when N particles occupy N different states respectively the combinations of spins of the particles are all possible. The total number of proper functions is

$$P_{0, N} = \sum_{m_1'=0 \text{ or } 1}^N (m_1'+1) d_{m_1', m_2'}, \quad N = 2m_2' + m_1',$$

where m_1' begins from 0 or 1 according to N even or odd.

(b) In the general case when there are m_2 pairs in each same state and m_1 singles in each different state the problem is entirely reduced to $N - 2m_2 = m_1$ bodies problem. The number of proper functions is $P_{m_2, m_1} = \sum_{m_1'=0 \text{ or } 1}^{m_1} (m_1'+1) d_{m_1', m_2'}$, $m_1 = 2m_2' + m_1'$ where $d_{m_1', m_2'} (m_1 = m_1' + 2m_2')$ denotes the dimension of the representation of symmetric group given by

$$m_1! (m_1'+1) / (m_1' + m_2' + 1)! m_2'! \\ = m_1! C_{m_1' + m_2'}^{m_1'} - m_1! C_{m_1' + m_2'}^{m_2' + 1}$$

by the virtue of Frobenius-Shur's dimension formula. P_{m_2, m_1} is simply reduced to the form $P_{m_2, m_1} = 2^{m_1} (N - 2m_2 + m_1)$.

The total number of the complexions C is $\sum P_{m_2, m_1}$.

Denoting the total energies of the ensemble as E and the occupation number in energy level ϵ_r as n_r , we add the conditions

$$\sum n_r = N, \quad 2m_2 + m_1 = N, \quad \sum n_r \epsilon_r = E. \quad (3)$$

Introducing the generating function $g(\epsilon_r) = 1 - 2x_2 \epsilon_r x_2^{-2} 2\epsilon_r$, we have the following distribution function after Fowler's method²⁾

$$W(\epsilon) = 2 \frac{e^{(\eta - \epsilon)/kT}}{1 + e^{(\eta - \epsilon)/kT}}, \quad (4)$$

which is precisely equal to 2 Fermi's distribution.

(B) Real gases. The distribution function of the entirely free ensemble of Fermi particles with spin 1/2 has shown nothing but the customary.

In the real ensemble with interactions, however, the circumstances suddenly change their aspects.

Because the total energies E of the ensemble depend upon the total spin momentum $S=m_1/2$ and moreover, each state with total spin $m_1/2$ has d_{m_1, m_2} different and m_1+1 fold total energies $E_{m_1, N}^i$ where i runs from 1 to d_{m_1, m_2} .

These facts are due to the exchange force caused by spin character. If we neglect the quantum mechanical force, for instance the exchange forces, the results will be the same as the case (A).

For instance, the existence of the pairs with the antiparallel spins in the same energy level comes to consequence to reduce the total binding energies of the ensemble, and the existence of the parallel spins in different energy levels comes to result to increase the total energies of the ensemble, which affects the ensemble to occupy a energy level with two particles more easily and with one particle less easily.

The total energies $E_{m_1, N}^i$ consist from the two parts E and $\epsilon_{m_1, N}^i$; $E_{m_1, N}^i = E + \epsilon_{m_1, N}^i$ where i runs from 1 to d_{m_1, m_2} , E denotes the usual total energies including the kinetic energy and coulomb energy etc., but exchange energy, and $\epsilon_{m_1, N}^i$ denotes the exchange energy only.

Therefore E must be expressed by the sum of the energy of individual particles; $E = \sum \epsilon_r$, ϵ_r denotes the usual energy of the r th particle.

To construct the formula analogous to Fowler's we may regard $z_{m_1, N}^i$ as the weight of that state of the ensemble, where Z is any how to be in fate to put $Z = e^{-1/KT}$ later, and so we shall take the weight as $e^{-\epsilon_{m_1, N}^i/KT}$.

(a') In the case corresponding to the case (a) in (A), the total weight is

$$P_{0, N} = \sum_{m_1'=0 \text{ or } 1}^N (m_1'+1) \left\{ \sum_{i=1}^{d_{m_1', m_2'}} e^{-\epsilon_{m_1', m_2'}^i/KT} \right\} \\ \times N = 2m_2' + m_1'.$$

(b') In the case corresponding to the case (b) in (A), the total weight is

$$P_{m_2', m_3'} = \sum_{m_1'=0 \text{ or } 1}^{m_1} (m_1'+1) \left\{ \sum_{i=1}^{d_{m_1', m_1}} e^{-\epsilon_{m_1', m_1}^i/KT} \right\}^*, \\ m_1 = m_1' + 2m_2'.$$

Then introducing two functions

$$g(\epsilon_i) = 1 + (x/u)z^{\epsilon_i} + (x/v)^2 z^{2\epsilon_i}$$

and

* $\epsilon_{m_1', m_2'}^i$ depends not only to m_1' , and m_2' but also N .

$$f(u, v^2) = P_{0, N} u^N + P_{1, N-2} v^2 u^{N-2} + \dots \\ + P_{m_2, m_1} v^{2m_2} u^{m_1} + \dots +$$

the total complexion C subject to the condition (3) is equal to just the coefficient of $x^N z^N$ in $\prod_i g(\epsilon_i) f(u, v)^2$.

Considering x, z, u, v as the complex number, and integrating a contour enclosing the origin, we have

$$C = \left(\frac{1}{2\pi i} \right)^4 \\ \times \iiint \prod_i g(\epsilon_i) f(u, v^2)^2 z^{E+1} x^{N+1} u^N v^N dx du dv. \quad (5)$$

The distribution function is obtained

$$\bar{n}_\epsilon = \frac{(\lambda/a)\theta^\epsilon + 2(\lambda^2/\beta^2)\theta^{2\epsilon}}{1 + (\lambda/a)\theta^\epsilon + (\lambda^2/\beta^2)\theta^{2\epsilon}}, \quad (6)$$

where $\lambda, \theta, a, \beta$, are unique positive real roots in following equations and correspond to x, z, u, v , respectively.

$$\Phi(x, z, u, v) = \prod_i (g(\epsilon_i)) f(u, v^2)^2 z^{E+1} x^{N+1} \\ \frac{\partial \Phi}{\partial x} = 0, \quad \frac{\partial \Phi}{\partial z} = 0, \quad \frac{\partial \Phi}{\partial u} = 0, \quad \frac{\partial \Phi}{\partial v} = 0. \quad (7)$$

θ is presumed to the function of temperature and put $\theta = e^{-1/KT}$ after Fowler.

By the suitable choice of η , and ζ function of T , we have

$$n_\epsilon = I^1(\epsilon) = \frac{e^{(\epsilon-\eta)/KT} - 2e^{2(\epsilon-\zeta)/KT}}{1 - e^{(\epsilon-\eta)/KT} - e^{2(\epsilon-\zeta)/KT}}. \quad (8)$$

In more general case n , similarly we have for the entirely free ensemble,

$$I^1(\epsilon) = \frac{\sum_{m=0}^n m {}_n C_m e^{-m(\epsilon-\eta)/KT}}{\sum_{m=0}^n {}_n C_m e^{-m(\epsilon-\eta)/KT}} \\ = -KT \frac{\partial}{\partial \epsilon} \log(1 + e^{-(\epsilon-\eta)/KT}) \\ = n \frac{e^{-(\epsilon-\eta)/KT}}{1 + e^{-(\epsilon-\eta)/KT}}, \quad (9)$$

which is just equal to n times Fermi's distribution.

And for real ensemble with interactions

$$W(\epsilon) = -KT \frac{\partial}{\partial \epsilon} \log[1 + e^{(\eta_1-\epsilon)/KT} - \dots - \\ - e^{n(\eta_n-\epsilon)/KT}] \quad (10)$$

where η_1, \dots, η_n are, in general,* function of T and

* At any rate, $\eta_m = {}_n C_m e^{-\bar{\epsilon}_m}$, where $\bar{\epsilon}_m$ means "the mean quantum mechanical" energy per one of m particles in the same state.

determined by the circumstances of the degenerations and interactions.

To stop his pen, the author expresses his thanks to Prof. M. Kobayasi.

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An Interpretation of Asymmetric Fission

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October 14, 1953

Roughly speaking, one of the characteristic features of nuclear fission may be that, when the energy of the incident neutron is low, fission occurs asymmetrically while in the case of high energy, it occurs symmetrically. Though this is a well-established experimental fact, a considerable time was spent in the theoretical study without satisfactory result. The first attempt was made by Present and Knipp¹⁾ who extended the potential energy series to higher order terms in a_2 and a_4 than the 4-th order in a_2 , including the coupling terms between even and odd harmonics. Such a homogeneous liquid drop model will fail in the interpretation of asymmetric fission was shown by Frankel and Metropolis.²⁾

Several other attempts were brought forward.³⁾⁻⁵⁾ But there were scarcely any dynamical theory. This is one present feature of the theory of asymmetric fission. Another feature may be that various theories resting on the assumption of uniformly distributed nuclear density seems to be unsuccessful to account for asymmetric fission. Thus it may be of significant

to try to interpret the asymmetric fission on the basis of hydrodynamics of an inhomogeneous liquid drop model, in spite of the fact that the drop model fails to interpret any reaction in detail, even fission itself.

Indeed it seems that apart from the prompt neutrons the droplet model, when it is described suitably by quantum mechanics, fails mainly in the incident channel. As for the prompt neutrons, the writers consider that this is the very problem to be handled by the use of "collective model". This situation may in a sense resemble to the theory of superconducting phenomena, in which the coupling between lattice vibration and motion of electrons plays an important role.

No numerical calculations have been made, since we cannot expect much from pure hydrodynamical considerations. A very simple model is taken, as otherwise the treatment will be very complicated. We do not consider the model taken in this article is essential, but consider it is only handy one. Here we want only to express that one course of asymmetric fission may be inhomogeneity of the nuclear matter. It is another problem to calculate the distribution. Perhaps we can do it from statistical considerations like Thomas and Fermi, taking into account of the influence of surface vibration on Fermi distribution.

In the case of uniformly distributed nuclear matter the deformation of the order one does not occur can be shown by the requirement of center of the gravity or by the expression¹⁰⁾ for frequencies:

$$\omega_n = \{n(n-1) [(n+2) - 10\gamma/(2n+1)] T' / (3r_0^2 MA)\}^{1/2}, \quad (1)$$

where

$$\gamma = (3/5) (e^2 Z^2 / r_0 T A)$$

and

$$\text{surface energy} = T A^{2/3}.$$

Then how will be the equation of motion when the mass distribution is not uniform?

For simplicity, we consider here the initial small vibration of "ocean" of uniform depth,¹¹⁾ covering a spherical core, radius of which is denoted by C . Here the calculation assumes that the core is fixed and the charge distribution is proportional to the mass distribution. This fixed core assumption may hold good in the initial small vibration, which will determine the shape for large deformation. For such a case we have an equation of motion:

$$\frac{d^2 a_n}{dt^2} + \frac{n(n+1) \{ (R_0/c)^n - (c/R_0)^{n+1} \}}{R_0^2 \{ (n+1) (R_0/c)^n + n(c/R_0)^{n+1} \}} \times \left\{ \frac{(n+2)(n-1)}{\rho} - \frac{T'}{4\pi r_0^3 A^{1/3}} \right\}$$

* Professor J. A. Wheeler remarked about the influence of the difference between the radius for the protons and the neutrons at the International Conference, held in Kyoto, Sept. 1953. He stated "Protons will undergo fission before neutrons undergo fission. And it is considerable that this circumstance gives an opportunity for shell effect to work in a way, which is not obvious from the ordinary liquid drop picture in which both particles treated in a same way."

$$-\frac{5}{3} \frac{R_0^2 \gamma T'}{r_0^2 M A} \left(1 - \frac{3}{2n+1} \frac{\rho}{\rho_0} \right) a_n = 0. \quad (2)$$

where ρ , ρ_0 and R_0 stands for the density of the surrounding part, the mean density of the core and the surrounding part combined and the mean radius respectively. For the pressure at the free surface we have

$$p = (n-1)(n+2) \left(\frac{\gamma T'}{4\pi r_0^3 A^{1/3}} \right) a_n P_n \quad (3)$$

from the calculation of radius of curvature. The potential energy due to the Coulomb force at the free surface is given by

$$\Omega = - \left(\frac{R_0^2}{2} \right) \left(\frac{10\gamma T'}{3r_0^2 M A} \right) \left(1 - \frac{3}{2n+1} \right) \times (\rho/\rho_0) a_n P_n \quad (4)$$

In this case, frequency for $n=1$ is given by

$$\omega_1 = \left[\frac{2 \{ (R_0/c) - (c/R_0)^2 \}}{2(R_0/c) + (c/R_0)^2} \left(\frac{\rho}{\rho_0} - 1 \right) \times \frac{5}{3} \frac{\gamma T'}{r_0^2 M A} \right]^{1/2} \quad (5)$$

Thus if $\rho_0 > \rho$, this deformation will increase exponentially,

$\rho_0 = \rho$, this deformation will never occur,

$\rho_0 < \rho$, this deformation will vibrate.

If in the course of vibration the core "melt" gradually, owing to the "conduction of heat" from surrounding part to the core, the deformation of the order one will stop. Then the deformation of the order 2 will become predominant. This process will give rise to asymmetric fission. If the excitation of the nucleus is so high that there is no core at the initial time, then the deformation of the nucleus will give rise to symmetric fission. Although here we assumed the existence of an spherical core, this is done so mainly for simplicity sake. As previously stated, what we want to express is that the asymmetric fission may come from a slight inhomogeneity of nucleus. From these considerations, one is suggested more or less to take into account of the influence of compressibility of the nucleus.

Now let us proceed to a non-concentric case with an axial symmetry. Notations are; radius of the core: C , distance between the center of gravity of the total system and the center of the core: r_0 . See Fig. 1. Velocity potential in this case for $C > r_0$, is given by

$$\phi = \sum_n \left(\sum_m a_{nm} r^m + (1/r^{n+1}) \right) B_n P_n(\mu), \quad (6)$$

where a_{nm} is the expansion coefficient of $\eta_n(\mu)$ by $\xi_m(\mu)$:

$$\eta_n(\mu) = \sum_m a_{nm} \xi_m(\mu), \quad (7)$$

η_n and ξ_m being given by

$$\begin{aligned} \eta_n(\mu) = & \{ (n+1) P_n(\mu) (P_1(\mu') P_1(\mu) \\ & + P_1^1(\mu') P_1^1(\mu)) - P_n^1(\mu) (P_1^1(\mu') P_1(\mu) \\ & - P_1^1(\mu) P_1(\mu')) \} f_{-(n+2)}(\mu), \end{aligned}$$

and

$$\begin{aligned} \xi_n(\mu) = & \{ n P_n(\mu) (P_1(\mu') P_1(\mu) \\ & + P_1^1(\mu') P_1^1(\mu)) + P_n^1(\mu) (P_1^1(\mu') P_1(\mu) \\ & - P_1^1(\mu) P_1(\mu')) \} f_{n-1}(\mu). \end{aligned}$$

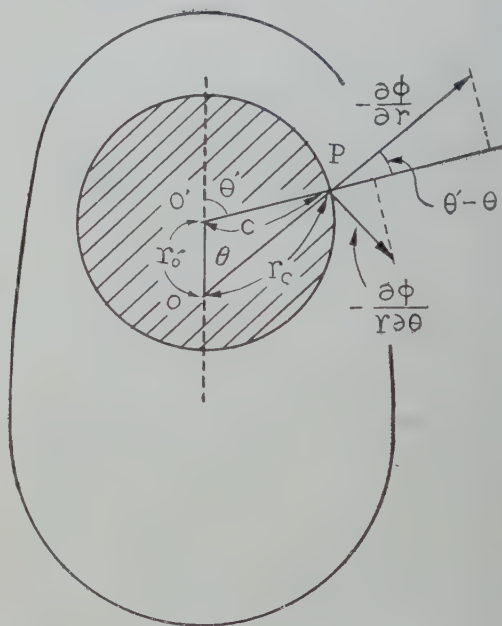


Fig. 1

Non-concentric axial symmetric deformation.

O : the center of gravity of the total system.

O' : the center of the core.

where $f_m(\mu)$ stands for $(r_0 \mu \pm \sqrt{(r_0 \mu)^2 - (r_0^2 - c^2)})^m$, and μ and μ' represents cosine of the polar angle, the origin being at the center of gravity of the total system, and at the center of the core respectively. It is easy to show that (6) satisfies the boundary condition at the core surface

$$-\frac{\partial \phi}{\partial n_o} = 0, \quad n_o: \text{the normal direction from the center of the core.} \quad (8)$$

Assuming the pressure divided by the density near the surface plus the Coulomb potential at the free surface, a particular case of which is given by eq. (3), plus eq. (4), is expressed by

$$P = \sum_n k_n a_n P_n, \quad (9)$$

we get the equation of motion in this case

$$\begin{aligned}
\frac{d^2 a_n}{dt^2} = & - \left\{ \sum_m m a_{mn} R_0^{m-1} - (n+1) R_0^{-(n+2)} \right\} \\
& \times k_n a_n / \left(\sum_m a_{mn} R_0^m + R_0^{-(n+1)} \right) \\
& + \sum_m \left\{ m - \left(\sum_m a_{mn} R_0^{m-1} - (n+1) R_0^{-(n+2)} \right) \right. \\
& \times R_0 / \left(\sum_m a_{mn} R_0^m + R_0^{-(n+1)} \right) \left. \right\} \\
& \times R_0^{m-1} (\partial a_{mn} / \partial t) \\
& \times (da_n / dt) / \left(\sum_m a_{mn} R_0^{m-1} - (n+1) R_0^{-(n+2)} \right).
\end{aligned} \quad (10)$$

If we can get explicit expressions for a_{mn} and k_n exactly or numerically, we know the manner of the surface vibration. The coefficient for da_n/dt of the last term of the above equation represents the coefficient of viscosity for the surface vibration.

When the core gradually move to the outside, the distance between the center of the core and that of gravity of the total system will be equal to the radius of the core. After this, the velocity potential becomes

$$\phi = \sum_{n=0}^{\infty} A_n r^n P_n(\mu). \quad (11)$$

The boundary condition (8) that the surrounding part must not flow into the core requires each A_n should vanish. This shows that the location of $C=r_0$ is a critical one and the nuclear surface vibrates no more. Thus for this location, the kinetic energy becomes zero. Note that the radius of the core may perhaps depends on the energy available. Excitation energy will be consumed partly to melt the core away, partly to climb up to the ridge of the barrier and partly to emit particles, neutrons and in some cases alpha

particles also. When the excitation of the nucleus is moderately high after emission of prompt neutrons, the melting process will proceed, causing asymmetric fission. Asymmetry thus depends upon the available energy i.e. the higher it becomes, the smaller asymmetry one gets. When the excitation of the nucleus is so high that the melting process occurs instantaneously, fission becomes symmetric.

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On the Duffin-Kemmer Algebra*

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(Received October 27, 1953)

Hitherto unnoticed algebraical features of the Duffin-Kemmer algebra, the so-called meson algebra, have been exhaustively brought to light in the general case of n dimensions. Stress is laid on treating it quite generally as an abstract algebra rather than as a matrix algebra, so that no reference is made to explicit representations. The main point consists in explicit construction of the $2n+1C_n$ linearly independent bases of the algebra each transforming as an antisymmetric tensor for all orthogonal transformations of the n -dimensional β -space. From the physical point of view this tensor character is of paramount importance, for it enables one to get an adequate apparatus for extracting any component from the wave function and accordingly to manifest the direct correspondence between the two alternative modes of formulation of meson theory, the particle formulation and the more usual wave formulation. Recently A. Klein has proposed an interesting procedure to extract a specific component from the meson wave function, which is now shown to be derivable from the present standpoint and at the same time a considerable amount of information is newly obtained in the Dirac algebra.

§ 1. Introduction

The present paper is devoted mainly to the investigation of the generalized Duffin-Kemmer algebra in n dimensions, that is, the enveloping algebra of n abstract quantities $\beta_1, \beta_2, \dots, \beta_n$ governed by the commutation relations

$$\beta_\mu \beta_\lambda \beta_\nu + \beta_\nu \beta_\lambda \beta_\mu = g_{\mu\lambda} \beta_\nu + g_{\nu\lambda} \beta_\mu, \quad (1)$$

together with the unit E . Since it was first announced by Duffin that the special case of four dimensions was quite adequate for synthetic presentation of the meson theories for spin one and spin zero,¹⁾ various attempts have been made by several authors to reveal its algebraic properties parallelly with the endeavours after an extensive use of this mathematical device for the meson theory. Each of these works, however, has its own limitation which is to be ascribed just to our insufficient knowledge concerning the structure of this algebra.

In his original work Kemmer developed the physical content implied in the four-dimensional β -formalism and alluded to some of its mathematical aspects.²⁾ Passing through Schrödinger's work treating the case of five dimensions,³⁾ Kemmer succeeded himself in

*) A preliminary treatment of this research concerning the special case of four dimensions is seen in Japanese journal: I. Fujiwara, *Soryushiron Kenkyu* 4, No. 4 (1952), 1. The main results of the present considerations were presented at the annual meeting of the Physical Society of Japan held at Kyoto, October 30, 1952.

fully elucidating the orderly structure of the meson algebra in the general case of n dimensions.⁴⁾ The difficulty in this line of approach lies however in that the analysis presupposes an *a priori* correspondence between the particle formulation in terms of the β -matrices and the more usual wave formulation. Properly speaking, the nature of the algebra should be determined by eq. (1) alone quite independently of any considerations of the linearized wave equation as the coefficients of which β -matrices were originally introduced into meson theory. Not only that, the latter formulation is to be derivable from the former on the basis of the fundamental commutation relations (1). Then with the very intention of manifesting the direct correspondence between the two alternative modes of formulation, the so-called I' -formalism was originated by Harish-Chandra.⁵⁾ Though we are not to dare to deny its usefulness for the case when the meson-nucleon interaction is taken into account, the introduction of I' -operators impairs considerably the elegance inherent in the particle formulation which is, strictly speaking, to be operated exclusively by the β -matrices. The present research is thus originally intended to replace such quantities introduced in order to extract any component from the wave function by a certain set of projectors constructed out of the β -matrices only. As was pointed out by the same author,⁶⁾ the primary feature of his formalism consists in that suitable combinations of I' -operators furnish a linearly independent basis of the algebra. This point was after extended to the general assertion that in the algebra of arbitrary spin the linearly independent elements can be chosen so that they transform as tensors for all orthogonal transformations of the β -space and in addition he determined the general conditions which they must satisfy.⁷⁾ In carrying out the above design we are therefore necessarily to conform to the lines laid out in these works and to enter into a more thorough investigation of the structure of the algebra.

For this purpose it is essential to observe that, as was emphasized by Madhavarao and others in the general case of arbitrary spin,⁸⁾ the commutative η -algebra plays a decisive part in determining the structure of the total algebra. Recently an ingenious treatment of this subalgebra has been given by Hönl and Boerner in constructing fusion-theoretically explicit representations of the β -matrices (four dimensions).⁹⁾ Then in his short note Peaslee has proposed a certain set of projectors which is incomplete as it stands because of a certain dichotomy in the treatment of the spin-zero and spin-one theories.¹⁰⁾ Suggested by these works the author has arrived at a final solution wherein all the defects of the preceding researches have been removed and whereby hitherto undiscovered features of the algebra have been exhaustively brought to light.

In this paper stress is laid on redering every stage of considerations as general as possible: First the algebra is treated in the general case of n dimensions, so that all the aspects have been revealed which were inaccessible within narrow limits of the usual four-dimensional treatment. Secondly it is regarded as an abstract algebra rather than as a matrix algebra, so that no reference is made to explicit representations and there is moreover no need for separate treatment of each irreducible representation. Of course the algebraic construction proceeds quite independently of any considerations of the Duffin-Kemmer wave equation.

In the next section we are first concerned with elucidating the general structure of the algebra. A full decomposition of the unit E into normal primitive idempotents is performed and their partial sum is taken so that it affords the unit element of each of the irreducible two-sided invariant subspaces or the simple algebras of which the original algebra is the direct sum. The orders of the simple algebras are then determined by the numbers of primitive idempotents comprised in corresponding unit elements and as their total sum the order of the whole algebra is given by $2n+1C_n$. Next § 3 is devoted to explicit construction of the linearly independent basis of each simple algebra and accordingly to a more thorough investigation of its structure. With the aid of a set of new elements $Q_{\mu;\nu}$ derived directly from eq. (1), we can construct a number of quantities each having the form of an antisymmetric tensor of certain rank. Then on multiplication by a suitable one of the above primitive idempotents each one of them yields the linearly independent element of the algebra transforming as an antisymmetric tensor for all orthogonal transformations $\beta'_\nu = \alpha_\nu^\lambda \beta_\lambda$ of the n -dimensional β -space.* In the case of odd dimensions we encounter here a certain anomaly coming from the existence of a pair of simple algebras termed "*twin algebras*" of which detailed accounts are given in § 4. In § 5 it is shown on the basis of the above transformation character that each linearly independent element works as the required projector and with the aid of this projector we can further establish the direct correspondence between the two alternative modes of formulation of meson theory. Incidentally we have here made use of an explicit expression for the well known spin transformation operator $S(a)$ such that $S^{-1} \beta_\nu S = \alpha_\nu^\lambda \beta_\lambda$ corresponding to the proper ones of the above transformations.

In a recent paper A. Klein has proposed an alternative procedure of extracting a specific component from the meson wave function which is based essentially on the fusion-theoretical standpoint.¹¹⁾ Although it may seem at first sight to have no bearing on the present solution, it is systematically derivable from ours as is shown in § 6. For this purpose it is necessary to make a closer investigation of the Dirac algebra and so we have contributed a considerable amount of new information on its properties. Above all it contains an alternative method for obtaining the explicit expression of $S(a)$ in the case of Dirac algebra.

We have thus taken a unified view of the orderly structure of the Duffin-Kemmer algebra and at the same time avoided the invocation of the Γ -formalism, but the whole plan of the present procedures forms essentially an exemplification of the abstract scheme due to Harish-Chandra in the special case of the meson algebra. Finally we hope that the present considerations afford a reliable guide for the future investigation of the algebras related to elementary particles.

*) Only the real orthogonal group is considered throughout in this paper, but the corresponding problem concerning other real groups (for example, the Lorentz group) associated with indefinite ground forms, is intimately connected with the present one, and has an almost identical solution (see reference 7). The $g_{\mu\nu}$ is therefore taken to be equal to one or zero according as $\mu=\nu$ or not and the $a_{\mu\nu}$ satisfies the orthogonality condition $a_\mu^\lambda a_\nu^\rho g_{\lambda\rho} = g_{\mu\nu}$.

§ 2. General structure of the Duffin-Kemmer algebra

In this section we shall be mainly concerned with elucidating the general structure of the generalized Duffin-Kemmer algebra in n dimensions. For this purpose it is most essential to observe that the commutative subalgebra termed η -algebra which is generated by n quantities $\eta_\rho = 2\beta_\rho^2 - E$ constitutes in fact the framework of the whole β -algebra. As is easily seen from eq. (1), the η_ρ commute with each other and on squaring every η_ρ we get the unit E . The unit E is thus fully decomposable into 2^n mutually normal idempotents

$$e(\epsilon) = (1/2^n) \prod_{\rho=1}^n [E + \epsilon_\rho \eta_\rho], \quad (2)$$

where ϵ stands for a set of n variables ϵ_ρ each assuming either of the two discrete values $+1$ and -1 . So far as the η -algebra is concerned, there is no question but that each idempotent $e(\epsilon)$ is primitive, that is, not to be split further into two normal idempotents. Therefore these together form a basis of this subalgebra. For example, in view of the relationship $\eta_\rho e(\epsilon) = \epsilon_\rho e(\epsilon)$ any element $F(\eta)$ defined as an arbitrary algebraic function of η_ρ may be expressed as a linear combination of the $e(\epsilon)$:

$$F(\eta) = F(\eta)E = \sum F(\eta)e(\epsilon) = \sum F(\epsilon)e(\epsilon),$$

where 2^n numbers $F(\epsilon)$ furnish the eigenvalues of $F(\eta)$. Among others the simplest one is the unit E for which every eigenvalue is equally unity.

Then we are further to investigate the nature of the above idempotents from a more general standpoint of the whole algebra. With the aid of the commutation relations

$$\eta_\rho \beta_\rho = \beta_\rho \eta_\rho = \beta_\rho \quad \text{and} \quad \eta_\lambda \beta_\rho + \beta_\rho \eta_\lambda = 0 \quad \text{for } \lambda \neq \rho$$

the following relations will readily be established. The $e(\epsilon)$ with negative ϵ_ρ vanishes when multiplied by β_ρ , but the $e(\epsilon)$ with positive ϵ_ρ satisfies

$$\beta_\rho e(\epsilon) = e(\epsilon') \beta_\rho \quad (3)$$

with $\epsilon'_\rho = \epsilon_\rho$ and $\epsilon'_\lambda = -\epsilon_\lambda$ for $\lambda \neq \rho$. According to the identity $\beta_\rho^2 = (1/2)[E + \eta_\rho]$ this can also be rewritten as $\beta_\rho e(\epsilon) \beta_\rho = e(\epsilon')$. The 2^n idempotents $e(\epsilon)$ can then be classified into $(n+1)$ classes according to how many of the n variables ϵ_ρ comprised in each of them are positive. If j be the common number of positive ϵ_ρ , there are ${}_nC_j$ ones subsumed in this class, of which the total sum will hereafter be referred to as R_j , and we have another decomposition of E into $(n+1)$ normal idempotents R_0, R_1, \dots, R_n . Especially the R_0 is nothing but the special $e(\epsilon)$ wherein every ϵ_ρ is negative, so that we have identically

$$\beta_\rho R_0 = R_0 \beta_\rho = 0; \quad (4)$$

it is commutable as well as anticommutable with every β_ρ and constitutes by itself the *trivial algebra* of order one. Then for $j \neq 0$, eq. (3) implies that

$$\beta_\rho R_j = R_{n+1-j} \beta_\rho, \quad (5)$$

so that $R_j - R_{n+1-j}$ anticommutes with every β_p . So far as the even case ($n=2m$) is concerned, $E_j = R_j + R_{n+1-j}$ ($1 \leq j \leq m$) together with $E_0 = R_0$ form the basis of the centrum and $E = E_0 + E_1 + \dots + E_m$ is the normal idempotent decomposition of E which stops short with the simple algebras of which the original algebra is the direct sum. In the odd case ($n=2m-1$), however, we encounter a strange situation. For eq. (5) shows first that the special R_j for $j=m$ commutes by itself with every β_p . Moreover as has already been pointed out by several authors, a pseudoscalar element

$$\omega = i^{n(n-1)/2} \{m!(n-m)!\}^{-1} \epsilon^{p_1 p_2 \dots p_n} \beta_{p_1} \beta_{p_2} \dots \beta_{p_n} \quad (6)^*$$

satisfies in both even and odd cases the two conditions $\omega R_j = R_j \omega = g_{jm} \omega$ and $\omega^2 = R_m$. Furthermore in the even case it can replace any one of the n elements β_p in eq. (1),** whereas in the odd case it is commutable with every β_p . Hence for odd n , $R_m (=E_m)$ can again be split into a pair of normal idempotents $E_m^{(\pm)} = (1/2)[R_m \pm \omega]$ which are commutable with every β_p , so that $E_0 = R_0$, $E_j = R_j + R_{n+1-j}$ ($1 \leq j \leq m-1$), $E_m^{(+)}$ and $E_m^{(-)}$ form the basis of the centrum. Now the decomposition of E into $E = E_0 + E_1 + \dots + E_{m-1} + E_m^{(+)} + E_m^{(-)}$ results in splitting up the total algebra into the irreducible two-sided invariant subspaces. Especially the pair of simple algebras generated by $E_m^{(+)}$ and $E_m^{(-)}$ which are derived from the single R_m shall hereafter be called the "twin algebras" according to Kemmer.⁴⁾ Above all the most striking difference between even and odd cases is that every $e(\epsilon)$ is primitive for even n , whereas this is not the case for odd n , where corresponding to the decomposition of R_m into two mutually normal parts each one of the nC_m idempotents $e(\epsilon)$ with m positive ϵ_p splits again into a pair of normal primitive idempotents:

$$e(\epsilon) = R_m e(\epsilon) = E_m^{(+)} e(\epsilon) + E_m^{(-)} e(\epsilon).$$

Finally the order $N(n)$ of the total algebra is obtained by summing up all the orders of the constituent simple algebras, each of which is inferred immediately from the number of primitive idempotents comprised in the corresponding unit element. For example, we find in the odd case the following result which agrees with that obtained by Kemmer:⁴⁾

$$N(n) = 1^0 + \sum_{j=1}^{m-1} (nC_j + nC_{j-1})^0 + 2(nC_m)^0 = \sum_{j=0}^n {}_{n+1}C_j \cdot nC_{n-j} = {}_{2n+1}C_n,$$

where use is made of the identities $nC_j + nC_{j-1} = {}_{n+1}C_j$ and $\sum_r {}_p C_r \cdot {}_q C_{s-r} = {}_{p+q}C_s$. It is interesting here to observe that in spite of all the above differences, the $N(n)$ is given by the same expression ${}_{2n+1}C_n$ for both even and odd cases. These considerations are in conformity with what is taught by the theory of abstract algebra or the ideal theory and concerning more details of algebraic treatments readers are requested to see, for example, the books cited in reference 14.

*) $\epsilon_{p_1 p_2 \dots p_n}$ is a tensor antisymmetric in every pair of indices with $\epsilon_{12 \dots n} = 1$. The summation convention for contracted indices is used throughout in this paper.

**) Strictly speaking this is true for the special simple algebra generated by E_m . Detailed discussions concerning all these facts are found in § 4.

In the followings we shall review some aspects of the above considerations with the aid of familiar (scalar) element $B = \beta_p \beta^p$. The application of this element to any $e(\epsilon)$ is equivalent to multiplying it numerically by the number of its positive ϵ_p , for if it be multiplied on either side by $\beta_p^2 = (1/2)[E + \eta_p]$, it survives only when its ϵ_p is positive and the β_p^2 is then absorbed in it. Therefore we find that $B \cdot R_j = j \cdot R_j$ and accordingly that

$$B = B \cdot E = B \sum_{j=0}^n R_j = \sum_{j=0}^n j \cdot R_j;$$

the B has thus $(n+1)$ eigenvalues $0, 1, 2, \dots, n$ and its minimum equation reads

$$\prod_{k=0}^n [B - k \cdot E] = \sum_{i=0}^n \left\{ \prod_{k=0}^n (i - k) \right\} R_i = 0.$$

If any one factor $[B - j \cdot E]$ is eliminated from this product, we have on symbolizing this procedure simply by $\prod_{k=0}^{n(j)}$,

$$\prod_{k=0}^{n(j)} [B - k \cdot E] / (j - k) = \sum_{i=0}^n \left\{ \prod_{k=0}^{n(j)} (i - k) / (j - k) \right\} R_i = R_j. \quad (7)$$

As is well known the most remarkable property of B is that

$$B\beta_p + \beta_p B = \sum_{\lambda=1}^n (1/2) \{ [E + \eta_\lambda] \beta_p + \beta_p [E + \eta_\lambda] \} = (n+1) \beta_p.$$

Accordingly $[(n+1)E - 2B]$ anticommutes with every β_p . This point agrees completely with the preceding considerations, for it can be rewritten as

$$[(n+1)E - 2B] = \sum_{j=0}^n (n+1-2j) R_j = (n+1) R_0 + \sum_{j=1}^n (n+1-2j) [R_j - R_{n+1-j}].$$

Of course its square

$$[(n+1)E - 2B]^2 = \sum_{j=0}^n (n+1-2j)^2 E_j$$

commutes with every β_p . More generally an element $[B - k \cdot E][B - (n+1-k)E]$ with any number k is commutable with every β_p :

$$\begin{aligned} 4[B - k \cdot E][B - (n+1-k)E] &= [(n+1)E - 2B]^2 - (n+1-2k)^2 E \\ &= \sum_{j=0}^n \{ (n+1-2j)^2 - (n+1-2k)^2 \} E_j = 4 \sum_{j=0}^m (j-k) \{ j - (n+1-k) \} E_j, \end{aligned}$$

where in the odd case it must be taken care of that $E_m = R_m$. Therefore we have

$$E_j = \prod_{k=0}^{m(j)} \frac{[B - k \cdot E][B - (n+1-k)E]}{(j-k) \{ j - (n+1-k) \}} \quad \text{for } 0 \leq j \leq m. \quad (8)$$

Moreover we have according to eq. (5)

$$\beta_p R_{n+1-j} \beta^p = \beta_p \beta^p R_j = B \cdot R_j = j \cdot R_j, \quad (9)$$

so that E_j can also be rewritten as

$$E_j = R_j + R_{n+1-j} = R_{n+1-j} + (1/j) \beta_p R_{n+1-j} \beta^p \quad (10)$$

excepting the special cases of the trivial algebra and the twin algebras. Finally we add the relations

$$R_j = \left\{ \frac{(n+1-j)-B}{(n+1-j)-j} \right\} E_j \quad \text{and} \quad R_{n+1-j} = \left\{ \frac{B-j}{(n+1-j)-j} \right\} E_j.$$

§ 3. Explicit construction of the linearly independent elements

To begin with we shall introduce a set of n^2 quantities

$$Q_{\mu;\nu} = g_{\mu\nu} E - \beta_\nu \beta_\mu, \quad (11)$$

each being specified by a pair of indices running through the integers 1, 2, ..., n . In terms of these elements eq. (1) can simply be rewritten as

$$\left. \begin{aligned} Q_{\lambda;\mu} \beta_\nu + Q_{\lambda;\nu} \beta_\mu &= 0 \\ \beta_\nu Q_{\mu;\lambda} + \beta_\mu Q_{\nu;\lambda} &= 0. \end{aligned} \right\} \quad (1')$$

or as

More generally we shall consider an element $Q_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j}$, a tensor of rank $2j$ having two independent sets of indices μ_k and ν_k ($k=1, 2, \dots, j$), which we shall hereafter abbreviate to $Q_{j;j}$. Then a sequence of such tensors with ascending values of j is generated by applying successively the recurrence formula

$$Q_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} = Q_{\mu_1 \mu_2 \dots \mu_{j-1}; \nu_1 \nu_2 \dots \nu_{j-1}} Q_{\mu_j; \nu_j} - \sum_{i=1}^{j-1} (-)^{j-1-i} g_{\nu_i \mu_j} Q_{\mu_1 \mu_2 \dots \mu_{j-1}; \nu_1 \nu_2 \dots (\nu_i) \dots \nu_j} \quad (12)^*$$

starting from the above $Q_{\mu;\nu}$ for $j=1$. Here we observe that this is equivalent to

$$Q_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} = Q_{\mu_1; \nu_1} Q_{\mu_2 \mu_3 \dots \mu_j; \nu_2 \nu_3 \dots \nu_j} - \sum_{i=2}^j (-)^{i-2} g_{\nu_1 \mu_i} Q_{\mu_1 \mu_2 \dots (\mu_i) \dots \mu_j; \nu_2 \nu_3 \dots \nu_j}; \quad (12')$$

for provided that it is true for a special value of j , it is readily seen that the same is also valid for another value of j larger by one, while in the initial case $j=2$ these equations reduce to one and the same relation

$$Q_{\mu\lambda;\nu\rho} = Q_{\mu;\nu} Q_{\lambda;\rho} - g_{\nu\lambda} Q_{\mu;\rho}$$

The most remarkable property of the $Q_{j;j}$ is that it is totally antisymmetric, that is, antisymmetric in every pair of indices in either of the two independent sets separated by the intervening semicolon. Provided that the $Q_{j-1;j-1}$ has such a property, eq. (12) shows that $Q_{j;j}$ is antisymmetric in $j-1$ indices $\mu_1, \mu_2, \dots, \mu_{j-1}$ as well as in the pair ν_1 and ν_j , whereas eq. (12') shows that it is also antisymmetric in $j-1$ indices $\nu_2, \nu_3, \dots, \nu_j$ as well as in the pair μ_{j-1} and μ_j . Therefore the same is also valid for $Q_{j;j}$. Now we have according to eqs. (1') and (11)

$$Q_{\mu\lambda;\nu\rho} = Q_{\mu;\nu} (g_{\lambda\rho} - \beta_\rho \beta_\lambda) - g_{\nu\lambda} Q_{\mu;\rho} = g_{\lambda\rho} Q_{\mu;\nu} + Q_{\mu;\rho} \beta_\nu \beta_\lambda - g_{\nu\lambda} Q_{\mu;\rho}$$

*) Parenthesized indices are to be eliminated.

$$= \mathcal{G}_{\lambda\rho} \mathcal{Q}_{\mu;\nu} - \mathcal{Q}_{\mu;\rho} \mathcal{Q}_{\lambda;\nu} = -\mathcal{Q}_{\mu\lambda;\rho\nu}$$

and in quite the same way

$$\mathcal{Q}_{\mu\lambda;\nu\rho} = -\mathcal{Q}_{\lambda\mu;\nu\rho},$$

which together prove the above statement. Eqs. (12) and (12') can thus be rewritten as

$$\mathcal{Q}_{\mu_1\mu_2\cdots\mu_j\mu;\nu_1\nu_2\cdots\nu_j\nu} = \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} \mathcal{Q}_{\mu;\nu} - \sum_{i=1}^j \mathcal{G}_{\nu_i\mu} \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots(\nu)\cdots\nu_j} \quad (13)$$

$$= \mathcal{Q}_{\mu;\nu} \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} - \sum_{i=1}^j \mathcal{G}_{\nu_i\mu_i} \mathcal{Q}_{\mu_1\mu_2\cdots(\mu_i)\cdots\mu_j;\nu_1\nu_2\cdots\nu_j}, \quad (13')$$

where the braces indicate that ν_i and μ_i are to be replaced by ν and μ respectively. Owing to this antisymmetry the $\mathcal{Q}_{j;j}$ vanishes identically for $j > n$, so that the said sequence ends with $j=n$.

Then putting in these equations $\mu=\nu=\nu_k$ and $\mu=\nu=\mu_k$ ($k=1, 2, \cdots, j$) respectively, we get

$$\mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} = \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} \mathcal{Q}_{\nu_k;\nu_k} = \mathcal{Q}_{\mu_k;\mu_k} \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j},$$

and therefore we have according to the identity $\mathcal{Q}_{\rho;\rho} = (1/2)[E - \eta_\rho]$

$$\mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} = \prod_{k=1}^j (1/2)[E - \eta_{\mu_k}] \cdot \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} \cdot \prod_{k=1}^j (1/2)[E - \eta_{\nu_k}]. \quad (14)$$

More specifically for $\mu_k=\nu_k$ the $\mathcal{Q}_{j;j}$ reduces to

$$\mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\mu_1\mu_2\cdots\mu_j} = \prod_{k=1}^j \mathcal{Q}_{\mu_k;\mu_k} = \prod_{k=1}^j (1/2)[E - \eta_{\mu_k}] \quad (15)$$

in view of the definition (12).

Now eqs. (4) and (5) show that the R_k is commutable with $\mathcal{Q}_{\mu;\nu}$, so that the same is in general valid for any $\mathcal{Q}_{j;j}$. Moreover since the R_{n-k} is the total sum of ${}_nC_k$ idempotents $e(\epsilon)$ with common number k of their negative ϵ_ρ , it follows from eq. (14) that

$$R_{n-k} \mathcal{Q}_{j;j} = \mathcal{Q}_{j;j} R_{n-k} = 0 \quad \text{for } k < j. \quad (16)$$

We don't get a nonvanishing product till k becomes equal to j . We thus define a further quantity $P_{j;j}$ by

$$P_{j;j} = R_{n-j} \mathcal{Q}_{j;j} = \mathcal{Q}_{j;j} R_{n-j} = R_{n-j} \mathcal{Q}_{j;j} R_{n-j}. \quad (17)$$

According to eq. (13) the $\mathcal{Q}_{j;j}$ for $j=0$ should be taken to be the unit E , so that the $P_{j;j}$ coincides with R_n for $j=0$. This can be expressed for convenience as $P=R_n$. Eq. (14) shows moreover that only a particular one can survive from among the ${}_nC_j$ idempotents $e(\epsilon)$ comprised in the above R_{n-j} standing on either side of $\mathcal{Q}_{j;j}$:

$$P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} = e(\epsilon) \mathcal{Q}_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} e(\epsilon'), \quad (18)$$

where ϵ_{μ_k} and ϵ'_{ν_k} ($k=1, 2, \cdots, j$) are negative and others are all positive. Then according to eq. (15) we have for $\mu_k=\nu_k$

$$P_{\mu_1\mu_2\cdots\mu_j;\mu_1\mu_2\cdots\mu_j} = e(\epsilon) \quad (19)$$

with the same ϵ_p as the above. The most remarkable properties of this $P_{j,j}$ are

$$P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} Q_{\mu;\nu} = \sum_{i=1}^j \mathcal{G}_{\nu_i\mu} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots(\nu_i)\cdots\nu_j} \quad (20)$$

and

$$Q_{\mu;\nu} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} = \sum_{i=1}^j \mathcal{G}_{\nu\mu_i} P_{\mu_1\mu_2\cdots(\mu_i)\cdots\mu_j; \nu_1\nu_2\cdots\nu_j}, \quad (20')$$

which are obtained at once by multiplying eqs. (13) and (13') respectively by R_{n-j} and taking into account eq. (16). In quite the same way we get from eqs. (11), (12) and (12')

$$P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} i^j \beta_{\nu_j} \beta_p = \sum_{i=1}^j (-)^{j-i} \mathcal{G}_{p\nu_i} P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots(\nu_i)\cdots\nu_j} \quad (21)$$

and

$$\beta_{\mu_i} \beta_{\mu_1} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_{j-1}} = \sum_{i=1}^j (-)^{i-1} \mathcal{G}_{p\mu_i} P_{\mu_1\nu_2\cdots(\mu_i)\cdots\mu_j; \nu_1\nu_2\cdots\nu_{j-1}}. \quad (21')$$

The $P_{j,j}$ multiplied on either side twice by $i^j \beta_p$ is trivial affording nothing new types of elements.

On the basis of $P_{j-1; j-1}$ we can therefore construct the following four types of elements which constitute together a subspace closed with respect to the operation of multiplication:

$$\left. \begin{aligned} P_{j-1; j-1} &= R_{n+1-j} Q_{j-1; j-1} \bar{R}_{n+1-j}, & (I) \\ P_{j-1; j-1} \beta_{\nu} &= R_{n+1-j} Q_{j-1; j-1} \beta_{\nu} R_j, & (II) \\ \beta_{\mu} P_{j-1; j-1} &= R_j i^j \beta_{\mu} Q_{j-1; j-1} R_{n+1-j} & (III) \\ \text{and } \beta_{\mu} P_{j-1; j-1} \beta_{\nu} &= R_j i^j \beta_{\mu} Q_{j-1; j-1} \beta_{\nu} R_j, & (IV) \end{aligned} \right\} \quad (22)$$

where use is made of eqs. (5) and (17). Then since we have from eq. (1')

$$Q_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} i^j \beta_{\nu} + Q_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_{j-1}} i^j \beta_{\nu_j} = 0$$

$$\text{and } \beta_{\mu} Q_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} + \beta_{\mu_1} Q_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} = 0,$$

they are all totally antisymmetric and the number of inequivalent nonvanishing ones belonging to each type is given respectively by

$$(I) = ({}_nC_{j-1})^2, \quad (II) = (III) = {}_nC_{j-1} \times {}_nC_j \quad \text{and} \quad (IV) = ({}_nC_j)^2.$$

Furthermore these $({}_{n+1}C_j)^2$ inequivalent elements are all linearly independent of each other, for according to eqs. (3) and (18) the above scheme can be written more precisely as

$$\left. \begin{aligned} P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} &= e(\epsilon) Q_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} e(\epsilon'), & (I) \\ P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} \beta_{\nu_j} &= e(\epsilon) Q_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} \beta_{\nu_j} e(\epsilon''), & (II) \\ \beta_{\mu_j} P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} &= e(\epsilon') \beta_{\mu_j} Q_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} e(\epsilon') & (III) \\ \text{and } \beta_{\mu_j} P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} \beta_{\mu_j} &= e(\epsilon') \beta_{\mu_j} Q_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} \beta_{\nu_j} e(\epsilon''), & (IV) \end{aligned} \right\}$$

where ϵ_{μ_k} and ϵ'_{ν_k} ($k=1, 2, \dots, j-1$) are negative and other ϵ_ρ and ϵ'_ρ are all positive, while ϵ'_{μ_k} and ϵ''_{ν_k} ($k=1, 2, \dots, j$) are positive and other ϵ'_ρ and ϵ''_ρ are all negative. There occur altogether $(n+1)$ such systems as j runs from 1 to $n+1$. In the final case $j=n+1$ the type (I) reduces however to the single element R_0 [see also eq. (27) below], so that all the other types vanish identically according to eq. (4). This is the trivial algebra alluded to in § 2, and we shall then investigate the nature of the remaining n ordinary subspaces.

Now we turn our attention to the special subspace composed of the four types of elements

$$\left. \begin{aligned} \beta_\mu P_{n-j; n-j} \beta_\nu &= R_{n+1-j} \beta_\mu Q_{n-j; n-j} \beta_\nu R_{n+1-j} & (I') \\ \beta_\mu P_{n-j; n-j} &= R_{n+1-j} \beta_\mu Q_{n-j; n-j} R_j, & (II') \\ P_{n-j; n-j} \beta_\nu &= R_j Q_{n-j; n-j} \beta_\nu R_{n+1-j} & (III') \\ P_{n-j; n-j} &= R_j Q_{n-j; n-j} R_j. & (IV') \end{aligned} \right\} \quad (23)$$

and

All the above arguments are also applicable to this subspace and we are led to conclude that this is essentially identical with the preceding one (22), each type of which corresponds of course to the present primed one. The only difference lies in the number of indices and the two subspaces may properly be said to be dual to each other.* In the even case the n subspaces are completely subdivided into m such dual pairs, whereas in the odd case besides $(m-1)$ ordinary dual pairs there remains one particular subspace for $j=m$ which is dual to itself and wherein independent types are reduced to only two in number: $(I) = (I') = (IV) = (IV')$ and $(II) = (II') = (III) = (III')$. This again corresponds to the case of twin algebras. Thus if j be restricted to the range $1 \leq j \leq m$, the dual pairs (22) and (23) exhaust in any case the sequence of n subspaces. Now by definition these subspaces, each comprising $(n+1)C_j$ ² inequivalent linearly independent elements, are both generated by $E_j = R_j + R_{n+1-j}$ and conversely the unit element E_j is included in either of them. For we have according to eq. (19)

$$R_{n+1-j} = \{1/(j-1)!\} P_{\rho_1 \rho_2 \dots \rho_{j-1}}{}^{\rho_1 \rho_2 \dots \rho_{j-1}} \quad (24)$$

and moreover according to eq. (9)

$$R_j = (1/j!) \beta_\rho R_{n+1-j} \beta^\rho = (1/j!) \beta_\rho P_{\rho_1 \rho_2 \dots \rho_{j-1}}{}^{\rho_1 \rho_2 \dots \rho_{j-1}} \beta^\rho, \quad (24')$$

so that the types (I) and (IV) comprise R_{n+1-j} and R_j respectively. Then on replacing the number j in these equations throughout by $n+1-j$ we see in the same way that R_{n+1-j} and R_j are comprised in (I') and (IV') respectively. We thus arrive at the final conclusion that the schemes (22) and (23) furnish the two alternative expressions of the linearly independent basis of the irreducible two-sided invariant subspace generated by E_j .

Before proceeding to explicit demonstration of the above duality, we shall now evaluate

* A more exact formulation of this duality will be given presently. See eqs. (30) below.

the product of two $P_{j;j}$'s. According to eq. (17) this can first be written in two alternative ways as $P_{j;j}Q_{j;j}$ or as $Q_{j;j}P_{j;j}$. Now that the $Q_{j;j}$ is sum of multiple products of $Q_{\mu;\nu}$, it is absorbed in the $P_{j;j}$ by successive application of eqs. (20) or (20') and the said product results in another $P_{j;j}$ with a numerical multiplier composed of the $g_{\mu\nu}$ alone. It is essential here to observe that the right (left) multiplication by $Q_{j;j}$ has nothing to do with the left (right) set of indices in $P_{j;j}$. Moreover it holds in general for multiplying $P_{j;j}$ by any element, for this can be expressed as a linear combination of the linearly independent basis $P_{j;j}$. Therefore we are naturally led to assume that

$$P_{\lambda_1\lambda_2\cdots\lambda_j; \mu_1\mu_2\cdots\mu_j} P_{\nu_1\nu_2\cdots\nu_j; \rho_1\rho_2\cdots\rho_j} = G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} P_{\lambda_1\lambda_2\cdots\lambda_j; \rho_1\rho_2\cdots\rho_j} \quad (25)$$

with a totally antisymmetric tensor $G_{j;j}$. Then in view of eq. (18) we infer that the $G_{j;j}$ should survive only when the two sets of indices form a certain permutation of each other and moreover that it should be equal to $+1$ or to -1 according as the permutation is even or odd. A more explicit definition of $G_{j;j}$ is however given as follows. On account of the total antisymmetry of the elements $P_{j;j} \beta_\mu$ and $\beta_\nu P_{j;j}$ we may be allowed to assume in terms of the same $G_{j;j}$ that

$$P_{\lambda_1\lambda_2\cdots\lambda_j; \mu_1\mu_2\cdots\mu_j} \beta_\mu \cdot \beta_\nu P_{\nu_1\nu_2\cdots\nu_j; \rho_1\rho_2\cdots\rho_j} = G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} P_{\lambda_1\lambda_2\cdots\lambda_j; \rho_1\rho_2\cdots\rho_j} \quad (25')$$

Then according to eqs. (20), (20') and (11) we get the recurrence formulas

$$G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} = g_{\mu\nu} G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} - \sum_{i=1}^j g_{\nu\mu_i} G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} \quad (26)$$

$$= g_{\mu\nu} G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} - \sum_{i=1}^j g_{\mu_i\nu} G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} \quad (26')$$

which are nothing but eqs. (13) and (13') wherein $Q_{j;j}$ are all replaced by $G_{j;j}$ and especially the $Q_{\mu;\nu}$ by $G_{\mu;\nu} = g_{\mu\nu}$. It is now evident that the $G_{j;j}$ is composed of the $g_{\mu\nu}$ alone and it can at once be obtained by eliminating all the β_p comprised in $Q_{j;j}$. Accordingly we have from eq. (4)

$$Q_{j;j} R_0 = G_{j;j} R_0,$$

which yields especially for $j=n$ in view of eqs. (14) and (17)

$$Q_{n;n} = P_{n;n} = G_{n;n} R_0. \quad (27)$$

Now from eq. (25) we infer that the $G_{j;j}$ is invariant for interchange of two sets of indices, which fact should also be derivable from the above recurrence formulas. As will be discussed later in the Appendix, the present $G_{j;j}$ is essentially a minor determinant taken from the n -dimensional identity matrix I .

Concerning the $G_{j;j}$ we add here the following two facts. First owing to its total antisymmetry it vanishes identically for $j=n+1$, so that we have according to eqs. (26) and (26')

$$\mathcal{G}_{\mu\nu} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} = \sum_{\ell=1}^n \mathcal{G}_{\nu\mu_\ell} G_{\mu_1\mu_2\cdots\{\mu\}\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \quad (28)$$

$$= \sum_{\ell=1}^n \mathcal{G}_{\mu\nu_\ell} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\{\nu\}\cdots\nu_n} \quad (28')$$

Secondly we have

$$(1/j!) G_{\lambda_1\lambda_2\cdots\lambda_j; \rho_1\rho_2\cdots\rho_j} F_{\rho_1\rho_2\cdots\rho_j} = F_{\lambda_1\lambda_2\cdots\lambda_j} \quad (29)$$

for any antisymmetric tensor $F_{\rho_1\rho_2\cdots\rho_j}$ of rank j .

Now we shall consider a totally antisymmetric tensor $W_{j;j}$ defined in terms of the above $G_{n;n}$ as

$$W_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j}^\rho = \{(n-j)!\}^{-2} G_{\mu_1\cdots\mu_j\mu_{j+1}\cdots\mu_n; \nu_1\cdots\nu_j\nu_{j+1}\cdots\nu_n} \beta_{\mu_n}^\rho P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\nu_n}^\nu,$$

and examine how the $Q_{\mu;\nu}$ operates on its either side. We have first from eq. (11)

$$\begin{aligned} & \{(n-j)!\}^2 W_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} Q_{\mu;\nu} \\ &= \mathcal{G}_{\mu\nu} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \beta_{\mu_n}^\mu P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\nu_n}^\nu \\ & - G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \beta_{\mu_n}^\mu P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\nu_n}^\nu \beta_{\mu}^\nu, \end{aligned}$$

wherein the second term can be rewritten according to eq. (21) as

$$\begin{aligned} & - G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \sum_{\ell=j+1}^n (-)^{n-\ell} \mathcal{G}_{\nu}^{\nu_\ell} \beta_{\nu_n}^\mu P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\mu}^{\nu_\ell} \\ &= \beta_{\mu_n}^\mu P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\nu_n}^\nu \sum_{\ell=j+1}^n \mathcal{G}_{\mu\nu_\ell} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\{\nu\}\cdots\nu_n}. \end{aligned}$$

Then applying eq. (28') to the first term we get

$$\begin{aligned} & W_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} Q_{\mu;\nu} \\ &= \{(n-j)!\}^{-2} \sum_{\ell=1}^j \mathcal{G}_{\mu\nu_\ell} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\{\nu\}\cdots\nu_n} \beta_{\mu_n}^\mu P^{\mu_{j+1}\cdots\mu_{n-1}; \nu_{j+1}\cdots\nu_{n-1}} \beta_{\nu_n}^\nu \\ &= \sum_{\ell=1}^j \mathcal{G}_{\mu\nu_\ell} W_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\{\nu\}\cdots\nu_j}. \end{aligned}$$

This is nothing but eq. (20) with all the $P_{j;j}$ replaced by $W_{j;j}$ and in quite the same way we can find that $Q_{\mu;\nu} W_{j;j}$ gives a result corresponding to eq. (20'). Furthermore the idempotent $e(\epsilon)$ to be attached to either side of $W_{j;j}$ is the same with that to be attached to $P_{j;j}$ and in addition we see that for $\mu_k = \nu_k$ ($k=1, 2, \cdots, j$) $W_{j;j}^\rho$ and $P_{j;j}$ reduce to one and the same $e(\epsilon)$. Thus we are led to identify the present $W_{j;j}$ with the previous $P_{j;j}$ itself:

$$P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} = \{(n-j)!\}^{-2} G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \beta_{\mu_n}^{\mu_{j+1}} P^{\mu_{j+2}\cdots\mu_n; \nu_{j+2}\cdots\nu_n} \beta_{\nu_n}^{\nu_{j+1}}, \quad (30\text{-I})$$

which then yields on account of eqs. (21) and (21') three more equations

$$P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j}\beta_{\nu_{j+1}} = \{(n-j)!(n-j-1)!\}^{-1} \\ \times G_{\mu_1\mu_2\cdots\mu_n;\nu_1\nu_2\cdots\nu_n}\beta_{\mu_{j+1}}P^{\mu_{j+2}\cdots\mu_n;\nu_{j+2}\cdots\nu_n} \quad (30\cdot\text{II})$$

$$\beta_{\mu_{j+1}}P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} = \{(n-j)!(n-j-1)!\}^{-1} \\ \times G_{\mu_1\mu_2\cdots\mu_n;\nu_1\nu_2\cdots\nu_n}P^{\mu_{j+2}\cdots\mu_n;\nu_{j+2}\cdots\nu_n}\beta_{\nu_{j+1}} \quad (30\cdot\text{III})$$

$$\beta_{\mu_{j+1}}P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j}\beta_{\nu_{j+1}} \\ = \{(n-j-1)!\}^{-2} G_{\mu_1\mu_2\cdots\mu_n;\nu_1\nu_2\cdots\nu_n}P^{\mu_{j+2}\cdots\mu_n;\nu_{j+2}\cdots\nu_n} \quad (30\cdot\text{IV})$$

Incidentally for any orthogonal substitution $\beta'_\nu = \alpha_\nu^\lambda \beta_\lambda$ the $Q_{\mu;\nu}$ transforms as a second ranked tensor: $Q'_{\mu;\nu} = g'_{\mu\nu}$, $E - \beta'_\nu \beta'_\mu = \alpha_\mu^\lambda \alpha_\nu^\rho Q_{\lambda;\rho}$, so that the general $Q_{j;j}$ transforms as $2j$ -ranked tensor by definition. Moreover the R_j is invariant for this substitution, because this is expressible as an algebraic function of the scalar element B [see eq. (7)]. Therefore we have

$$P'_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} = \alpha_{\mu_1}^{\lambda_1} \alpha_{\mu_2}^{\lambda_2} \cdots \alpha_{\mu_j}^{\lambda_j} \alpha_{\nu_1}^{\rho_1} \alpha_{\nu_2}^{\rho_2} \cdots \alpha_{\nu_j}^{\rho_j} P_{\lambda_1\lambda_2\cdots\lambda_j;\rho_1\rho_2\cdots\rho_j} \\ = (1/j!)^2 A_{\mu_1\mu_2\cdots\mu_j;\lambda_1\lambda_2\cdots\lambda_j} A_{\nu_1\nu_2\cdots\nu_j;\rho_1\rho_2\cdots\rho_j} P_{\lambda_1\lambda_2\cdots\lambda_j;\rho_1\rho_2\cdots\rho_j} \quad (31)$$

according to eqs. (17), (29) and (A.1).

Finally we shall briefly allude to the problem of matrix representations. As is well known (see reference 14), if the complete set of normal primitive idempotents is once established, this can be settled quite easily. To the simple algebra generated by E_j there corresponds in general an irreducible representation of degree ${}_{n+1}C_j$, wherein each linearly independent element constructed above is represented by a matrix whose only one non-vanishing element is situated in a suitable position as is indicated in the scheme

$$\left(\begin{array}{ccc} \text{(I)} & P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} & \vdots \\ & \cdots & \cdots \\ \text{(III)} & \beta_\mu P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} & \vdots \end{array} \right) \begin{array}{c} \text{(II)} \\ \vdots \\ \text{(IV)} \end{array} \left(\begin{array}{c} P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} \beta_\nu \\ \vdots \\ \beta_\mu P_{\mu_1\mu_2\cdots\mu_j;\nu_1\nu_2\cdots\nu_j} \beta_\nu \end{array} \right)$$

The idempotent $e(\epsilon)$ is of course represented by a diagonal matrix. Especially the trivial algebra consisting of the isolated element R_0 alone permits a one-rowed representation wherein R_0 itself is represented by unity and every β_ν by zero according to eq. (4). In ordinary simple algebras we have from eq. (5)

$$E_j \beta_\nu = R_{n+1-j} \beta_\nu + \beta_\nu R_{n+1-j},$$

which comprises ${}_{n-1}C_{j-1}$ linearly independent elements of either of the types (II) and (III). Hence in the ordinary irreducible representation of degree ${}_{n+1}C_j$ the single β_ν is represented by a matrix in which $2{}_{n-1}C_{j-1}$ nonvanishing elements are equally distributed over the off-diagonal rectangles (II) and (III) in the above scheme. In the special case of four dimensions we thus get exactly the same matrix representation of the single β_ν as was

obtained by Kemmer.²⁾

Then we shall take up the traces of these representation matrices. One can easily find that all the elements belonging to the types (II) and (III) have identically vanishing traces, while the traces of the elements (I) and (IV) are given respectively by

$$\text{tr}(P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j}) = \text{tr}(\beta_{\mu_j} P_{\mu_1\mu_2\cdots\mu_{j-1}; \nu_1\nu_2\cdots\nu_{j-1}} \beta_{\nu_j}) = G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j},$$

where the ones with nonvanishing traces are of course the idempotents $e(\epsilon)$. It will be worthwhile to observe that the present equations also afford the recurrence formulas (26) and (26') for the tensor $G_{j,j}$.

§ 4. The twin algebras

We shall now embark upon a more thorough investigation of the structure of the twin algebras wherein, as has been alluded to in § 2, a special quantity ω plays a decisive part. We are thus before every thing to relate full details of its properties. First of all we take up an element $\beta_{\rho_1}\beta_{\rho_2}\cdots\beta_{\rho_n}$, a product of n different β_{ρ} wherein the indices $\rho_1, \rho_2, \cdots, \rho_n$ are any permutation of n figures 1, 2, \cdots , n . All possible elements of this type are apparently $n!$ in number, but the commutation relations $\beta_{\mu}\beta_{\lambda}\beta_{\nu} = -\beta_{\nu}\beta_{\lambda}\beta_{\mu}$ for all different indices show that they are classified into ${}_nC_m$ sets of $m!(n-m)!$ elements identical with each other but for signs; inequivalent ones are thus ${}_nC_m$ in number. Then according to the relationships

$$\beta_{\rho} = \beta_{\rho}^3 = (1/2)[E + \eta_{\rho}] \beta_{\rho} = \beta_{\rho}(1/2)[E + \eta_{\rho}]$$

and

$$\beta_{\lambda}\beta_{\rho} = (1/2)[E - \eta_{\rho}] \beta_{\lambda}\beta_{\rho} = \beta_{\lambda}\beta_{\rho}(1/2)[E - \eta_{\lambda}] \quad \text{for } \lambda \neq \rho,$$

we can attach to the left of this product the $e(\epsilon)$ wherein only $\epsilon_{\rho_{2k-1}}$ ($k=1, 2, \cdots, m$) are positive. It is the same $e(\epsilon)$ with this that can be attached to the right of this product for odd n , whereas for even n it is the one with positive $\epsilon_{\rho_{2k}}$. This difference is noteworthy in connection with the fact that for odd n the said product is invariant (but for sign) for the reversal of the order of all the β_{ρ} , whereas this is not the case for even n . In any case it is true that we can attach to either side of the product a certain $e(\epsilon)$ with m positive ϵ_{ρ} . It is therefore invariant for the multiplication by R_m on either side. Moreover these ${}_nC_m$ inequivalent products are linearly independent of each other and their total sum is nothing but the ω defined by eq. (6) wherein signs are suitably adjusted by introducing an antisymmetric tensor $\epsilon_{\rho_1\rho_2\cdots\rho_n}$. Hence we conclude that

$$\omega R_j = R_j \omega = g_{jm} \omega.$$

Next on squaring the ω there are apparently produced $({}_nC_m)^2$ elements. But surviving ones are only ${}_nC_m$ in number [see below eq. (25)]. Among others the typical product that furnishes nonvanishing result is of the form

$$\epsilon_{\rho_1\rho_2\cdots\rho_n} \beta_{\rho_1} \beta_{\rho_2} \cdots \beta_{\rho_n} \cdot \epsilon_{\rho_n\cdots\rho_2\rho_1} \beta_{\rho_n} \cdots \beta_{\rho_2} \beta_{\rho_1} =$$

$$\begin{aligned}
&= (-)^{n(n-1)/2} (\epsilon_{\rho_1 \rho_2 \dots \rho_n})^2 \beta_{\rho_1} \beta_{\rho_2} \dots \beta_{\rho_n} \beta_{\rho_n} \dots \beta_{\rho_2} \beta_{\rho_1} \\
&= (-)^{n(n-1)/2} \beta_{\rho_1}^2 (E - \beta_{\rho_2}^2) \beta_{\rho_3}^2 \times \dots \times \begin{cases} \dots (E - \beta_{\rho_{n-1}}^2) \beta_{\rho_n}^2 & \text{for odd } n; \\ \dots \beta_{\rho_{n-1}}^2 (E - \beta_{\rho_n}^2) & \text{for even } n. \end{cases}
\end{aligned}$$

This is the same thing as the $\epsilon(\epsilon)$ with positive $\epsilon_{\rho_{2k-1}}$ ($k=1, 2, \dots, m$) and the definition (6) shows that

$$\omega^2 = R_m.$$

Incidentally we have $\omega^3 = \omega$.

In the even case ($n=2m$) we find with the aid of eq. (12) and the identity $\beta_\nu \beta_\mu = -Q_{\mu;\nu}$ for $\mu \neq \nu$ that

$$\begin{aligned}
\epsilon_{\rho_1 \rho_2 \dots \rho_{2m}} \beta_{\rho_1} \beta_{\rho_2} \dots \beta_{\rho_{2m}} &= (-)^m \epsilon_{\rho_1 \rho_2 \dots \rho_{2m}} R_m Q_{\rho_2; \rho_1} Q_{\rho_4; \rho_3} \times \dots \times Q_{\rho_{2m}; \rho_{2m-1}} \\
&= (-)^{m(m-1)/2} \epsilon_{\rho_2 \rho_4 \dots \rho_{2m} \rho_1 \rho_3 \dots \rho_{2m-1}} P_{\rho_2 \rho_4 \dots \rho_{2m}; \rho_1 \rho_3 \dots \rho_{2m-1}}
\end{aligned}$$

and in the same way that

$$\begin{aligned}
\epsilon_{\rho_1 \rho_2 \dots \rho_{2m}} \beta_{\rho_1} \beta_{\rho_2} \dots \beta_{\rho_{2m}} &= (-)^{m-1} \epsilon_{\rho_1 \rho_2 \dots \rho_{2m}} R_m \beta_{\rho_1} Q_{\rho_3; \rho_2} \times \dots \times Q_{\rho_{2m-1}; \rho_{2m-2}} \beta_{\rho_{2m}} \\
&= -(-)^{m(m-1)/2} \epsilon_{\rho_1 \rho_3 \dots \rho_{2m-1} \rho_2 \rho_4 \dots \rho_{2m}} \beta_{\rho_1} P_{\rho_3 \rho_5 \dots \rho_{2m-1}; \rho_2 \rho_4 \dots \rho_{2m-2}} \beta_{\rho_{2m}}.
\end{aligned}$$

The ω can therefore be expressed in two alternative ways as

$$\omega = c(m) \epsilon^{\lambda_1 \lambda_2 \dots \lambda_m \rho_1 \rho_2 \dots \rho_m} P_{\lambda_1 \lambda_2 \dots \lambda_m; \rho_1 \rho_2 \dots \rho_m} \quad (32)$$

or as
$$\omega = -c(m) \epsilon^{\lambda_1 \lambda_2 \dots \lambda_m \rho_1 \rho_2 \dots \rho_m} \beta_{\lambda_1} P_{\lambda_2 \lambda_3 \dots \lambda_m; \rho_1 \rho_2 \dots \rho_{m-1}} \beta_{\rho_m} \quad (32')$$

with $c(m) = i^{m^2} (m!)^{-2}$. Now n is even, so that we have according to eq. (5)

$$\omega \beta_\rho \omega = \omega R_m \beta_\rho R_m \omega = \omega R_m R_{m+1} \beta_\rho \omega = 0$$

and
$$\omega^2 \beta_\rho + \beta_\rho \omega^2 = R_m \beta_\rho + \beta_\rho R_m = R_m \beta_\rho + R_{m+1} \beta_\rho = E_m \beta_\rho.$$

Then eqs. (21), (21') and (32') yield together

$$\beta_\lambda \omega \beta_\rho = -m^2 c(m) \epsilon_{\lambda \lambda_1 \dots \lambda_{m-1} \rho_1 \dots \rho_{m-1} \rho} P^{\lambda_1 \lambda_2 \dots \lambda_{m-1}; \rho_1 \rho_2 \dots \rho_{m-1}},$$

so that
$$\beta_\lambda \omega \beta_\rho + \beta_\rho \omega \beta_\lambda = 0.$$

Using eqs. (32), (20), (25), (A. 9), (30. IV), (29), (24) and (5) successively we have moreover

$$\omega Q_{\lambda; \rho} \omega = \beta_\lambda R_{m+1} \beta_\rho = R_m \beta_\lambda \hat{\beta}_\rho,$$

which yields $Q_{\lambda; \rho} \omega = \omega \beta_\lambda \beta_\rho$ on left multiplication by ω . On account of eq. (11) this can be rewritten as

$$\omega \beta_\lambda \beta_\rho + \beta_\rho \beta_\lambda \omega = g_{\lambda \rho}^{\omega \omega} \omega.$$

We have thus confirmed the statement made in § 2 that the ω can replace any one of the n elements β_ρ in eq. (1) in the special simple algebra generated by L_m for even n .

Now in the odd case ($n=2m-1$) we have in quite the same way as the above

$$\begin{aligned} & \epsilon_{\rho_1 \rho_2 \dots \rho_{2m-1}} \beta_{\rho_1} \beta_{\rho_2} \dots \beta_{\rho_{2m-1}} \\ &= -(-)^{m(m+1)/2} \epsilon_{\rho_2 \rho_4 \dots \rho_{2m-2} \rho_1 \rho_3 \dots \rho_{2m-1}} P_{\rho_2 \rho_4 \dots \rho_{2m-2}} \beta_{\rho_1 \rho_3 \dots \rho_{2m-1}} \\ &= -(-)^{m(m+1)/2} \epsilon_{\rho_1 \rho_3 \dots \rho_{2m-1} \rho_2 \rho_4 \dots \rho_{2m-2}} \beta_{\rho_1} P_{\rho_2 \rho_4 \dots \rho_{2m-2}} \beta_{\rho_3 \rho_5 \dots \rho_{2m-1}} \end{aligned}$$

so that the ω can be rewritten with $d(m) = i^{m^2-1} \{m!(m-1)!\}^{-1}$ as

$$\omega = d(m) \epsilon_{\lambda_1 \lambda_2 \dots \lambda_{m-1} \rho_1 \rho_2 \dots \rho_m} P_{\lambda_1 \lambda_2 \dots \lambda_{m-1}} \beta_{\rho_1 \rho_2 \dots \rho_m} \quad (33)$$

$$= d(m) \epsilon_{\lambda_1 \lambda_2 \dots \lambda_m \rho_1 \rho_2 \dots \rho_{m-1}} \beta_{\lambda_1} P_{\lambda_2 \lambda_3 \dots \lambda_m} \beta_{\rho_1 \rho_2 \dots \rho_{m-1}} \quad (33')$$

Then eqs. (21) and (21') show that if the former transcription is multiplied by any β_ρ on the right, it yields exactly the same result as is obtained from the latter on left multiplication by β_ρ :

$$\omega \beta_\rho = \beta_\rho \omega = m d(m) \epsilon_{\lambda_1 \lambda_2 \dots \lambda_{m-1} \rho_1 \rho_2 \dots \rho_m} P_{\lambda_1 \lambda_2 \dots \lambda_{m-1}} \beta_{\rho_1 \rho_2 \dots \rho_{m-1}} \beta_\rho \quad (34)$$

the ω is commutable with every β_ρ in the present odd case. The unit elements of the twin algebras are therefore given respectively by

$$E_m^{(+)} = (1/2)[R_m + \omega] \quad \text{and} \quad E_m^{(-)} = (1/2)[R_m - \omega],$$

in terms of which we construct the following two types of elements that inherit all the properties possessed by ordinary $P_{j;j}$:

$$P_{m-1; m-1}^{(\pm)} = E_m^{(\pm)} P_{m-1; m-1} \quad (35)$$

The $({}_nC_m)^2$ inequivalent nonvanishing elements of either type are evidently linearly independent of each other and further constitute a subspace closed with respect to the operation of multiplication. For multiplying eq. (34) by $E_m^{(\pm)}$ we have according to the relationships $E_m^{(\pm)} \omega = \pm E_m^{(\pm)}$

$$E_m^{(\pm)} \beta_\rho = \pm m d(m) \epsilon_{\lambda_1 \lambda_2 \dots \lambda_{m-1} \rho_1 \rho_2 \dots \rho_m} P_{\lambda_1 \lambda_2 \dots \lambda_{m-1}} \beta_{\rho_1 \rho_2 \dots \rho_{m-1}} \beta_\rho \quad (36)$$

and then on left and right multiplications by any $P_{m-1; m-1}^{(\pm)}$ these equations yield

$$\left. \begin{aligned} P_{\mu_1 \mu_2 \dots \mu_{m-1}; \nu_1 \nu_2 \dots \nu_{m-1}}^{(\pm)} \beta_\rho &= \pm m! d(m) \epsilon_{\nu_1 \nu_2 \dots \nu_{m-1} \rho_1 \rho_2 \dots \rho_{m-1}} P_{\mu_1 \mu_2 \dots \mu_{m-1}}^{(\pm)} \beta_{\rho_1 \rho_2 \dots \rho_{m-1}} \\ \beta_\rho P_{\mu_1 \mu_2 \dots \mu_{m-1}; \nu_1 \nu_2 \dots \nu_{m-1}}^{(\pm)} &= \pm m! d(m) \epsilon_{\lambda_1 \lambda_2 \dots \lambda_{m-1} \mu_1 \mu_2 \dots \mu_{m-1}} P_{\lambda_1 \lambda_2 \dots \lambda_{m-1}}^{(\pm)} \beta_{\nu_1 \nu_2 \dots \nu_{m-1}} \end{aligned} \right\} \quad (37)$$

where use is made of eqs. (25) and (29). Among others the special ones with identical sets of indices reduce to the normal primitive idempotents $E_m^{(\pm)} e(\epsilon)$ [see eq. (19)], of which the total sums afford the unit elements $E_m^{(+)}$ and $E_m^{(-)}$ in view of eq. (24):

$$\{1/(m-1)!\} P_{\rho_1 \rho_2 \dots \rho_{m-1}}^{(\pm)} = E_m^{(\pm)} R_m = E_m^{(\pm)}.$$

We thus see that the $P_{m-1; m-1}^{(+)}$ and $P_{m-1; m-1}^{(-)}$ form respectively the complete linearly independent bases of the twin algebras.

Finally the identity

$$a_{\lambda_1}^{\rho_1} a_{\lambda_2}^{\rho_2} \dots a_{\lambda_n}^{\rho_n} F_{\rho_1 \rho_2 \dots \rho_n} = \det(\alpha) F_{\lambda_1 \lambda_2 \dots \lambda_n}$$

holding for any antisymmetric tensor of rank n including $\epsilon_{\rho_1 \rho_2 \dots \rho_n}$, shows that for any orthogonal substitution $\beta'_\rho = \alpha_\rho^\lambda \beta_\lambda$ the ω transforms as $\omega' = \det(\alpha) \omega$; the ω is a pseudo-scalar quantity changing sign for improper transformations of the n -dimensional β -space. Therefore the unit elements $E_m^{(+)}$ and $E_m^{(-)}$ and accordingly the twin algebras interchange for improper transformations. In terms of the matrix representations, the same fact can also be stated according to eq. (36) as, whereas for all other irreducible representations the matrix corresponding to a single β_ρ is invariant for both proper and improper transformations, for the twin representations this is so only for proper transformations; for improper transformations the matrix corresponding to every β_ρ changes its sign.

§ 5. Identification of projectors

In the absence of interaction the Duffin-Kemmer wave equation is written in the form

$$\{\partial^\rho \beta_\rho + \chi\} \psi = 0, \quad (38)$$

where ∂_ρ stands for the differential operator $i\partial/\partial x^\rho$, and the β_ρ are fully reducible set of square matrices operating on the ψ which itself is a one-column matrix. The components of this wave function are extracted respectively by means of the normal primitive idempotents each represented by a diagonal matrix with only one nonvanishing element which is unity, so that the number of components is determined by that of all possible normal primitive idempotents. More generally each linearly independent element constructed in the preceding sections works as a "projector" wherein the right set of indices specifies the particular component to be extracted, while the left one indicates the position to which the extracted component is to be shifted. So far as we are concerned with explicit representations, we remain completely ignorant of the transformation character of the component thus extracted. In the followings we shall therefore be engaged in clarifying this point.

If the orthogonal substitution $x'_\rho = \alpha_\rho^\lambda x_\lambda$ for fixed β_ρ corresponds to the transformation

$$\psi'(x') = S(\alpha) \psi(x), \quad (39)$$

the invariance of the system of eqs. (38) requires, as is well known, that there exists a similarity transformation of the β_ρ having the property

$$\beta'_\rho = \alpha_\rho^\lambda \beta_\lambda = S(\alpha)^{-1} \beta_\rho S(\alpha). \quad (40)$$

In terms of this spin transformation operator $S(\alpha)$ we can write $P'_{j,j}$ as $S^{-1} P_{j,j} S$. Now that the left (right) multiplication by $S(\alpha)^{-1}$ [$S(\alpha)$] has nothing to do with the

right (left) set of indices in $P_{j;j}$ [see above eq. (25)], eq. (31) at once gives

$$\begin{aligned} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} S(a) &= \tau v_j(a) \alpha_{\nu_1}^{p_1} \alpha_{\nu_2}^{p_2} \cdots \alpha_{\nu_j}^{p_j} P_{\mu_1\mu_2\cdots\mu_j; p_1p_2\cdots p_j} \\ &= (\tau v_j/j!) A_{\nu_1\nu_2\cdots\nu_j; p_1p_2\cdots p_j} P_{\mu_1\mu_2\cdots\mu_j; p_1p_2\cdots p_j} \end{aligned} \quad (41)$$

and
$$\begin{aligned} S^{-1} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} &= (1/\tau w_j) \alpha_{\mu_1}^{\lambda_1} \alpha_{\mu_2}^{\lambda_2} \cdots \alpha_{\mu_j}^{\lambda_j} P_{\lambda_1\lambda_2\cdots\lambda_j; \nu_1\nu_2\cdots\nu_j} \\ &= (1/j! w_j) A_{\mu_1\mu_2\cdots\mu_j; \lambda_1\lambda_2\cdots\lambda_j} P_{\lambda_1\lambda_2\cdots\lambda_j; \nu_1\nu_2\cdots\nu_j} \end{aligned} \quad (41')$$

with a yet unknown numerical factor $\tau w_j(a)$ depending on a . Now for $0 \leq j \leq n$ eq. (41) gives according to eq. (24)

$$R_{n-j} S(a) = (1/j!)^2 \tau w_j(a) A^{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j} P_{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j} \quad (42)$$

and since $R_{n-j} = R'_{n-j} = S^{-1} R_{n-j} S$, we get in the same way from eq. (41')

$$R_{n-j} S^{-1} = \{1/j! \tau w_j\} (a^T)^{\lambda_1 p_1} (a^T)^{\lambda_2 p_2} \cdots (a^T)^{\lambda_j p_j} P_{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j}, \quad (42')$$

where T denotes the transpose. The total $S(a)$ is thus given by

$$S(a) = \sum_{j=0}^n R_{n-j} S(a).$$

Moreover for $0 \leq j \leq n-1$ we have according to eqs. (24'), (40) and (41)

$$\begin{aligned} R_{j+1} S(a) &= \{1/(j+1)!\}^2 \tau w_j(a) A^{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j} \beta_{\lambda}^{\mu} P_{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j} i^{\beta_p} \\ &= \{\tau w_j/(j+1)!\} a^{\lambda_p} a^{\lambda_1 p_1} \cdots a^{\lambda_j p_j} \beta_{\lambda}^{\mu} P_{\lambda_1\lambda_2\cdots\lambda_j; p_1p_2\cdots p_j} \beta_p. \end{aligned} \quad (43)$$

Then we can easily assure that the present establishments of $S(a)$ actually realize the relationship (40), for $R_{n-j} S^{-1} \beta_p S = R_{n-j} S^{-1} \cdot \beta_p \cdot R_{j+1} S$ gives $a^{\lambda_p} R_{n-j} \beta_{\lambda}$ according to eqs. (42'), (43) and other equations. Here it is evident that the numerical factor $\tau w_j(a)$ does not produce any effect at all in this calculation. Then is there really any need for introducing such an extra factor? Although it takes in fact no part in realizing eq. (40) and accordingly can not be determined by this equation, it is quite essential for unambiguous establishment of $S(a)$, for without one eqs. (42) and (43) afford $S(a)$'s different in sign in the case of improper transformations.

Before entering into thorough discussion on this point we must note that in the case of odd dimensions there exists no such $S(a)$ that can satisfy eq. (40) for improper a , because we have in this case a pseudoscalar element ω which is commutable with all possible elements in the algebra. As far as the proper transformations are concerned, we may naturally put $\tau w_j(a) = 1$ for both even and odd cases, whereas for improper transformations we are now going to take up only the $\tau w_j(a)$ for even dimensions.

Now according to eqs. (30·I), (40) and (41) we get

$$\begin{aligned} P_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} S(a) &= \{1/(n-j)!\}^2 \tau v_{n-j-1}(a) G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} \\ &\quad \times a^{\nu_{j+1} p_{j+1}} a^{\nu_{j+2} p_{j+2}} \cdots a^{\nu_{n-1} p_{n-1}} \beta^{\mu_{j+1}} P^{\mu_{j+2} \cdots \mu_n; p_{j+2} \cdots p_n} \beta^{p_{j+1}} \end{aligned}$$

$$= \det(a) w_{n-j-1}(a) a_{v_1}^{p_1} a_{v_2}^{p_2} \dots a_{v_j}^{p_j} P_{\mu_1 \mu_2 \dots \mu_j; p_1 p_2 \dots p_j},$$

where use is made of the identity

$$\begin{aligned} & G_{\mu_1 \mu_2 \dots \mu_n; v_1 v_2 \dots v_n} a^{v_{j+1} p_{j+1}} a^{v_{j+2} p_{j+2}} \dots a^{v_n p_n} \\ &= a_{v_1}^{p_1} a_{v_2}^{p_2} \dots a_{v_j}^{p_j} G_{\mu_1 \mu_2 \dots \mu_n; \lambda_1 \lambda_2 \dots \lambda_n} a^{\lambda_1 p_1} a^{\lambda_2 p_2} \dots a^{\lambda_j p_j} a^{\lambda_{j+1} p_{j+1}} a^{\lambda_{j+2} p_{j+2}} \dots a^{\lambda_n p_n} \\ &= \det(a) a_{v_1}^{p_1} a_{v_2}^{p_2} \dots a_{v_j}^{p_j} G_{\mu_1 \mu_2 \dots \mu_n; p_1 p_2 \dots p_n}. \end{aligned}$$

Then if we drop the factor $w_{n-j-1}(a)$ in the above equation we get for improper a a result different in sign from that of eq. (41), so that the $w_j(a)$ must satisfy

$$w_j(a) = \det(a) w_{2m-j-1}(a).$$

This suggests that the $w_j(a)$ may be a function of $\det(a)$ alone and we may put

$$w_j(a) = \{\det(a)\}^j \quad \text{or} \quad w_j(a) = \{\det(a)\}^{j+1}$$

as the simplest solutions for even n . The ambiguity in the exponent here corresponds to the well known fact that there is an essential indefiniteness in the sign of $S(a)$ for improper a .

As a typical example we shall take up the case of reversal of the λ -th axis such that $\beta_{\lambda}' = -\beta_{\lambda}$ and $\beta_{\rho}' = \beta_{\rho}$ for $\rho \neq \lambda$. Then eq. (41) yields according to eq. (18) [see also below eq. (2)]

$$P_{\mu_1 \mu_2 \dots \mu_j; v_1 v_2 \dots v_j} e(\epsilon') S(\text{reversal}) = \pm (-)^j \epsilon_{\lambda}' P_{\mu_1 \mu_2 \dots \mu_j; v_1 v_2 \dots v_j} e(\epsilon'),$$

so that we have

$$S(\text{reversal})^{-1} = S(\text{reversal}) = \pm H \eta_{\lambda},$$

where H is the product of all the γ_{ρ} and accordingly anticommutable with every β_{ρ} . With the factor $w_j(a)$ eqs. (42) and (43) afford the same result $\pm H \eta_{\lambda}$, but without one the former reduces to η_{λ} whereas the latter to $-\eta_{\lambda}$. It should however be noted that in realizing eq. (40) this difference in sign turns out to be quite effective, because one has

$$\eta_{\lambda} \beta_{\rho} (-\eta_{\lambda}) = (\pm H \eta_{\lambda}) \beta_{\rho} (\pm H \eta_{\lambda}) = a_{\rho}^{\tau} \beta_{\tau}.$$

It is interesting in addition to observe that eq. (42) shows that $R_{n-j} S(a)$ is represented by a $n C_j$ -dimensional matrix each element of which is a minor of order j of $\det(a)$.

On left multiplication by any $P_{j;j}$ which itself is fixed for the above transformation, eq. (39) yields with the aid of eq. (41)

$$P_{\mu_1 \mu_2 \dots \mu_j; v_1 v_2 \dots v_j} \psi'(x') = w_j(a) a_{v_1}^{p_1} a_{v_2}^{p_2} \dots a_{v_j}^{p_j} P_{\mu_1 \mu_2 \dots \mu_j; p_1 p_2 \dots p_j} \psi(x).$$

From this we immediately see that the component extracted by means of the projector $P_{j;j}$ transforms as a tensor of rank j . In other words the projector $P_{\mu_1 \mu_2 \dots \mu_j; v_1 v_2 \dots v_j}$ extracts an antisymmetric tensor $F_{v_1 v_2 \dots v_j}(x)$ taking over its indices from the right set

of the corresponding projector. Moreover eqs. (30) show that the tensors extracted by means of the projectors (22) are respectively dual to those extracted by means of the projectors (23).

Now in the followings we shall be concerned with explicit demonstration of eq. (41) in the case of proper transformations. For this purpose we must first find out an explicit expression of the spin transformation operator $S(a)$ defined by eq. (40). In terms of n -dimensional proper real orthogonal matrix a we define

$$z(a) = \log(a) = \sum_{k=1}^{\infty} (-)^{k-1} (a-I)^k / k, \quad (44)^*$$

where I is the n -dimensional identity matrix. Then according to the orthogonality condition $a^T a = I$ this $z(a)$ turns out to be skew-symmetric:

$$z(a)^T + z(a) = 0. \quad (44')$$

In addition we introduce a set of nC_2 quantities $J_{\mu\nu}$ antisymmetric in μ and ν defined by two types of commutation relations

$$[\beta_\rho, J_{\mu\nu}] = g_{\rho\mu} \beta_\nu - g_{\rho\nu} \beta_\mu \quad (45)$$

and

$$[J_{\lambda\rho}, J_{\mu\nu}] = -g_{\lambda\mu} J_{\rho\nu} + g_{\lambda\nu} J_{\rho\mu} + g_{\rho\mu} J_{\lambda\nu} - g_{\rho\nu} J_{\lambda\mu}. \quad (46)$$

Then with the aid of $(1/2)z^{\mu\nu}J_{\mu\nu}$ for which we have $[\beta_\rho, (1/2)z^{\mu\nu}J_{\mu\nu}] = z_\rho^\lambda \beta_\lambda$, the required operator $S(a)$ can at once be given by

$$S(a) = E \cdot \exp[(1/2)z^{\mu\nu}J_{\mu\nu}], \quad (47)$$

for this satisfies in fact the defining equation (40):

$$S^{-1} \beta_\rho S = \exp[-z^{\mu\nu}J_{\mu\nu}/2] \cdot \beta_\rho \cdot \exp[z^{\mu\nu}J_{\mu\nu}/2] = (\exp z)_\rho^\lambda \beta_\lambda = a_\rho^\lambda \beta_\lambda = \beta'_\rho.$$

Moreover we have from eq. (46) $[J_{\lambda\rho}, (1/2)z^{\mu\nu}J_{\mu\nu}] = z_{\lambda}^\tau J_{\tau\rho} - J_{\lambda\tau} z^\tau_\rho$, which can simply be rewritten as $[J, (1/2)z^{\mu\nu}J_{\mu\nu}] = [z, J]$ by regarding $J_{\mu\nu}$ as an element of n -dimensional skew-symmetric matrix J . Taking account of further relationships $[[J, (1/2)z^{\mu\nu}J_{\mu\nu}], (1/2)z^{\mu\nu}J_{\mu\nu}] = [z, [z, J]]$ and so on, we arrive at

$$\exp[-z^{\mu\nu}J_{\mu\nu}/2] \cdot J \cdot \exp[z^{\mu\nu}J_{\mu\nu}/2] = \exp(z) \cdot J \cdot \exp(-z);$$

which reads more precisely

$$S(a)^{-1} J_{\mu\nu} S(a) = (a J a^T)_{\mu\nu} = a_\mu^\lambda a_\nu^\rho J_{\lambda\rho};$$

the $J_{\mu\nu}$ transforms as an antisymmetric tensor. In order to complete the present consideration we are necessarily to establish the relationship $S(b)S(a) = S(ba)$. It has been shown in the author's recent note¹²⁾ that for any noncommuting operators Y and Z , $\log(e^Y e^Z)$ gives a "Lie-element", that is, is expressed as $Y + Z + \text{multiple commutators of } Y \text{ and } Z$, so that we have in this case

*) For any proper real orthogonal matrix a the existence of $z(a)$ is ensured. See reference 15.

$$\log [\exp (y^{\mu\nu} J_{\mu\nu}/2) \cdot \exp (z^{\mu\nu} J_{\mu\nu}/2)] = (1/2) \{ \log (e^y e^z) \}^{\mu\nu} J_{\mu\nu},$$

for eq. (46) gives $[(1/2)y^{\mu\nu} J_{\mu\nu}, (1/2)z^{\mu\nu} J_{\mu\nu}] = (1/2) \{ [y, z] \}^{\mu\nu} J_{\mu\nu}$ with $y(b) = \log(b)$ being another n -dimensional skew-symmetric matrix. Incidentally we have owing to $\text{tr}(J_{\mu\nu}) = 0$

$$\det [S(a)] = \det [\exp (z^{\mu\nu} J_{\mu\nu}/2)] = \exp [\text{tr}(z^{\mu\nu} J_{\mu\nu}/2)] = 1.$$

Finally it is important to observe that the present consideration is valid in general for the algebra of arbitrary spin.

As an example we shall first take up the case of the four-dimensional Dirac algebra generated by well known commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} E$, wherein $J_{\mu\nu}$ is given by $J_{\mu\nu} = (1/4)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu)$. Now according to the identity

$$4J_{\lambda\rho} J_{\mu\nu} = -(g_{\lambda\mu} g_{\rho\nu} - g_{\lambda\nu} g_{\rho\mu}) E + \epsilon_{\lambda\rho\mu\nu} \gamma_5 \\ + 2(-g_{\lambda\mu} J_{\rho\nu} + g_{\lambda\nu} J_{\rho\mu} + g_{\rho\mu} J_{\lambda\nu} - g_{\rho\nu} J_{\lambda\mu}) \quad (48)$$

the square of $(1/2)z^{\mu\nu} J_{\mu\nu}$ is given by $(z^{\mu\nu} J_{\mu\nu}/2)^2 = M^2$ in terms of an auxiliary quantity $M = sE + t\gamma_5$ where the numbers s and t are defined by

$$st = (1/32) \epsilon^{\lambda\rho\mu\nu} z_{\lambda\rho} z_{\mu\nu} \quad \text{and} \quad s^2 + t^2 = (1/8) \text{tr}(z^2). \quad (48')$$

Then on account of the relationship $M^{-1} = [sE - t\gamma_5]/(s^2 - t^2)$ eq. (47) yields

$$S(a) = \cosh(M) + (1/2) z^{\mu\nu} J_{\mu\nu} M^{-1} \sinh(M) \\ = \cosh(s) \cosh(t) E + \sinh(s) \sinh(t) \gamma_5 \\ + \{1/2(s^2 - t^2)\} z^{\mu\nu} J_{\mu\nu} [\{s \cdot \sinh(s) \cosh(t) - t \cdot \cosh(s) \sinh(t)\} E \\ + \{s \cdot \cosh(s) \sinh(t) - t \cdot \sinh(s) \cosh(t)\} \gamma_5] \quad (49)$$

with $(1/2)z^{\mu\nu} J_{\mu\nu} \gamma_5 = -(1/4) \epsilon^{\lambda\rho\mu\nu} z_{\lambda\rho} J_{\mu\nu}$. Although the present transcription of $S(a)$ rather seems to be lacking in elegance and simplicity, an alternative form will be given later in § 6.

Returning to our subject, the $J_{\mu\nu}$ is written in the form

$$J_{\mu\nu} = \beta_\mu \beta_\nu - \beta_\nu \beta_\mu = Q_{\mu;\nu} - Q_{\nu;\mu}$$

in the n -dimensional Duffin-Kemmer algebra, so that we have

$$S(a) = E \cdot \exp [z^{\mu\nu} \beta_\mu \beta_\nu] = E \cdot \exp [z^{\mu\nu} Q_{\mu;\nu}]. \quad (50)$$

Then according to eq. (20) one has

$$P_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} z^{\mu\nu} Q_{\mu;\nu} \\ = \sum_{i=1}^j g_{\nu_1}^{p_1} g_{\nu_2}^{p_2} \dots g_{\nu_{i-1}}^{p_{i-1}} z_{\nu_i}^{p_i} g_{\nu_{i+1}}^{p_{i+1}} \dots g_{\nu_j}^{p_j} P_{\mu_1 \mu_2 \dots \mu_j; \rho_1 \rho_2 \dots \rho_j},$$

which is now symbolized simply as

$$= \sum_{i=1}^j Z_{\nu_1 \nu_2 \dots \nu_j; \rho_1 \rho_2 \dots \rho_j}^{(i)} P_{\mu_1 \mu_2 \dots \mu_j; \rho_1 \rho_2 \dots \rho_j}$$

in terms of $Z^{(i)} = I \times I \times \cdots \times I \times S \times I \times \cdots \times I$, a direct product of j n -dimensional matrices of which only the i -th one is $S(\alpha)$ and others are all identity matrices I . Incidentally we have $[Z^{(i)}, Z^{(k)}] = 0$. Now we find

$$\begin{aligned} P_{\mu_1 \mu_2 \cdots \mu_j; \nu_1 \nu_2 \cdots \nu_j} S(\alpha) &= P_{\mu_1 \mu_2 \cdots \mu_j; \nu_1 \nu_2 \cdots \nu_j} \exp [\mathcal{Z}^{\mu\nu} Q_{\mu; \nu}] \\ &= \left\{ \exp \left[\sum_{i=1}^j Z^{(i)} \right] \right\}_{\nu_1 \nu_2 \cdots \nu_j; \nu_1 \nu_2 \cdots \nu_j} P_{\mu_1 \mu_2 \cdots \mu_j; \mu_1 \mu_2 \cdots \mu_j} \\ &= \left\{ \prod_{i=1}^j \exp [Z^{(i)}] \right\}_{\nu_1 \nu_2 \cdots \nu_j; \nu_1 \nu_2 \cdots \nu_j} P_{\mu_1 \mu_2 \cdots \mu_j; \mu_1 \mu_2 \cdots \mu_j} \\ &= (e^{\alpha} \times e^{\alpha} \times \cdots \times e^{\alpha})_{\nu_1 \nu_2 \cdots \nu_j; \nu_1 \nu_2 \cdots \nu_j} P_{\mu_1 \mu_2 \cdots \mu_j; \mu_1 \mu_2 \cdots \mu_j} \\ &= a_{\nu_1}^{p_1} a_{\nu_2}^{p_2} \cdots a_{\nu_j}^{p_j} P_{\mu_1 \mu_2 \cdots \mu_j; \mu_1 \mu_2 \cdots \mu_j} \end{aligned}$$

which is quite the same thing as eq. (41) for proper α .

For $j=0$ eq. (42) reads simply $PS(\alpha) = P$ (remember $R_n = P$), which can also be derived from eq. (43) on putting $j=n-1$ and using eq. (30·IV). The same result can at once be obtained from the latter transcription in eq. (50) together with the equation $PQ_{\mu; \nu} = 0$ which is the very eq. (16) for $k=0 < j=1$. In the same way for $j=n$ eq. (42) leads us to $R_0 S(\alpha) = R_0$ in view of eq. (27), which result can also be obtained from the former transcription in eq. (50) together with eq. (4).

Finally it will be worthwhile to examine how our projectors operate on the wave equation (38), for this enables one to establish the direct correspondence between the two alternative modes of formulation of meson theory. First on multiplying eq. (38) by R_0 and taking into account eq. (4) we get the trivial scalar equation

$$\chi R_0 \psi = 0.$$

Then in the ordinary irreducible representations we get by means of $P_{j-1; j-1}$

$$\partial^p P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots \nu_{j-1}} \beta_p \psi + \chi P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots \nu_{j-1}} \psi = 0$$

and by means of $P_{j-1; j-1} \beta_{\nu_j}$

$$\sum_{i=1}^j (-)^{j-i} \partial_{\nu_i} P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots (\nu_i) \cdots \nu_j} \psi + \chi P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots \nu_{j-1}} \beta_{\nu_j} \psi = 0,$$

where use is made of eq. (21). As was pointed out by Kemmer,²⁾ we can derive from eq. (38)

$$\partial_\lambda \psi = \partial^p \beta_p \beta_\lambda \psi \quad \text{or} \quad \partial^p Q_{\lambda; p} \psi = 0. \quad (38')$$

On left multiplication by $P_{j-1; j-1}$ the latter yields according to eq. (20)

$$\partial^p P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots \nu_{j-2} p} \psi = 0,$$

while on left multiplication by $P_{j-1; j-1} \beta_{\nu_j}$ the former yields according to eq. (21)

$$\partial_\lambda P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots \nu_{j-1}} \beta_{\nu_j} \psi = \sum_{i=1}^j (-)^{j-i} \partial_{\nu_i} P_{\mu_1 \mu_2 \cdots \mu_{j-1}; \nu_1 \nu_2 \cdots (\nu_i) \cdots \nu_j} \beta_\lambda \psi,$$

which can further be rewritten by putting $\lambda = \nu_{j+1}$ as

$$\sum_{i=1}^{j+1} (-)^{j+1-i} \partial_{\nu_i} P_{\mu_1 \mu_2 \dots \mu_{j-1}; \nu_1 \nu_2 \dots (\nu_i) \dots \nu_j} \beta_{\nu_{j+1}} \psi = 0.$$

Especially in the twin representations eq. (38) reduces with the aid of eq. (37) to

$$m! d(m) \epsilon_{\nu_1 \nu_2 \dots \nu_{m-1} \rho_1 \rho_2 \dots \rho_{m-1}} \partial^\rho P_{\mu_1 \mu_2 \dots \mu_{m-1}}^{(\pm)} \cdot \rho_1 \rho_2 \dots \rho_{m-1} \psi \\ \pm \chi P_{\mu_1 \mu_2 \dots \mu_{m-1}; \nu_1 \nu_2 \dots \nu_{m-1}}^{(\pm)} \psi = 0,$$

and the latter in eq. (38') to

$$\partial^\rho P_{\mu_1 \mu_2 \dots \mu_{m-1}; \nu_1 \nu_2 \dots \nu_{m-1}}^{(\pm)} \psi = 0.$$

§ 6. Derivation of Klein's procedure

As has already been touched upon in the introduction, the present section is intended to set up a linkage between Klein's procedure and ours. The former is characterized by appealing throughout to the fusion-theoretical standpoint from the very beginning, so that we may safely expect that it is derivable from ours by adopting a special reducible representations $\gamma_\nu = \gamma'_\nu \times \gamma''_\nu$ in the final result of our projectors. Here we shall confine ourselves to the ordinary case of four dimensions.

To begin with we shall enter into the mathematics of the Dirac algebra generated by the commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I$. Almost all of its essential aspects have been revealed up by Pauli in his celebrated work,¹³⁾ so that we are anyhow to follow the lines laid out in his paper. In the followings we shall however proceed somewhat differently.

A linearly independent basis of this algebra is formed by the 16 elements

$$\gamma_A = \{I, \gamma_\rho, \sigma_{\mu\nu} = (i/2)(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) = 2iJ_{\mu\nu}, i\gamma_\rho \gamma_5 \text{ and } \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4\}$$

each having the square $(\gamma_A)^2 = I$, in terms of which the generic element of the algebra is expressed as $X = \sum_A x_A \gamma_A$. The whole argumentation of this section pivotes upon a newly introduced element termed "surrounded" X which is defined by

$$\langle X \rangle = (1/16) \sum_A \gamma_A X \gamma_A = (1/8) \{ [X] + \gamma_\rho [X] \gamma^\rho + (1/4) \sigma_{\mu\nu} [X] \sigma^{\mu\nu} \} \quad (51)$$

with $[X] = (1/2) \{ X + \gamma_5 X \gamma_5 \}$. Then the relationships

$$\begin{aligned} \gamma_\rho \gamma^\rho &= 4E, \quad (1/4) \sigma_{\mu\nu} \sigma^{\mu\nu} = 3E; \\ \gamma_\rho \sigma_{\mu\nu} \gamma^\rho &= 0, \quad (1/4) \sigma_{\lambda\rho} \sigma_{\mu\nu} \sigma^{\lambda\rho} = -\sigma_{\mu\nu}; \\ \gamma_\rho \gamma_5 \gamma^\rho &= -4\gamma_5, \quad (1/4) \sigma_{\mu\nu} \gamma_5 \sigma^{\mu\nu} = 3\gamma_5 \end{aligned}$$

show that

$$\langle E \rangle = E \text{ and } \langle \gamma_A \rangle = 0 \text{ for } \gamma_A \neq E. \quad (52)$$

We thus get $\langle X \rangle = \sum_A x_A \langle \gamma_A \rangle = x_E \cdot E$ and more generally $\langle X \gamma_A \rangle = \sum_B x_B \langle \gamma_B \gamma_A \rangle = x_A \cdot E$. Therefore the X can also be expressed as $X = \sum_A \langle X \gamma_A \rangle \gamma_A$ and we have

accordingly $\langle XY \rangle = \sum_A \langle X \gamma_A \rangle \langle \gamma_A Y \rangle$. It is essential here to observe that the surrounded element is closely related to the trace. As is well known the traces of γ_A are given by

$$\text{tr}(E) = 4 \quad \text{and} \quad \text{tr}(\gamma_A) = 0 \quad \text{for} \quad \gamma_A \neq E,$$

which yields at once $\text{tr}(X) = \sum_A x_A \text{tr}(\gamma_A) = 4x_E$. Hence we see that

$$\langle X \rangle = (1/4) \text{tr}(X) \cdot E \quad (53)$$

and moreover that $X = (1/4) \sum_A \text{tr}(X \gamma_A) \gamma_A$.

Then according to Pauli we shall make a slight extension of the above surrounded element. If a second system of the Dirac algebra generated by four quantities γ'_ν is pre-supposed, how can we find out a nonsingular matrix S such that $\gamma'_\nu = S^{-1} \gamma_\nu S$? The answer to this question is simple enough. For if in terms of the linearly independent element $(\gamma')_A = S^{-1} \gamma_A S$ of the primed system the surrounded element is extended to

$$\langle X \rangle' = (1/16) \sum_A \gamma_A X (\gamma')_A = (1/8) \{ [X'] + \gamma^\rho [X]' \gamma'_\rho + (1/4) \sigma^{\mu\nu} [X]' \sigma_{\mu\nu}' \} \quad (54)$$

with $[X]' = (1/2) \{ X + \gamma_5 X \gamma_5' \}$, this turns out to be a numerical multiple of the required transformation operator S :

$$\langle X \rangle' = (1/16) \sum_A \gamma_A X S^{-1} \gamma_A S = \langle X S^{-1} \rangle S = (1/4) \text{tr}(X S^{-1}) \cdot S. \quad (55)$$

Incidentally for $X=S$ this reduces to $\langle S \rangle' = S$; the S itself is invariant for extended surrounding. Above all the simplest and in addition the most interesting example is that for any orthogonal transformation $\gamma'_\nu = a_\nu^\lambda \gamma_\lambda$ of the four-dimensional γ -space. Among various possibilities of specifying the yet undetermined X , the only workable way is to put $X=E$ for which we have $\langle E \rangle' = \langle S^{-1} \rangle S = (1/4) \text{tr}(S^{-1}) \cdot S$. The spin transformation operator $S(a)$ is thus given by

$$S(a) = \{4/\text{tr}(S^{-1})\} \langle E \rangle'. \quad (56)$$

On account of the well known relationship $\gamma_5' = \det(a) \gamma_5$, the quantity $[E]' = (1/2) \{ E + \gamma_5 E \gamma_5' \}$ and accordingly the $\langle E \rangle'$ vanish identically for improper transformations, so that eq. (56) is applicable to the case of proper transformations alone. But since there certainly exists a non-zero operator $S(a)$ in either case, this situation is simply caused by vanishing of $\text{tr}(S^{-1})$ for improper transformations, as can also be seen from Racah's explicit establishments of $S(a)$. Now for proper transformations we have according to $[E]' = E$, $\gamma_\mu \gamma_\nu = g_{\mu\nu} E - i \sigma_{\mu\nu}$ and eq. (48) with $\sigma_{\mu\nu} = 2i J_{\mu\nu}$

$$\begin{aligned} \langle E \rangle' &= (1/8) \{ E + \gamma^\rho \gamma'_\rho + (1/4) \sigma^{\mu\nu} \sigma_{\mu\nu}' \} \\ &= (1/8) \{ E + a^{\rho\lambda} \gamma_\rho \gamma_\lambda + (1/4) a^{\mu\lambda} a^{\nu\rho} \sigma_{\mu\nu} \sigma_{\lambda\rho} \} \\ &= (1/8) \{ [1 + \text{tr}(a)/2]^2 - \text{tr}(a^2)/4 \} E + (1/4) \epsilon^{\mu\nu\lambda\rho} a_{\mu\nu} a_{\lambda\rho} \gamma_5 \\ &\quad - i \{ [1 + \text{tr}(a)/2] a^{\mu\nu} - (a^2)^{\mu\nu}/2 \} \sigma_{\mu\nu}, \end{aligned} \quad (57)$$

of which the trace is

$$\text{tr}(\langle E \rangle') = (1/4) \text{tr}(S^{-1}) \text{tr}(S) = (1/2) \{ [1 + \text{tr}(a)/2]^2 - \text{tr}(a^2)/4 \}. \quad (58)$$

Next in terms of the alternative mode of extension

$${}'\langle X \rangle = \Sigma_A (\gamma')_A X \gamma_A = S^{-1} \langle SX \rangle = (1/4) \text{tr}(SX) \cdot S^{-1}, \quad (55')$$

we get the inverse operator $S(a)^{-1}$ (of course for proper transformations):

$$S(a)^{-1} = \{4/\text{tr}(S)\} \cdot {}'\langle E \rangle. \quad (56')$$

Here the ${}'\langle E \rangle$ is simply given by replacing the matrix a in the above $\langle E \rangle'$ throughout by its transpose a^T :

$$\begin{aligned} {}'\langle E \rangle &= (1/8) \{E + \gamma'_\rho \gamma^\rho + (1/4) \sigma_{\mu\nu}' \sigma^{\mu\nu}\} \\ &= (1/8) \{E + (a^T)^{\rho\lambda} \gamma_\rho \gamma_\lambda + (1/4) (a^T)^{\mu\lambda} (a^T)^{\nu\rho} \sigma_{\mu\nu} \sigma_{\lambda\rho}\}. \end{aligned} \quad (57')$$

That is to say, the ${}'\langle E \rangle$ is obtained by reversing the sign of the third term containing $\sigma_{\mu\nu}$ in the final transcription of eq. (57). Comparing eqs. (56) and (56') we infer that we shall arrive at a transformation operator such that $S(a)^{-1} = S(a^T) = S(a^{-1})$ by imposing on $S(a)$ the condition $\text{tr}(S) = \text{tr}(S^{-1})$. Hence we have from eq. (58)

$$\text{tr}(S^{-1}) = \text{tr}(S) = 2\{\text{tr}(\langle E \rangle')\}^{1/2}. \quad (59)$$

The present expression of $S(a)$ is quite satisfactory being written in terms of the original matrix a , but this is not identifiable with the previous solution (49) owing to tremendous computations. For example we should at least be able to establish the equality $\cosh(s) \times \cosh(t) = \{\text{tr}(\langle E \rangle')/4\}^{1/2}$ obtained by comparing the traces of both expressions.

Now we should return to our original topics. In the ordinary case of four dimensions the Duffin-Kemmer wave function comprises $2^4 = 16$ components which are extracted respectively by means of the 16 normal primitive idempotents $e(\epsilon)$ defined by eq. (2). Among others the special $e(\epsilon)$ wherein every ϵ_ρ is positive, that is, $P = R_4$ [see below eq. (17)] can be rewritten with the aid of the special reducible representations $\eta_\rho = \gamma_\rho \times \gamma_\rho$ (here we have made a special choice $\gamma'_\rho = \gamma''_\rho$ in $\eta_\rho = \gamma'_\rho \times \gamma''_\rho$ and primes have been omitted for simplicity) as

$$P = (1/16) (E \times E + \gamma_1 \times \gamma_1) (E \times E + \gamma_2 \times \gamma_2) (E \times E + \gamma_3 \times \gamma_3) (E \times E + \gamma_4 \times \gamma_4). \quad (60)$$

Then on the basis of this element we can set up one-to-one correspondences between these idempotent projectors $e(\epsilon)$ and the 16 linearly independent elements γ_A of the Dirac algebra in such a way that

$$e(\gamma_A) = (\gamma_A \times E) P (\gamma_A \times E). \quad (61)$$

Of course $\gamma_A = E$ affords the P itself; $e(E) = P$. The specification $\gamma_A = \gamma_5$ corresponds to the $e(\epsilon)$ with all its ϵ_ρ being negative, that is, to R_0 which extracts the trivial scalar component. Furthermore it is easily seen that $\gamma_A = (E \text{ and } \gamma_\rho)$ correspond to the usual scalar theory, while $\gamma_A = (\sigma_{\mu\nu} \text{ and } i\gamma_\rho \gamma_5)$ to the vector theory.

As the next step we shall specify each one of the 16 components of the wave function ψ by a pair of indices μ and ν , and thus regard it as an element of four-dimensional matrix \mathcal{U} situated in μ -th row and ν -th column. If we assume that a 16-dimensional

matrix C is expressible as direct product of two four-dimensional matrices A and B in such a way that $C=A \times B$, the $\mu\nu$ -component of the wave function $C\psi$ can be rewritten in terms of the above Ψ as

$$(C\psi)_{\mu\nu} = ([A \times B]\psi)_{\mu\nu} = A_{\mu}^{\lambda} B_{\nu}^{\rho} \psi_{\lambda\rho} = (A\Psi B^T)_{\mu\nu}.$$

In this connection we notice that there exists a nonsingular matrix T such that $\gamma_{\rho}^T = T^{-1}\gamma_{\rho}T$. Moreover $(\gamma_A)^T$ is not necessarily the same as $(\gamma^T)_A = T^{-1}\gamma_A T$ and the 16 linearly independent elements γ_A are subdivided into two classes according as $(\gamma_A)^T = (\gamma^T)_A$ or $(\gamma_A)^T = -(\gamma^T)_A$. The former corresponds to the six elements E , γ_{ρ} and γ_5 , while the latter to the remaining ten elements $\sigma_{\mu\nu}$ and $i\gamma_{\rho}\gamma_5$.

Now eq. (60) can further be rewritten as

$$\begin{aligned} P &= (1/16) \{E \times E + \gamma_{\rho} \times \gamma^{\rho} - (1/2) \sigma_{\mu\nu} \times \sigma^{\mu\nu} - i\gamma_{\rho}\gamma_5 \times i\gamma^{\rho}\gamma_5 + \gamma_5 \times \gamma_5\} \\ &= (1/16) \Sigma_B \pm \gamma_B \times \gamma_B, \end{aligned} \quad (60')$$

where the negative sign corresponds to the above second class, so that according to all the preceding considerations we arrive at the final conclusion that

$$\begin{aligned} \{v(\gamma_A)\psi\}_{\mu\nu} &= \{(\gamma_A \times E)P(\gamma_A \times E)\psi\}_{\mu\nu} = (1/16) \Sigma_B \pm \{(\gamma_A \times E)(\gamma_B \times \gamma_B)(\gamma_A \times E)\psi\}_{\mu\nu} \\ &= (1/16) \Sigma_B \pm \{\gamma_A \gamma_B \gamma_A \Psi (\gamma_B)^T\}_{\mu\nu} = (1/16) \Sigma_B \{\gamma_A \gamma_B \gamma_A \Psi (\gamma^T)_B\}_{\mu\nu} \\ &= (1/16) \Sigma_B \{\gamma_A \gamma_B \gamma_A \Psi T^{-1} \gamma_B T\}_{\mu\nu} = \{\gamma_A \langle \gamma_A \Psi T^{-1} \rangle T\}_{\mu\nu} \\ &= (1/4) i \nu (T^{-1} \gamma_A \Psi) \cdot (\gamma_A T)_{\mu\nu}. \end{aligned} \quad (62)$$

In the last expression the first member provides essentially the same thing as that of Klein's. The only difference lies in the fact that instead of the present T exclusive use is made of the charge-conjugating matrix $C = \gamma_5 T$ in his paper. This comes directly from the standpoint regarding R_0 as the basic projector instead of the present P . The second numerical factor $(\gamma_A T)_{\mu\nu}$ confirms the celebrated fact that the wave function $\{v(\gamma_A)\psi\}_{\mu\nu}$ is symmetric or antisymmetric in its indices according as it refers to the vector theory or to the trivial scalar and usual scalar theories.

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Appendix

In terms of the $G_{j;j}$ constructed from the $g_{\mu\nu}$ alone we define a new totally anti-symmetric tensor $A_{j;j}$ by

$$\begin{aligned} A_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} &= G_{\mu_1 \mu_2 \dots \mu_j; \rho_1 \rho_2 \dots \rho_j} a^{\rho_1}_{\nu_1} a^{\rho_2}_{\nu_2} \dots a^{\rho_j}_{\nu_j} \\ &= G_{\lambda_1 \lambda_2 \dots \lambda_j; \nu_1 \nu_2 \dots \nu_j} a_{\mu_1}^{\lambda_1} a_{\mu_2}^{\lambda_2} \dots a_{\mu_j}^{\lambda_j}, \end{aligned} \quad (A.1)^*$$

* Strictly speaking the equivalence of these two alternative definitions of $A_{j;j}$ follows from the recurrence formulas (A. 2).

where α is an arbitrary n -dimensional matrix. From the very nature of $G_{j,j}$ [see below eq. (25)] it will easily be seen that this $A_{j,j}$ corresponds to what is called in the theory of determinant the minor of $\det(\alpha)$ wherein the rows $(\mu_1, \mu_2, \dots, \mu_j)$ and the columns $(\nu_1, \nu_2, \dots, \nu_j)$ are taken from the original matrix α . Then eqs. (26) and (26') show that the $A_{j,j}$ is generated by the following recurrence formulas quite identical in structure with those for $G_{j,j}$:

$$\left. \begin{aligned} A_{\mu_1 \mu_2 \dots \mu_j \mu; \nu_1 \nu_2 \dots \nu_j \nu} &= a_{\mu \nu} A_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} - \sum_{i=1}^j a_{\mu \nu_i} A_{\mu_1 \mu_2 \dots (\mu) \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} \\ &= a_{\mu \nu} A_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} - \sum_{i=1}^j a_{\mu \nu_i} A_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots (\nu) \dots \nu_j} \end{aligned} \right\} \quad (\text{A} \cdot 2)$$

with $A_{\mu; \nu} = a_{\mu \nu}$. Now if we put $a_{\mu \nu} = g_{\mu \nu}$, the $A_{j,j}$ reduces to $G_{j,j}$ itself, so that we can say that this is a minor determinant taken from the n -dimensional identity matrix I . Incidentally eq. (A.1) yields for $j=n$

$$A_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} = G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} \det(\alpha). \quad (\text{A} \cdot 3)$$

The cofactor or the complementary determinant of $A_{j,j}$ is further given by

$$\left. \begin{aligned} \bar{A}_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} &= \{1/(n-j)!\} G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} a^{\mu_{j+1} \nu_{j+1}} a^{\mu_{j+2} \nu_{j+2}} \dots a^{\mu_n \nu_n} \\ &= \{1/(n-j)!\} {}^2 G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} A^{\mu_{j+1} \mu_{j+2} \dots \mu_n; \nu_{j+1} \nu_{j+2} \dots \nu_n} \end{aligned} \right\} \quad (\text{A} \cdot 4)$$

and reduces for $j=0$ and $=n$ respectively to $\bar{A} = \det(\alpha)$ and to

$$\bar{A}_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} = G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n}. \quad (\text{A} \cdot 5)$$

Then according to eq. (28') we get

$$\begin{aligned} & a_{\mu \nu} \bar{A}_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} - \sum_{i=1}^j a_{\mu \nu_i} \bar{A}_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots (\nu) \dots \nu_j} \\ &= \{1/(n-j)!\} a_{\mu}^{\lambda} a^{\mu_{j+1} \nu_{j+1}} a^{\mu_{j+2} \nu_{j+2}} \dots a^{\mu_n \nu_n} \\ & \quad \times \{g^{\lambda \nu} G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots \nu_n} - \sum_{i=1}^j g^{\lambda \nu_i} G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots (\nu) \dots \nu_n}\} \\ &= \{1/(n-j)!\} a_{\mu}^{\lambda} a^{\mu_{j+1} \nu_{j+1}} a^{\mu_{j+2} \nu_{j+2}} \dots a^{\mu_n \nu_n} \sum_{i=j+1}^n g^{\lambda \nu_i} G_{\mu_1 \mu_2 \dots \mu_n; \nu_1 \nu_2 \dots (\nu) \dots \nu_n} \\ &= \{1/(n-j-1)!\} a_{\mu}^{\lambda} a^{\rho \lambda} G_{\mu_1 \mu_2 \dots \mu_{n-1}; \nu_1 \nu_2 \dots \nu_{n-1}} a^{\mu_{j+1} \nu_{j+1}} a^{\mu_{j+2} \nu_{j+2}} \dots a^{\mu_{n-1} \nu_{n-1}} \\ &= (a a^T)_{\mu}^{\rho} \bar{A}_{\mu_1 \mu_2 \dots \mu_j \rho; \nu_1 \nu_2 \dots \nu_j \nu} \end{aligned}$$

Provided that $aa^T = I$, this is quite identical in structure with the second one in eq. (A.2) and on comparing eqs. (A.3) and (A.5) we get the identity

$$A_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} = \det(\alpha) \cdot \bar{A}_{\mu_1 \mu_2 \dots \mu_j; \nu_1 \nu_2 \dots \nu_j} \quad (\text{A} \cdot 6)$$

for an orthogonal α such that $\det(\alpha) = \pm 1$. Especially for $\alpha_{\mu\nu} = \delta_{\mu\nu}$ this gives together with eq. (A.4)

$$G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} = \bar{G}_{\mu_1\nu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j} \\ = \{1/(n-j)!\} G_{\mu_1\mu_2\cdots\mu_j\rho_1\rho_2\cdots\rho_{n-j}; \nu_1\nu_2\cdots\nu_j}^{\rho_1\rho_2\cdots\rho_{n-j}}. \quad (\text{A.7})$$

Moreover we have from eqs. (A.1), (A.4), (29) and (A.7)

$$(1/j!) \bar{A}_{\mu_1\mu_2\cdots\mu_j; \rho_1\rho_2\cdots\rho_j} A_{\nu_1\nu_2\cdots\nu_j}^{\rho_1\rho_2\cdots\rho_j} \\ = \{j!(n-j)!\}^{-1} G_{\mu_1\mu_2\cdots\mu_j\lambda_{j+1}\lambda_{j+2}\cdots\lambda_n; \rho_1\rho_2\cdots\rho_j\rho_{j+1}\rho_{j+2}\cdots\rho_n} G_{\nu_1\nu_2\cdots\nu_j; \lambda_1\lambda_2\cdots\lambda_j} \\ \times a^{\lambda_1\rho_1} a^{\lambda_2\rho_2} \cdots a^{\lambda_j\rho_j} a^{\lambda_{j+1}\rho_{j+1}} a^{\lambda_{j+2}\rho_{j+2}} \cdots a^{\lambda_n\rho_n} \\ = \{j!(n-j)!\}^{-1} \det(\alpha) G_{\mu_1\mu_2\cdots\mu_j\lambda_{j+1}\lambda_{j+2}\cdots\lambda_n; \lambda_1\lambda_2\cdots\lambda_j\lambda_{j+1}\lambda_{j+2}\cdots\lambda_n} G_{\nu_1\nu_2\cdots\nu_j; \lambda_1\lambda_2\cdots\lambda_j} \\ = \det(\alpha) \cdot G_{\mu_1\mu_2\cdots\mu_j; \nu_1\nu_2\cdots\nu_j}, \quad (\text{A.8})$$

which furnishes a generalized expression for Laplace's development of $\det(\alpha)$.

Finally it is inferred naturally from the properties of $G_{n,n}$ that this is expressible simply as

$$G_{\mu_1\mu_2\cdots\mu_n; \nu_1\nu_2\cdots\nu_n} = \epsilon_{\mu_1\mu_2\cdots\mu_n} \epsilon_{\nu_1\nu_2\cdots\nu_n}, \quad (\text{A.9})$$

and in view of eq. (A.7) we immediately see that the $G_{j,j}$ is invariant in general for interchange of two sets of indices. The transcription (A.9) shows in addition that eqs. (28) and (28') reduce to a single relationship

$$\mathcal{G}_{\lambda\rho} \epsilon_{\rho_1\rho_2\cdots\rho_n} = \sum_{\delta=1}^n \mathcal{G}_{\lambda\rho_\delta} \epsilon_{\rho_1\rho_2\cdots\{\rho\}\cdots\rho_n}. \quad (\text{A.10})$$

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On the Transition Matrix and the Green Function in the Quantum Field Theory

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The J. Schwinger's formalism of the Green function in the quantum electrodynamics is applied to the transition problem of the state. It is shown that the many body kernel in the Heisenberg representation involves the information about the transition of the state and this is directly represented by the repeated use of the one body kernel G , \mathcal{G} and the vertex operator Γ_μ defined by J. Schwinger. Further, the renormalization is discussed without use of the usual perturbation theory, although there remains the difficulty associated with the b -divergence.

§ 1. Introduction

The propagation of the interaction effects between the fields is usually represented by the Green functions in the present quantum field theory. In the interaction representation these Green functions are the well-known Δ_F , S_F , etc. The matrix describing the transition of the state is constructed by the repeated use of these Feynman's functions. As was shown by Stückelberg¹⁾, introducing the Feynman's functions and the usual Dyson's S -matrix follows as a logical consequence of the requirements of the causality for the propagation of the interaction effects. This fact suggests that the Green function is one of the fundamental basis of the current quantum field theory.

The importance of these Green function becomes clearer in the Heisenberg representation, because there the state vector is time-independent and the temporal development of the system is described by the field operators and so it can be expected that the Green function involves the information about the transition of the state. On the other hand, many authors^{2), 3)} have expected that the Green function of the n -body problem involves also the information about the stationary state of the n -body problem.

It is the aim of this paper to investigate the detailed property of the Green function in the Heisenberg representation. A crucial method to treat the Green function in the Heisenberg representation has been proposed by J. Schwinger.²⁾ In this paper we shall treat our problem along the same line as he has done and restrict ourselves only to the quantum electrodynamics.

In the usual perturbation theory, the Green function of the one body problem has a special importance. As Dyson⁴⁾ has shown in the quantum electrodynamics, the S -matrix element is obtained through the substitution of S'_F , D'_F , and I' for the electron line, photon line, and the vertex part in any possible irreducible graphs corresponding to the given transition process.

These S'_F , D'_F correspond to the Green functions of the one body problem. In § 4, we show, without use of the usual perturbation theory, that the matrix element of any transition is written by an adequate graph which contains the Green function of the one body problem and I'_μ as for the internal line and vertex part, respectively. Further, we will treat the renormalization problem in our method without use of the perturbation theory. However, on account of the difficulty associated with the b -divergence, the completion of the discussion is confined to be left in future.

§ 2. On the transition matrix and the Green function of the many body problem

We treat the problem in the quantum electrodynamics, whose Lagrange density is of the form

$$L = -1/2 \cdot \bar{\psi} \{ \gamma_\mu (\partial_\mu - ie A_\mu) + \kappa \} \psi + 1/2 \cdot \{ \bar{\psi} \gamma_\mu + \bar{\tilde{\psi}} \gamma_\mu \} + h.c. \\ - 1/4 \cdot F_{\mu\nu}^2 - 1/2 \cdot (\partial_\mu A_\mu)^2 - J_\mu A_\mu, \quad (2.1)$$

where ψ is a spinor source which anticommutes with ψ and $\bar{\tilde{\psi}}$, $\tilde{\psi}$ is given by $\gamma^* \gamma_1$ and J_μ is a c-number source current and further $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.*

From (2.1), the well-known equations of motion in the Heisenberg representation are obtained:

$$\{ \gamma_\mu (\partial_\mu - ie A_\mu) + \kappa \} \psi = \eta, \quad (2.2)$$

$$\square A_\mu = -J_\mu - j_\mu, \quad (2.3)$$

$$j_\mu \equiv -ie/2 \bar{\psi} \gamma_\mu \psi + h.c. \quad (2.4)$$

Now let us introduce the following notation for the operators in the Heisenberg representation, $A(x_1)$, $B(x_2)$, ...,

$$\langle A(x_1), B(x_2), \dots \rangle_+ \equiv (\Psi_0, P(A(x_1), B(x_2), \dots) \Psi_0), \quad (2.5)$$

where Ψ_0 is a vacuum state vector defined as the minimum energy state in this representation. Further, we use the interaction representation which coincides with the Heisenberg representation at $\sigma(x) = 0$ and denote the minimum energy state in this representation as ϕ_0 . If the quantities in the Heisenberg representation are denoted by $A(x_1)$, $B(x_2)$, ..., corresponding to $A(x_1)$, $B(x_2)$, ..., respectively, then there are following relations between both quantities:

$$A(x_1) = U(\sigma(x_1), 0) A(x_1) U^{-1}(\sigma(x_1), 0), \text{ etc.} \quad (2.6)$$

As Gell-Mann and Low⁽¹⁾ has shown, Ψ_0 may be written as

$$\Psi_0 = \frac{1}{c} U^{-1}(\pm \infty, 0) \phi_0 / (\phi_0, U^{-1}(\pm \infty, 0) \phi_0), \quad (2.7)$$

where c is a normalization constant.

* The star expresses the hermitian conjugate.

Using (2.7), (2.5) is rewritten as

$$\langle A(x_1), B(x_2), \dots \rangle_+ = (\Phi_0, U(\infty, 0) P(A(x_1), B(x_2), \dots) \\ \times U^{-1}(-\infty, 0) \Phi_0) / (\Phi_0, U(\infty, -\infty) \Phi_0), \quad (2.8)^*$$

where we used the following relation ;

$$U(\sigma, 0) U^{-1}(\sigma', 0) = U(\sigma, \sigma'). \quad (2.9)$$

Since the annihilation and creation of α -particle are described by the positive and negative frequency parts $Q_\alpha^\pm(x)$ of α -field operator $Q_\alpha(x)$, respectively, we have from (2.8) the next theorem. Namely, the transition matrix S_η between the initial state, where α -particle with the energy-momentum k_μ and spin state r etc. exists, and the final state, where α -particle with the energy-momentum k'_μ and spin state r' etc. exists, is

$$S_\eta = a I_{i(k_\mu, r; \dots)}^{f(k'_\mu, r'; \dots)} \langle Q_\alpha(\bar{x}_1), \dots, Q_\alpha(x_1), \dots \rangle_+, \quad (2.10)^{**}$$

where $I_{i(k_\mu, r; \dots)}^{f(k'_\mu, r'; \dots)}$ is the operator which takes out the Fourier's amplitudes referring to " k_μ , r and the negative frequency part at the world point x_1 " and " k'_μ , r' and the positive frequency part at the world point x_1 ". a is a normalization constant and x_1 and \bar{x}_1 are the coordinates of the particles in the initial and final state, respectively, and so their time components are $-\infty$ and $+\infty$, respectively. If we bring out the quantities of \bar{x}_1 in front of the one of x_1 , we have the following relation :

$$\langle Q_\alpha(x_1), \dots, Q_\alpha(\bar{x}) \dots \rangle_+ = (\Phi_0, Q_\alpha(x_1), \dots, U(\infty, -\infty) Q_\alpha(\bar{x}_1) \dots \Phi_0) \\ \times 1 / (\Phi_0, U(\infty, -\infty) \Phi_0),$$

which gives the proof of (2.10).

Using the creation and annihilation operators q_r^\pm , the Fourier transform of Q_α is expressed in the form

$$Q_\alpha(x) = \lim_{p \rightarrow \infty} \sum_k V^{-1/2} [d_{\alpha r}(k) q_r^+(k) \exp i(kr - k_0 t) \\ + d_{\alpha r}^*(k) q_r^-(k) \exp -i(kr - k_0 t)]. \quad (2.11)$$

The normalization constant in (2.10) is expressed by the reciprocal of the product of

* If we denote the interaction Hamiltonian density as $H(x)$ and apply the usual perturbation theory, we have

$$\langle A(x_1), B(x_2), \dots \rangle_+ = (\Phi_0, \sum_{n=0}^{\infty} (-i)^n / n! \int \dots \int dy_1 \dots dy_n \\ \times P(A(x_1), B(x_2), \dots H(y_1) \dots H(y_n)) \Phi_0) / (\Phi_0, U(\infty, -\infty) \Phi_0). \quad (2.8)'$$

The appearance of the denominator in (2.8)' corresponds to the procedure in the usual perturbation which leaves out of consideration the isolated diagram, whose initial and final states are the vacuum states.

** (2.10) has been applied to the problem of the multiple production of meson ; H. Umezawa et al., Phys. Rev. **85** (1952), 505.

$$(d_{\alpha r'}^*(\mathbf{k}') \dots, d_{\alpha r}(\mathbf{k}) \dots)^*.$$

Now we define the Green functions for many body problems as follows:
electron Green functions

$$\begin{aligned} G_{\eta}(x_1, x'_1) &\equiv \partial/\partial\eta(x'_1) \langle \psi(x_1) \rangle, \\ G_{\eta}(\bar{x}_1, x_2, x'_1) &\equiv \partial/\partial\bar{\eta}(x_2) \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+ \epsilon, \\ G_{\eta}(x_1, x'_1, x'_2) &\equiv \partial/\partial\eta(x'_2) \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+ \epsilon, \\ G_{\eta}(x_1, x_2, x'_1, x'_2) &\equiv \partial/\partial\eta(x'_2) \langle \psi(x_1), \bar{\psi}(x'_1) \psi(x_2) \rangle_+ \epsilon \text{ etc.} \end{aligned} \quad (2.12)^{**}$$

photon Green functions

$$\begin{aligned} \mathfrak{G}_{J,\nu}^{\mu}(\xi_1, \xi_2) &\equiv \partial/\partial J_{\nu}(\xi_2) \langle A_{\mu}(\xi_1) \rangle, \\ \mathfrak{G}_{J,\nu}^{\mu,\sigma}(\xi_1, \xi_2, \xi_3) &\equiv \partial/\partial J_{\nu}(\xi_3) \langle A_{\mu}(\xi_1), A_{\sigma}(\xi_2) \rangle_+, \\ \mathfrak{G}_{J,\nu}^{\mu,\sigma,\rho}(\xi_1, \xi_2, \xi_3, \xi_4) &\equiv \partial/\partial J_{\nu}(\xi_4) \langle A_{\mu}(\xi_1), A_{\sigma}(\xi_2), A_{\rho}(\xi_3) \rangle_+ \text{ etc.} \end{aligned} \quad (2.13)$$

mixed Green functions

$$\begin{aligned} K_{\eta J,\nu}(x_1, x'_1, \xi_1) &\equiv \partial/\partial J_{\nu}(\xi_1) \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+, \\ K_{\eta J,\nu}^{\mu}(x_1, x'_1, \xi_1, \xi'_1) &\equiv \partial/\partial J_{\nu}(\xi'_1) \langle \psi(x_1), \bar{\psi}(x'_1), A_{\mu}(\xi_1) \rangle_+. \end{aligned} \quad (2.14)$$

According to the calculation rule given by J. Schwinger, we have

$$\begin{aligned} G_{\eta}(x_1, x'_1) &= i \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+ \epsilon - i \langle \psi(x_1) \rangle \langle \bar{\psi}(x'_1) \rangle, \\ G_{\eta}(x_1, x_2, x'_1) &= i \langle \psi(x_1), \bar{\psi}(x'_1), \psi(x_2) \rangle_+ \epsilon \\ &\quad - i \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+ \langle \psi(x_2) \rangle \epsilon, \\ &\dots\dots\dots \\ \mathfrak{G}_J(\xi_1, \xi_2) &= i \langle A(\xi_1), A(\xi_2) \rangle_+ - i \langle A(\xi_1) \rangle \langle A(\xi_2) \rangle, \\ &\dots\dots\dots \\ K_{\eta J}(x_1, x'_1, \xi_1) &= i \langle \psi(x_1), \bar{\psi}(x'_1), A(\xi_1) \rangle_+ \epsilon \\ &\quad - i \langle A(\xi_1) \rangle \langle \psi(x_1), \bar{\psi}(x'_1) \rangle_+ \epsilon, \\ &\dots\dots\dots \end{aligned} \quad (2.15)$$

* (2.11) for the case of the field with the arbitrary spin has been discussed by Y. Takahashi and H. Umezawa; Prog. Theor. Phys. 9(1953), 14.

** ϵ in the expression $\langle \rangle_+ \epsilon$ means the sign function as to the coordinates appearing in $\langle \rangle_+$. For instance, for the case of $\langle \psi(x_1), \psi(x_2), \bar{\psi}(x'_1), \bar{\psi}(x'_2) \rangle_+ \epsilon$, ϵ is given by

$$\epsilon = \epsilon(x_1, x_2) \epsilon(x'_1, x'_2) \epsilon(x_1, x'_2) \epsilon(x_2, x'_1) \epsilon(x_2, x'_2) \epsilon(x_1, x'_1),$$

where

$$\begin{aligned} \epsilon(x, y) &= 1 & \text{for } x_0 > y_0, \\ &= -1 & \text{for } x_0 < y_0. \end{aligned}$$

In the following discussion we shall eliminate this symbol as far as it does not give rise to mistake.

Since $U(\sigma, \sigma')$ has non vanishing matrix element only for the transition satisfying the charge conservation law, we have the following relation for F , in which the numbers of the ψ and $\bar{\psi}$ are different each other;

$$\langle F \rangle_{\eta \rightarrow 0} = 0. \quad (2.16)$$

For example, we have

$$\langle \psi \rangle_{\eta \rightarrow 0} = \langle \bar{\psi} \rangle_{\eta \rightarrow 0} = 0. \quad (2.17)$$

From the invariance of the theory under the charge conjugation, we have the Furry's theorem, $\langle A_\mu \rangle_{J \rightarrow 0} = 0$.

From these relation and (2.10), (2.15), we find that the many body Green function at the limit ($J \rightarrow 0, \eta \rightarrow 0$) corresponds to the transition matrix element. Hereafter, we denote these quantities with ($J \rightarrow 0, \eta \rightarrow 0$) by those with the super suffix 0.* For example, $G_\eta^0(x_1, \bar{x}_2, \bar{x}_1, \bar{x}_2)$, $\mathcal{G}_J(\xi_1, \bar{\xi}_2, \bar{\xi}_1, \bar{\xi}_2)$, and $K_{\eta J}^0(x_1, \bar{x}_2, \bar{\xi}_1, \bar{\xi}_2)$ correspond to the Möller scattering of the two electrons, photon-photon scattering, and Compton-scattering, respectively.

One body Green function is connected with the current $j_\mu(x)$ as follows:

$$\langle \hat{j}_\mu(x) \rangle = -T_r \{ \gamma_\mu G_\eta(x, x) \}, \quad (2.18)$$

where

$$G_\eta(x, x) \equiv \lim_{x' \rightarrow x} [G_\eta(x, x') + G_\eta(x', x)]/2. \quad (2.19)$$

Hereafter, the matrix representation for the coordinates of electron and photon is used and so the matrices 1 , ∂_μ , ψ , $\bar{\psi}$, A_μ , G_η and $\mathcal{G}_{J, \nu}^\mu$ are $\partial(x-x')$, $\partial(x-x')\partial_\mu$, $\psi(x)\partial(x-x')$, $\bar{\psi}(x)\partial(x-x')$, $A_\mu(x)\partial(x-x')$, $G_\eta(x, x')$ and $\mathcal{G}_{J, \nu}^\mu(x, x')$, respectively.

From the equations of motion (2.2), (2.3), we have for G_η , $\mathcal{G}_{J, \nu}^\mu$ the equations,

$$\{ \gamma_\mu (\partial_\mu - ie \langle A_\mu \rangle) + \bar{M} \} G_\eta = 1 + ie \gamma_\mu^1 \langle \psi \rangle \partial / \partial J_\mu \langle \bar{\psi} \rangle, \quad (2.20)^{**}$$

$$\{ \square - \bar{P} \} \mathcal{G}_{J, \nu}^\mu = \delta_{\mu\nu} 1. \quad (2.21)$$

In the above expression the mass operator \bar{M} and polarization operator \bar{P} are the matrices whose elements are given by $\bar{M}(x, x')$, $\bar{P}(\xi, \xi')$ defined as follows:

$$\int dx'' \bar{M}(x, x'') G_\eta(x'', x') \equiv (x - e \gamma_\mu \partial / \partial J_\mu(x)) G_\eta(x, x'), \quad (2.22)$$

$$\int d\xi'' \bar{P}(\xi, \xi'') \mathcal{G}_{J, \nu}^\mu(\xi'', \xi') \equiv e T_r \{ \gamma_\mu \partial / \partial J_\nu(\xi') G_\eta(\xi, \xi) \}. \quad (2.23)$$

* The limiting process ($J \rightarrow 0, \eta \rightarrow 0$) should be taken after the variational operation of (2.12), (2.13) and (2.14).

** The product of the two matrices A and B is defined by

$$(x|AB|x') \equiv \int dy (x|A|y) (y|B|x').$$

Further the symbol \cdot in the expression \mathcal{A} or $\bar{\mathcal{A}}$ denotes the right or left coordinate of the matrix A .

As was shown by J. Schwinger, we have

$$\partial/\partial J_\nu(\xi) G_\eta|_{J,\eta \rightarrow 0} = e G_\eta^0 \int d\xi' I_\mu^0(\xi') G_\eta^0 \mathfrak{G}_{J,\nu}^0(\xi', \xi), \quad (2.24)$$

where the vertex operator $I_\mu^0(\xi)$ is defined as follows²⁾:

$$ie\gamma\partial/\partial J_\nu(\xi) \langle A_\mu(x) \rangle \delta(x-x') - \frac{\partial \bar{M}}{\partial J_\nu(\xi)} \Big|_{J,\eta \rightarrow 0} = e \int d\xi' I'(\xi'; x, x') \mathfrak{G}_{J,\nu}^0(\xi', \xi). \quad (2.25)$$

From (2.22) and (2.24), we have

$$\begin{aligned} \bar{M} = & x1 + ie\gamma_\mu \langle A_\mu \rangle - e^2 \gamma_\mu \int d\xi' G_\eta I_\nu(\xi') \mathfrak{G}_{J,\nu}^0(\xi', \cdot) \\ & - e\gamma_\mu G_\eta \partial/\partial J_\mu \cdot \{ie\gamma_\nu \langle \psi \rangle \partial/\partial J_\nu \langle \psi \rangle G_\eta^{-1}\} G_\eta, \end{aligned} \quad (2.26)$$

and so

$$\bar{M}^0 \equiv \bar{M}|_{J,\eta \rightarrow 0} = x1 - e^2 \gamma_\nu \int d\xi' G_\eta^0 I_\mu^0(\xi') \mathfrak{G}_{J,\nu}^0(\xi', \cdot). \quad (2.27)$$

§ 3. Note on the renormalization theory

In this section a short note on the relation of the above theory with Dyson's one is given.

The differential equation (2.21) can be transformed into the integral form;

$$\begin{aligned} G_\eta^0(x, x') = & S_F(x-x') - \delta x \int dy S_F(x-y) G_\eta^0(y, x') \\ & - \int dy dy' S_F(x-y) \Sigma^0(y, y') G_\eta^0(y', x'), \end{aligned} \quad (3.1)$$

where

$$\Sigma^0 = \bar{M}^0 - x'1, \quad x' = x + \delta x.$$

Using (2.27) we have

$$\begin{aligned} G_\eta^0(x, x') = & S_F(x-x') - \delta x \int dy S_F(x-y) G_\eta^0(y, x') \\ & - e^2 \int dy dy' dy'' d\xi' S_F(x-y) G_\eta^0(y, y') \\ & \times I_\nu^0(\xi'; y', y'') G_\eta^0(y'', x') \mathfrak{G}_{J,\nu}^0(\xi', y). \end{aligned} \quad (3.2)$$

If we replace G_η^0 by S'_F in (3.1), this equation corresponds to the integral equation given by Dyson and Σ^0 amounts to the total contribution from the self-energy graph. (3.2) agrees with the final integration of the self-energy graph given by Dyson and corresponds to Fig. 1, in which the electron line, photon line and vertex δ correspond to G_η^0 , \mathfrak{G}_J^0 and I_μ^0 , respectively.



Fig. 1



Fig. 2 (i)



Fig. 2 (ii)

The fact that γ of the point a is not replaced by Γ_μ^0 corresponds to the Dyson's argument on the b -divergence, because we must take into account only one of the equivalent graphs (i), (ii) in Fig. 2.

In the quantum electrodynamics we can normalize ϕ and A_μ so that in the high energy region all possible quantities with dimension of the length are the momenta p_μ of particles^{(6)*}. In this case the dimensions of G_η^0 and \mathfrak{G}_J^0 agree with those of S_F and D_F , respectively, and Γ_μ^0 and γ_μ are the dimensionless quantities.

Let us separate the infinite constants from G_η^0 , \mathfrak{G}_J^0 and Γ_μ^0 as follows :

$$\begin{aligned} G_\eta^0 &= Z_2 G_{\eta 1}^0, \\ \mathfrak{G}_J^0 &= Z_3 \mathfrak{G}_{J 1}^0, \\ \Gamma_\mu^0 &= Z_1^{-1} \Gamma_{\mu 1}^0, \end{aligned} \quad (3.3)$$

where $G_{\eta 1}^0$, $\mathfrak{G}_{J 1}^0$ and $\Gamma_{\mu 1}^0$ are free from infinity. Substituting (3.3) into (3.2), we have

$$\begin{aligned} G_\eta^0(x, x') &= \frac{1}{Z_2} S_F(x - x') - \delta x \int dy S_F(x - y) G_{\eta 1}^0(y, x') \\ &\quad - e^2 Z_2^2 Z_1^{-1} Z_3 \int dy dy' dy'' d\xi' \gamma_\mu S_F(x - y) G_{\eta 1}^0(y, y') \\ &\quad \times \Gamma_{\nu 1}^0(\xi'; y', y'') G_{\eta 1}^0(y'', x') \mathfrak{G}_{J, \nu 1}^0(\xi', x). \end{aligned} \quad (3.4)$$

In the above equation, while the integrand of the third term is finite, its integration may be divergent. Since this divergence comes from the contribution of the high energy region, the following discussion shows that this integral is at most linearly or logarithmically divergent. Z_1 , Z_2 , and Z_3 are the function of the upper limit $p \rightarrow \infty$ of the integration concerning the internal momentum and so the dimension of the divergent Z_1 , Z_2 , and Z_3 should be zero or negative power of the length. Therefore, the dimension of $G_{\eta 1}^0$, $\mathfrak{G}_{J 1}^0$ and $\Gamma_{\mu 1}^0$ are $[L^{-n}]$ ($n \leq 3$, $n \leq 2$, $n \leq 0$), respectively. Since the integral (3.4) is most strongly divergent in the case of $n=3, 2, 0$, it is sufficient to treat this case for the consideration of the highest degree of divergence. Then the integral has a dimension $[L^{12-3-3-3-2}] = [L]$, and can be written as follows :

$$\begin{aligned} &\int dy dy' dy'' d\xi' \gamma_\mu S_F(x - y) G_{\eta 1}^0(y, y') \Gamma_{\nu 1}^0(\xi'; y', y'') G_{\eta 1}^0(y'', x') \mathfrak{G}_{J, \nu 1}^0(\xi', x) \\ &= Z_1^{-1} [A + B(\gamma_\mu \partial_\mu + x) + C(\gamma_\mu \partial_\mu + x)^2 + \dots] \\ &\quad \times \int dy S_F(x - y) G_{\eta 1}^0(y, x'), \end{aligned} \quad (3.5)$$

where A and B are linearly and logarithmically divergent and C is a finite quantity.

As Fig. 1 is symmetric in association with the two vertices a and b , it is expected that a infinite constant factor Z_1^{-1} appears from the vertex a as well as the vertex b after

* In the interaction of the second kind, the situation is not so simple as in this case, because the coupling constant has the dimension of the length.

the integration is analogous to Dyson's argument. However, the consistent proof of this situation is not yet verified in our method. This defect which is due to the asymmetrical treatment of the two vertices makes it difficult to compare our method with Dyson's one also in the discussion of the skelton approximation in the next section.*

Substituting (3.5) into (3.4), we obtain the following relations as a necessary condition for the convergence of the right side of (3.4) :

$$\begin{aligned}\delta x &= -ie_1^2 AZ_2^{-1}, \\ Z_2 &= 1 - ie_1^2 B, \\ e_1 &= Z_1^{-1} Z_2 Z_3^{-1} e.\end{aligned}\tag{3.6}$$

Here e_1 and x' should be considered as a finite and observable electric charge and mass, respectively. Then (3.4) becomes as follows :

$$G_\eta^0(x, x') = S_F(x - x') - e_1^2 C(\gamma_\mu \partial_\mu + x) G_{\eta_1}^0(x, x') + \dots \tag{3.7}$$

(3.6) is in agreement with the condition as to the renormalization constant given by Dyson. Further it is easily found by the same dimensional argument as in the above (3.5) that the divergent quantities are restricted only to self-energy parts G_η^0 , \mathfrak{G}_J^0 and vertex part Γ_μ^0 .

Therefore, the theory is entirely free from divergence, provided that it is shown that any Feynman diagram of S -matrix is expressed by the irreducible skelton in which internal lines, vertex, and charge correspond to $G_{\eta_1}^0$, $\mathfrak{G}_{J_1}^0$, $\Gamma_{\mu_1}^0$, and e_1 , respectively. In this paper the method in which any transition matrix is represented entirely by the $G_{\eta_1}^0$, $\mathfrak{G}_{J_1}^0$, and Γ_μ^0 is called the skelton approximation. If we have the formulation of the skelton approximation, any vertex in Fig. 3 is of the form $e(G_\eta^0 \mathfrak{G}_J^0 G_\eta^0)^{1/2} \Gamma_\mu^0$ which can be rewritten as follows ;

$$e Z_1^{-1} Z_2 Z_3^{1/2} (G_{\eta_1}^0 \mathfrak{G}_{J_1}^0 G_{\eta_1}^0)^{1/2} \Gamma_{\mu_1}^0 = e_1 (G_{\eta_1}^0 \mathfrak{G}_{J_1}^0 G_{\eta_1}^0) \Gamma_{\mu_1}^0, \tag{3.8}$$

so that it turns out that there exists no more any infinite quantity in the theory.

Thus it is necessary for completion of our procedure only to investigate the possibility of the above skelton approximation and this is the aim of the next section. As will be shown there, unfortunately, we can not yet find the complete formulation of the skelton approximation and this problem is left to be investigated in future.

§ 4. The skelton approximation

In this section we shall prove that the transition matrix element is obtained through substituting the Green function of the one body problem and Γ_μ into any internal lines and the vertex part of the adequate graph corresponding to its process. However this graph is not completely equivalent to the skelton and we must often use the uncorrected

* If this defects are get rid of, one could set up a non-singular theory by using the Lagrangian given by G. Takeda²⁷⁾ and applying the variational method in § 2.

vertex γ_μ , although it is needless to use S_F and D_F .

According to (2.10), it turns out that the problem is how to express all the many body Green function by the one body Green function and the vertex function Γ_μ ; i.e., the corrected function approximation. In the following we discuss on this problem in some examples, i.e., the Möller scattering of two electrons and the photon-photon scattering.

(i) Möller scattering

According to (2.10), the transition matrix element for the Möller scattering corresponds to $G_\eta^0(x_1, x_2, x'_1, x'_2)$. Using (2.12), we have

$$G_\eta^0(x_1, x_2, x'_1, x'_2) = -\partial^2/\partial\eta(x'_2)\partial\eta(x_2) G_\eta(x_1, x'_1)|_{J,\eta\rightarrow 0} + G_\eta^{(0)}(x_1, x_2, x'_1, x'_2), \quad (4.1)$$

where

$$G_\eta^{(0)}(x_1, x_2, x'_1, x'_2) = iG_\eta^0(x_1, x'_1)G_\eta^0(x_2, x'_2) - iG_\eta^0(x_1, x'_2)G_\eta^0(x_2, x'_1). \quad (4.2)$$

This is represented by the diagrams denoted in Fig. 3. In this diagram and hereafter it should be noted that the straight and waved line, the vertex and vertex with circle correspond to G_η^0 , \mathbb{G}_η^0 , γ_μ , and I_μ^0 , respectively. The second term of (4.1) expresses two independent electrons scattering without the real interaction. The effect of the true Möller scattering due to the real interaction of two electrons is involved in the first terms of (4.1). From (2.21), we have

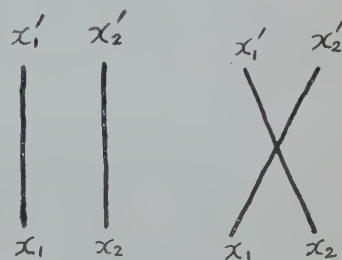


Fig. 3

$$\begin{aligned} \frac{\delta}{\partial\bar{\eta}(x_2)} G_\eta &= -ieG_\eta\gamma_\mu\langle\psi^*\rangle\left\{\frac{\delta}{\partial J_\mu}\langle\bar{\psi}\rangle\right\}\frac{\delta}{\partial\bar{\eta}(x_2)} G_\eta \\ &+ G_\eta\left(ie\gamma_\mu\frac{\delta}{\partial\bar{\eta}(x_2)}\langle A_\mu\rangle - \frac{\delta}{\partial\bar{\eta}(x_2)}\bar{M}\right)G_\eta \\ &+ ieG_\eta\gamma_\mu\frac{\delta}{\partial\bar{\eta}(x_2)}\left\{\langle\bar{\psi}^*\rangle\frac{\delta}{\partial J_\mu}\langle\bar{\psi}\rangle\right\}. \end{aligned} \quad (4.3)$$

Further using (2.17) and the Furry's theorem, we obtain

$$\begin{aligned} \frac{\partial^2}{\partial\eta(x'_2)\partial\bar{\eta}(x_2)} G_\eta|_{J,\eta\rightarrow 0} &= G_\eta^0\frac{\partial^2}{\partial\eta(x'_2)\partial\bar{\eta}(x_2)} \{ie\gamma_\mu\langle A_\mu\rangle - \bar{M}\} G_\eta^0 \\ &+ ieG_\eta\gamma_\mu\left\{\frac{\delta}{\partial\eta(x'_2)}\langle\psi^*\rangle\right\}\left\{\frac{\partial^2}{\partial J_\mu\partial\bar{\eta}(x_2)}\langle\bar{\psi}\rangle\right\}. \end{aligned} \quad (4.4)$$

Substituting the following relation

$$\frac{\partial^2}{\partial\eta(x'_2)\partial\bar{\eta}(x_2)}\langle A_\mu(x)\rangle = \frac{\delta}{\delta J_\mu(x)}G(x_2, x'_2) \quad (4.5)$$

into (4.4), we have

$$\frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} G_\eta|_{J, \eta \rightarrow 0} = G^{(2)}(x_2, x'_2) - G_\eta^0 \left\{ \frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \bar{M} \right\} G_\eta^0, \quad (4.6)$$

where

$$\begin{aligned} G^{(2)}(x_2, x'_2) = & -e^2 [G_\eta \gamma_\mu G_\eta(x_2)] \int \Gamma'_\mu(\xi') G_\eta(x'_2) \mathfrak{G}_J(\xi', \cdot) G_\eta \\ & + G_\eta \gamma G_\eta(\cdot, x'_2) G_\eta(x_2) \int \Gamma(\xi') \mathfrak{G}_J(\xi', \cdot) G_\eta] |_{J, \eta \rightarrow 0}. \end{aligned} \quad (4.7)$$

This is represented by the diagrams in Fig. 4, and this diagrams correspond to ϵ^2 -skelton.

It can be shown that the second term of (4.6) contributes to the skelton higher than the fourth order. Using (2.22), we have

$$\begin{aligned} \frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \bar{M} &= \frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} (\bar{M} G_\eta G_\eta^{-1}) \\ &= \frac{\partial}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \left[\left(z - e \gamma_\mu \frac{\partial}{\partial J_\mu} G_\eta \right) G_\eta^{-1} \right]. \end{aligned} \quad (4.8)$$

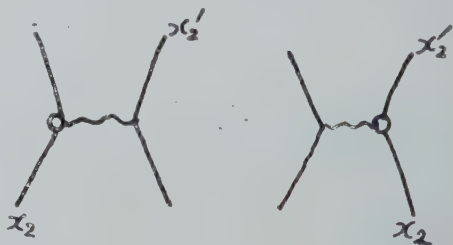


Fig. 4

After the tedious calculation, (4.8) is rewritten in the form;

$$\frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \bar{M}|_{J, \eta \rightarrow 0} = -e \gamma G_\eta \frac{\partial}{\partial J} \left[G_\eta^{-1} \frac{\partial^2 G_\eta}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \right] G_\eta^{-1} |_{J, \eta \rightarrow 0}. \quad (4.9)$$

Therefore, from (4.6) we have

$$\frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} G_\eta|_{J, \eta \rightarrow 0} = G^{(2)}(x_2, x'_2) + e G_\eta^0 \gamma G_\eta^0 \frac{\partial}{\partial J} \left[G_\eta^{-1} \frac{\partial^2 G_\eta}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \right] G_\eta^{0-1} |_{J, \eta \rightarrow 0}. \quad (4.10)$$

From (4.3), (2.17), (4.5), we can obtain the relation

$$\begin{aligned} \frac{\partial^3}{\partial J(\xi) \partial \eta(x'_2) \partial \bar{\eta}(x_2)} G_\eta|_{J, \eta \rightarrow 0} &= \frac{\partial}{\partial J(\xi)} G^{(2)}(x_2, x'_2) |_{J, \eta \rightarrow 0} \\ &\quad - \frac{\partial}{\partial J(\xi)} \left[G_\eta \left\{ \frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} \bar{M} \right\} G_\eta \right] |_{J, \eta \rightarrow 0}, \end{aligned} \quad (4.11)$$

where $G^{(2)}$ is defined by the left side of (4.7) without superscript 0. The second term of (4.4) gives to (4.10) the skelton higher than the ϵ^6 -approximation, and we write this part as $O(\epsilon^6)$. Using (4.10), we can rewrite (4.10) as follows:

$$\frac{\partial^2}{\partial \eta(x'_2) \partial \bar{\eta}(x_2)} G_\eta|_{J, \eta \rightarrow 0} = G^{(2)}(x_2, x'_2) + G^{(4)}(x_2, x'_2) + O(\epsilon^6). \quad (4.12)$$

The second term of (4.12) corresponds to e^4 -skelton, and is given by

$$\begin{aligned}
 & G_{\eta}^{(4)}(x_2, x'_2) \\
 &= -ie^4 G_{\eta} \gamma G_{\eta} [\gamma G_{\eta}(x_2,) \int \Gamma(\xi'') G_{\eta}(\xi'', \cdot) \int \Gamma(\xi'') G(\cdot, x'_2) \mathfrak{G}_J(\xi', :) G_{\eta}(:,) \\
 &+ \gamma G_{\eta}(x_2,) \int \Gamma(\xi'') G_{\eta} \int \Gamma(\xi') G_{\eta}(\cdot, x'_2) \mathfrak{G}_J(\xi', :) \mathfrak{G}_J(\xi'', :) G_{\eta}(:,) \\
 &+ \gamma G_{\eta}(x_2,) \int \Gamma(\xi'') G_{\eta}(\cdot, x'_2) \mathfrak{G}_J(\xi'', :) G_{\eta}(:,) \int \Gamma(\xi') G_{\eta} \mathfrak{G}_J(\xi', \cdot)]_{J, \eta > 0} \\
 &- ie^4 G_{\eta} \gamma G_{\eta} [\gamma G_{\eta}(:,) \int \Gamma(\xi'') G_{\eta}(\cdot, x'_2) \mathfrak{G}_J(\xi'', \cdot) G_{\eta}(x_2,) \int \Gamma(\xi') G_{\eta} \mathfrak{G}_J(\xi', :) \\
 &+ \gamma G_{\eta}(:, x'_2) G_{\eta}(x_2,) \int \Gamma(\xi'') G_{\eta}(\cdot, x'_2) \mathfrak{G}_J(\xi'', \cdot) \int \Gamma(\xi') G_{\eta} \mathfrak{G}_J(\xi', :) \\
 &+ \gamma G_{\eta}(:, x'_2) G_{\eta}(\cdot, x_2) \int \Gamma(\xi'') G_{\eta} \mathfrak{G}_J(\xi'', :) \int \Gamma(\xi') G_{\eta} \mathfrak{G}_J(\xi', \cdot)]_{J, \eta > 0}. \quad (4.13)
 \end{aligned}$$

This is represented by the diagrams in Fig. 5.

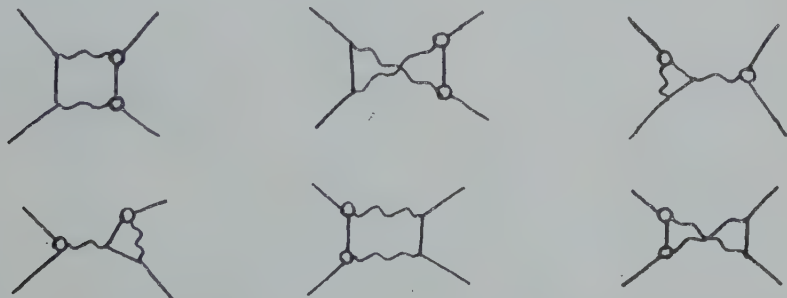


Fig. 5

As stated in the preceding section, it should be noted here that only some part of all vertices are replaced by Γ_{μ} . For instance, in Fig. 4, only one vertex among two is replaced by Γ_{μ} . Thus the contribution which corresponds to Fig. 6 in the usual perturbation theory is not involved in this diagram, but in Fig. 5 which belongs to e^4 -skelton. Of course, if we take into account infinitely higher order terms in the present approximation, then the both vertices will be completely corrected and so by Γ_{μ} , and then the skelton approximation may be obtained. However such a procedure is of a perturbation theoretical concept. This unfavourable situation of our method is due to the fact that, in the course of obtaining the higher order skelton by applying (2.24) to one body Green function, only one vertex among two of the self-energy part is replaced by Γ_{μ} .

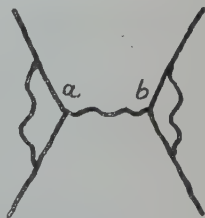


Fig. 6

(ii) Photon-photon scattering

Now we discuss briefly on the photon-photon scattering. This transition matrix ele-

ment corresponds to

$$\mathbb{G}_J^0(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{\partial^2}{\partial J(\xi_4) \partial J(\xi_3)} \mathbb{G}_J(\xi_1, \xi_2) \Big|_{J \rightarrow 0} + \mathbb{G}_J^{(0)}(\xi_1, \xi_2, \xi_3, \xi_4), \tag{4.14}$$

where

$$\begin{aligned} \mathbb{G}_J^{(0)}(\xi_1, \xi_2, \xi_3, \xi_4) = & -i \left[\mathbb{G}_J^0(\xi_1, \xi_2) \mathbb{G}_J^0(\xi_3, \xi_4) + \mathbb{G}_J^0(\xi_1, \xi_3) \mathbb{G}_J^0(\xi_2, \xi_4) \right. \\ & \left. + \mathbb{G}_J^0(\xi_1, \xi_4) \mathbb{G}_J^0(\xi_2, \xi_3) \right]. \end{aligned} \tag{4.15}$$

This is represented by the diagrams in Fig. 7. After the analogous calculation as for

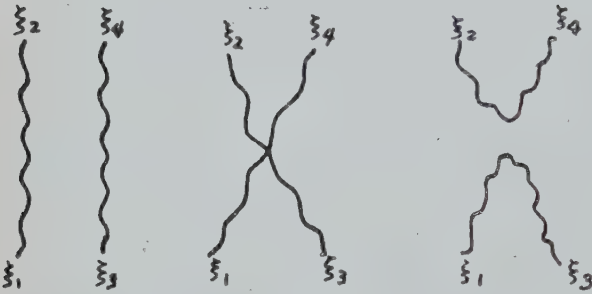


Fig. 7

$\frac{\partial^2}{\partial \eta \partial \bar{\eta}} G_\eta$, the first term is transformed into the following form:

$$\begin{aligned} & \frac{\partial^2 \mathbb{G}_J}{\partial J(\xi_4) \partial J(\xi_3)} \Big|_{J \rightarrow 0} \\ &= -\mathbb{G}_J^0 \frac{\partial^2 \bar{P}}{\partial J(\xi_4) \partial J(\xi_3)} \Big|_{J \rightarrow 0} \mathbb{G}_J^0. \end{aligned} \tag{4.16}$$

Using (2.23), we can rewrite

$\frac{\partial^2 P}{\partial J \partial \bar{J}}$ as follows:

$$\begin{aligned} \frac{\partial^2 \bar{P}}{\partial J(\xi_4) \partial J(\xi_3)} \Big|_{J, \eta \rightarrow 0} = & -e T_\tau \left[\gamma \frac{\partial^3 G_\eta}{\partial J \partial J(\xi_4) \partial J(\xi_3)} \right] \mathbb{G}_J^{-1} \Big|_{J, \eta \rightarrow 0} \\ & + e T_\tau [\gamma \partial / \partial J G_\eta] \mathbb{G}_J^{-1} \frac{\partial^2 \mathbb{G}_J}{\partial J(\xi_4) \partial J(\xi_3)} \mathbb{G}_J^{-1} \Big|_{J, \eta \rightarrow 0}. \end{aligned} \tag{4.17}$$

Further, we have

$$\begin{aligned} \frac{\partial^3 G_\eta}{\partial J \partial J(\xi_4) \partial J(\xi_3)} = & \frac{\partial^2 G_\eta}{\partial J \partial J(\xi_4)} \left(i e \gamma \frac{\partial \langle A \rangle}{\partial J(\xi_3)} - \frac{\partial \bar{M}}{\partial J(\xi_3)} \right) G + (\xi_4 \leftrightarrow \xi_3) \\ & + \frac{\partial G}{\partial J(\xi_4)} \left(i e \gamma \frac{\partial \langle A \rangle}{\partial J(\xi_3)} - \frac{\partial \bar{M}}{\partial J(\xi_3)} \right) \frac{\partial G_\eta}{\partial J} + (\xi_4 \leftrightarrow \xi_3) \\ & + G \left(i e \gamma \frac{\partial \langle A \rangle}{\partial J(\xi_3)} - \frac{\partial \bar{M}}{\partial J(\xi_3)} \right) \frac{\partial^2 G}{\partial J(\xi_4) \partial J} \\ & + O(\epsilon^5), \end{aligned} \tag{4.18}$$

where the symbol $(x \leftrightarrow y)$ means the term obtained by the exchange of x and y in the preceding term. Thus the final result is given by

$$\frac{\partial^2 \mathbb{G}_\eta}{\partial J(\xi_4) \partial J(\xi_3)} \Big|_{J \rightarrow 0} = \mathbb{G}_J^0(\cdot, \cdot, \xi_3, \xi_4) + O(\epsilon^6), \tag{4.19}$$

where

$$\begin{aligned}
 & \mathfrak{G}_J^{(4)}(\cdot, \cdot, \xi_3, \xi_4) \\
 &= -ic^4 \mathfrak{G}_J^0 T_r [\gamma G_\eta \int \Gamma(\xi') \mathfrak{G}_J(\xi', \xi_4) G_\eta \int \Gamma(\xi'') \mathfrak{G}_J(\xi'', \cdot) G_\eta \int \Gamma(\xi''') \mathfrak{G}_J(\xi''', \xi_3) G_\eta \\
 & \quad + \gamma G_\eta \int \Gamma(\xi') \mathfrak{G}_J(\xi', \xi_4) G_\eta \int \Gamma(\xi'') \mathfrak{G}_J(\xi'', \xi_3) G_\eta \int \Gamma(\xi''') \mathfrak{G}_J(\xi''', \cdot) G_\eta \\
 & \quad + \gamma G_\eta \int \Gamma(\xi') \mathfrak{G}_J(\xi', \xi_3) G_\eta \int \Gamma(\xi'') \mathfrak{G}_J(\xi'', \cdot) G_\eta \int \Gamma(\xi''') \mathfrak{G}_J(\xi''', \xi_4) G_\eta]_{J, \eta \rightarrow 0} \\
 & \quad + (\xi_4 \leftrightarrow \cdot),
 \end{aligned}$$

These correspond to c^4 -skelton approximation and are represented by diagrams in Fig. 8.

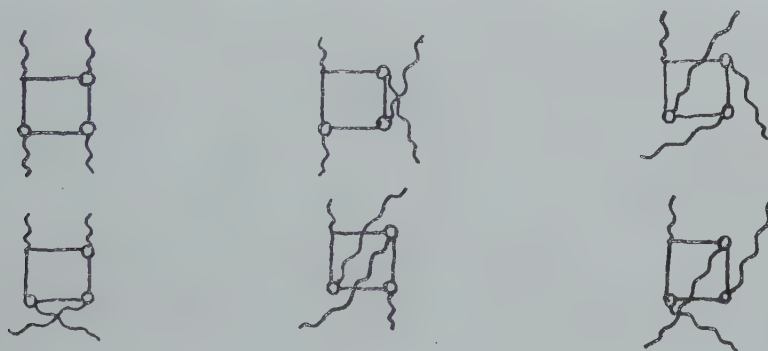


Fig. 8

All other processes be treated in the similar method as to the above two examples.

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β -spectra of Fe^{59} , Rb^{87} , Tc^{99} , Cs^{137} and the Coupling Constants of Scalar and Tensor Interactions in β -decay

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The β -spectra of Fe^{59} , Rb^{87} , Tc^{99} and Cs^{137} are investigated with the linear combination of scalar and tensor interactions in the Fermi theory of β -decay, and it is inferred that the relative sign between two coupling constants of scalar and tensor interactions is minus.

§ 1. Introduction

Recently many facts have been known about the interaction types of the Fermi theory of β -decay. First, the investigation of allowed transitions (He^6 , C^{10} , O^{14} etc.) showed that both of the selection rules, Fermi type and Gamow-Teller type, are necessary. On the other hand, according to the experiments on the electron-neutrino angular correlations of He^6 and P^{32} , the Gamow-Teller type selection rule must be due to T^{11} . (We write the five interaction types of the Fermi theory of β -decay as S , V , TA and P hereafter.) Then, taking into account the Fierz conditions which exclude the SV (this notation indicates the linear combination of S and V , where the coupling constants of S and V are not yet specified) and TA combinations, the full interaction type is one of the following four combinations, ST , STP , VT and VTP . According to the investigation of Konopinski and Mahmoud²⁾ VT and VTP are unlikely. (The authors think that their conclusion is not definite but probable only.) Therefore, there remains only the problem concerning with the coupling constants of S , T and P . In these three types P , which appears on account of the selection rules only in the first forbidden transition with no spin change, is not considered here, because it has been discussed in many other papers. In this paper the coupling constants of S and T , especially their relative sign, are investigated.

Information about the absolute magnitudes of the coupling constants G_S and G_T (of course we assume real coupling constants³⁾) of S and T can be obtained from the analysis of allowed transitions. The result obtained until now is that the absolute magnitudes of them are about the same or $|G_S|$ is a little smaller than $|G_T|$. According to Blatt⁴⁾

$$0.29 < |G_S/G_T|^2 < 1.04. \quad (1)$$

On the other hand, we cannot decide the sign of G_S/G_T by the study of allowed transitions (including electron-neutrino angular correlation), because no interference term between S and T appears in the allowed transition. To do this we must examine parity-unfavoured forbidden transitions (transitions whose spin changes are not larger than the orders of for-

biddeness). In a parity-unfavoured forbidden transition several nuclear matrix elements appear, and at present we cannot accurately calculate these nuclear matrix elements. Therefore, strictly speaking, these nuclear matrix elements should be treated as adjustable parameters. If so, however, we cannot deduce any conclusion, and so we use the shell model for the calculation of ratios between some nuclear matrix elements. It seems that the assignment of the shell for an even mass nucleus is more difficult than that for an odd mass nucleus, so that we restrict our investigation to odd mass nuclei only. Then the first forbidden transitions supply no information since they all show the allowed shape spectra. The elements available for us are the following four, Fe^{59} , Rb^{87} , Tc^{99} and Cs^{137} . In the next section we examine the β -spectra of these four elements with ST . Some of them have been already analysed with T only, but the analysis with ST has not yet been carried out.

§ 2. Analysis of the β -spectra of Fe^{59} , Rb^{87} , Tc^{99} and Cs^{137}

First, we shall show the procedure of our analysis. Generally, parity-unfavoured transitions are very complicated. Therefore, it is very difficult to deduce any conclusion, if we do not neglect various corrections, e.g. the finite nuclear size effect.⁵⁾ Accordingly we neglect these corrections, and make use of the tables calculated by Rose, Perry and Dismuke.⁶⁾ * Owing to the neglect of corrections, our analysis is not accurate for each individual case, and the conclusion is deduced from rather statistical standpoint.

In the case of ST , four nuclear matrix elements, $Q_n(\beta r, r)$, $Q_n(\beta \sigma, r)$, $Q_n(\beta \sigma \times r, r)$ and $Q_{n+1}(\beta \sigma, r)$ in Greuling's⁷⁾ notation, appear generally in the parity-unfavoured n -th forbidden transition, where $n \geq 2$. Among these nuclear matrix elements, the last is a tensor of one rank higher than the others, and no interference exists in spectrum between tensors with different ranks. The effect of $Q_{n+1}(\beta \sigma, r)$ is relatively small, and we neglect this nuclear matrix element in the following calculation. We are interested only in the spectrum shapes now, and to analyse them it is necessary to know the relative magnitudes of the remaining three nuclear matrix elements. Among them the ratio of $Q_n(\beta r, r)$ to $Q_n(\beta \sigma \times r, r)$ is easily calculated, if we use the Mayer's single particle shell model. This ratio is also common to the special j - j coupling shell model in which outer neutrons and protons enter into the same shells respectively. For the calculation of this ratio only the angular part of the wave function is required, and the radial part can be dropped as a common factor. In view of the above fact this ratio will perhaps be fairly reliable. We calculate for the four cases in which G_S/G_T takes the values $\pm 1/2$ and ± 1 , which have been chosen considering (1). Next, we take for the maximum electron energy the value usually accepted, then the ratio between $Q_n(\beta \sigma, r)$ and $Q_n(\beta \sigma \times r, r)$ is decided for each value of G_S/G_T so as to make the Kurie plot straight at three electron energies, one of which is chosen as the maximum energy. Lastly, the Kurie plot thus obtained is

*) In these tables the necessary values of Z (nuclear charge) and A (mass number) are not always found. In these cases the nearest values are taken. This will perhaps not change the general tendencies.

examined for all electron energies.

i) Fe^{59}

The β -spectrum of Fe^{59} was measured by Metzger⁸⁾, and he proposed the decay scheme shown in Fig. 1. The two lower energy β -ray groups are allowed transitions, and our object is the highest energy group. The experimental results are given only down to the electron energy $W=2.6mc^2$ by the correction factor which is necessary to make the Kurie plot straight. The experiment of the lower electron energy part seems difficult owing to the Compton electron.

If we make use of the single particle shell model, this β -decay takes place with the transition of a neutron in $p_{3/2}$ state into a proton in $f_{7/2}$ state, and this transition is second forbidden. The calculation of the ratio between $Q_n(\beta\nu, \nu)$ and $Q_n(\beta\sigma \times \nu, \nu)$ (in this case $n=2$) can be performed easily using the components of the polarized solid harmonics⁹⁾, $Y_{2,2}(\beta\nu, \nu)$ and $Y_{2,2}(\beta\sigma \times \nu, \nu)$, which correspond to $Q_2(\beta\nu, \nu)$ and $Q_2(\beta\sigma \times \nu, \nu)$ respectively. Then the ratio is written as

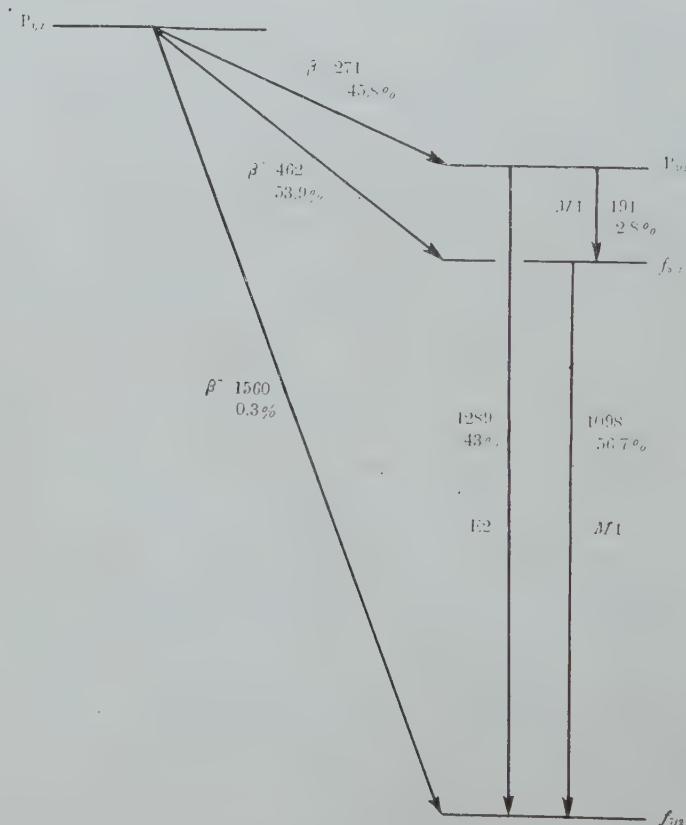


Fig. 1. Disintegration scheme of Fe^{59} proposed by Metzger⁸⁾. Transition energies are given in kev

$$\frac{Q_2(\beta \mathbf{r}, \mathbf{r})}{Q_2(\beta \boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r})} = \frac{\int d\mathbf{r} \psi^*_{f\frac{1}{2}\frac{7}{2}} Q \mathcal{Y}_{22}(\beta \mathbf{r}, \mathbf{r}) \psi_{p\frac{3}{2}\frac{3}{2}}}{\int d\mathbf{r} \psi^*_{f\frac{1}{2}\frac{7}{2}} Q \mathcal{Y}_{22}(\beta \boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) \psi_{p\frac{3}{2}\frac{3}{2}}}, \tag{2}$$

where $\psi_{f\frac{1}{2}\frac{7}{2}}$ and $\psi_{p\frac{3}{2}\frac{3}{2}}$ represent the wave functions of the initial neutron and the final proton respectively, in which the last suffixes represent the magnetic quantum numbers, and Q is an operator which changes a neutron into a proton. We use the non-relativistic approximation to nucleons, then the calculation of the right-hand side of (2) can be easily carried out, and we find

$$Q_2(\beta \mathbf{r}, \mathbf{r})/Q_2(\beta \boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) = -i. \tag{3}$$

For simplicity we introduce the following real numbers¹⁰⁾ x and Λ :

$$Q_n(\beta \mathbf{r}, \mathbf{r})/Q_n(\beta \boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) = -ix, \quad nQ_n(\beta \mathbf{a}, \mathbf{r})/Q_n(\beta \boldsymbol{\sigma} \times \mathbf{r}, \mathbf{r}) = \alpha Z/2\rho \cdot \Lambda, \tag{4}$$

where α is the fine structure constant and ρ is the nuclear radius. From (3) $x=1$ in this case. Λ is fixed so as to make the Kurie plot straight at the following three points, the maximum energy $W_0=4.059mc^2$, $W=3.64mc^2$ (electron momentum $p=3.5mc$) and $W=2.8mc^2$ ($p=2.6mc$). We have calculated Λ for the four cases in which the ratios of the coupling constants G_S/G_T are $\pm 1/2$ and ± 1 . Generally the values of Λ are obtained in pair for each ratio of coupling constants, and they are shown in Table I. For these values of Λ the Kurie plots are given in Fig. 2. For simplicity we represent the assumption with G_S/G_T and Λ by $(G_S/G_T, \Lambda)$. (1, -2.53) fits the experiment completely, and $(-1/2, -1.70)$ and $(1/2, -2.25)$ fit the experiment for the small deviations at the lowest energy to which the experimental data are available. However, the experimental data for this energy will be perhaps not so exact on account of the Compton electron, and these small deviations cannot be used to exclude $(-1/2, -1.70)$ and $(1/2, -2.25)$. Therefore, we think that (1, -2.53), $(-1/2, -1.70)$ and $(1/2, -2.25)$ fit the

Table I

The values of Λ (eq. (4)) to make the Kurie plot straight at the three electron energies, one of which is chosen as the maximum energy. The mark \bigcirc following the value of Λ indicates that the Kurie plot is straight on the whole; \times indicates that the Kurie plot deviates from straight line so much that the case is completely excluded; \triangle indicates that the Kurie plot is not so straight as the case of mark \bigcirc but cannot definitely be excluded.

elements	G_S/G_T	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1
Fe ⁵⁹ ($x=1$)		-0.278 \triangle	0.714 \times	2.56 \triangle	3.53 \triangle
		-1.38 \triangle	-1.70 \bigcirc	-2.25 \bigcirc	-2.53 \bigcirc
Rb ⁸⁷ ($x=-1$)		2.47 \times	1.87 \times	0.64 \times	no real value
		1.54 \bigcirc	1.16 \bigcirc	0.42 \triangle	
Tc ⁹⁹ ($x=1$)		-0.039 \times	0.742 \bigcirc	2.27 \triangle	3.04 \triangle
		-0.586 \triangle	-2.22 \triangle	-5.61 \triangle	-7.31 \triangle
Cs ¹³⁷ ($x=-1$)		2.40 \bigcirc	1.82 \triangle	no real value	no real value
		1.99 \triangle	1.49 \bigcirc		

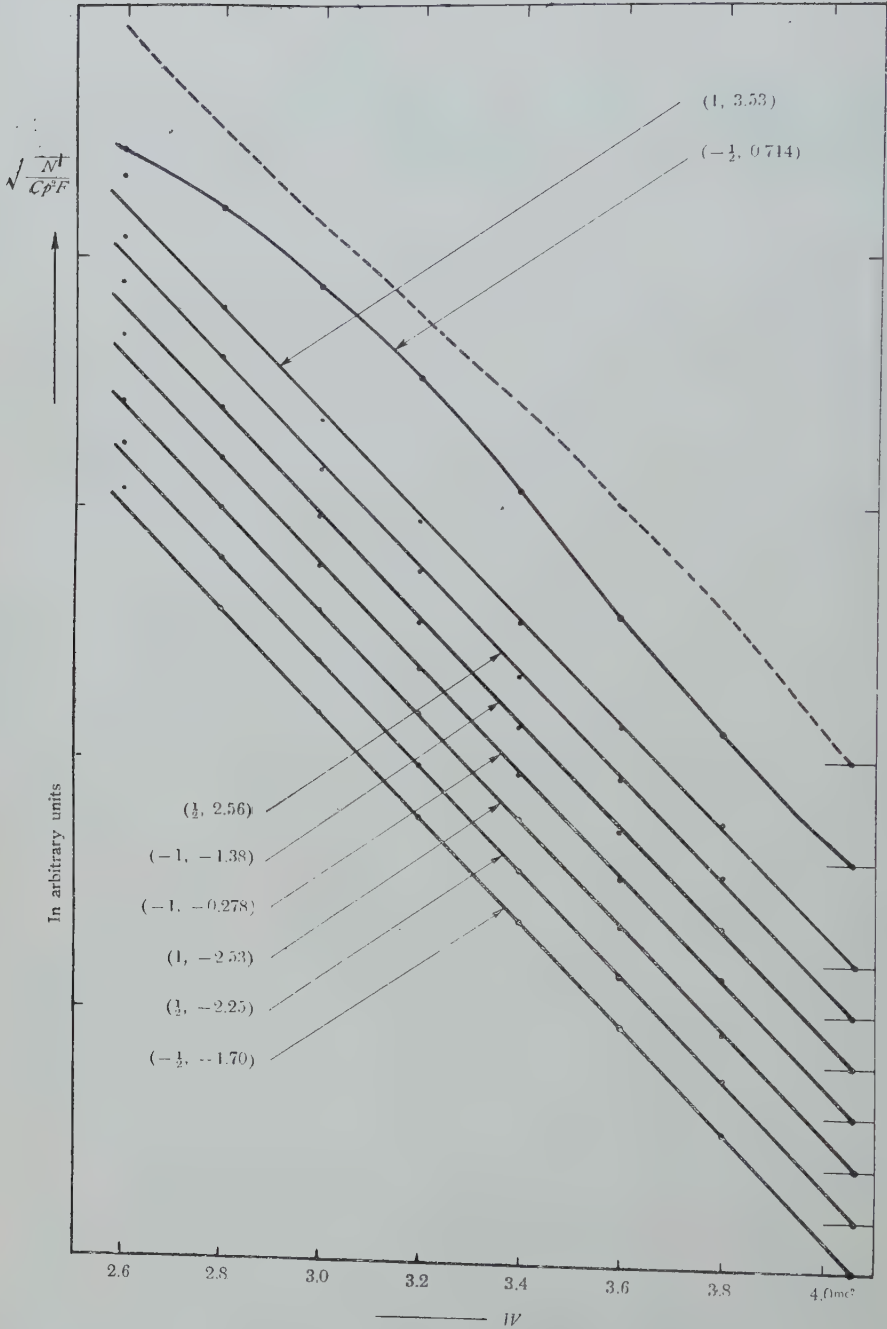


Fig. 2. Kurie plots for Fe^{59} . The sets of numbers in the figure are $(G_S/G_T, A)$. The dotted curve is the uncorrected Kurie plot

experiment well, and they are marked \bigcirc in Table I. The fits of $(-1, -0.278)$, $(-1, -1.38)$, $(1/2, 2.56)$ and $(1, 3.53)$ to the experiment are somewhat worse than the above three, but they cannot definitely be excluded considering the neglect of many corrections. They are marked \triangle in Table I. The Kurie plot for $(-1/2, 0.714)$ deviates from straight line so much that this choice is excluded definitely, and is marked \times in Table I.

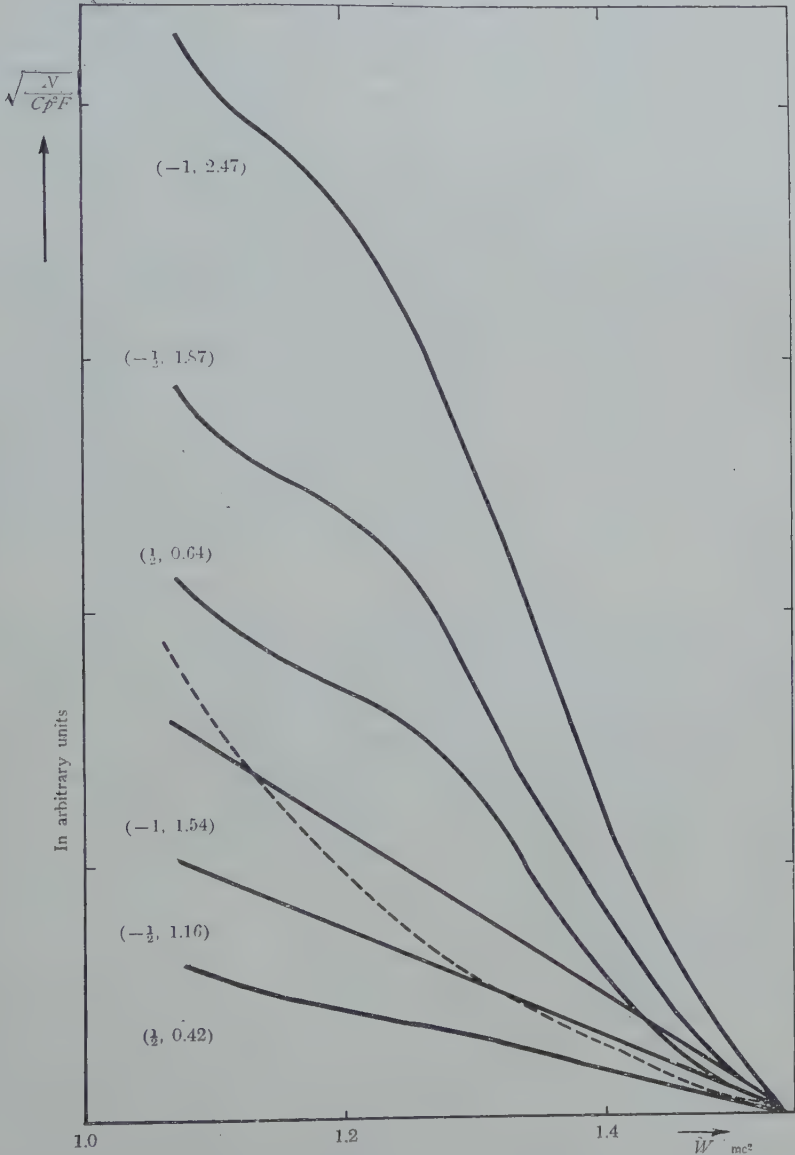


Fig. 3. Kurie plots for Rb^{87} . The sets of numbers in the figure are $(G_S/G_T, A)$. The dotted curve is the uncorrected Kurie plot.

Finally we mention about the corrected $f_c t$ -value for $Q_2(\beta\sigma \times \nu, \nu)/2!$ (i.e. $f_c t = (2\pi^3/G_T^2) \log 2/\sum |Q_2(\beta\sigma \times \nu, \nu)/2!|^2$) briefly. The uncorrected $\log f t$ -value is 10.9^5 in this case, and the corrected $f_c t$ -value for $Q_2(\beta\sigma \times \nu, \nu)/2!$, which represents the magnitude of this nuclear matrix element, is of the order of $10^{12} \sim 10^{14}$ for each case in Table I. This $f_c t$ -value $10^{12} \sim 10^{14}$ suits well for the second forbidden coordinate type nuclear matrix element.

ii) Rb^{87}

According to the single particle shell model, this β^2 -decay takes place with the transition of a neutron in $g_{9/2}$ state into a proton in $p_{3/2}$ state, and this is a third forbidden transition. The calculated ratio of $Q_3(\beta\nu, \nu)$ to $Q_3(\beta\sigma \times \nu, \nu)$ is i ($x = -1$), and it has the opposite sign to that of Fe^{59} . The analysis is carried out using the experimental data of Lewis¹¹⁾ in the same way as Fe^{59} . We calculate the values of λ so as to make the Kurie plot straight at the following three points, maximum energy $W_0 = 1.54mc^2$, $W = 1.35mc^2$ ($p = 0.9mc$) and $W = 1.17mc^2$ ($p = 0.6mc$), and they are indicated in Table I. In the case of $G_S/G_T = 1$ no real value of λ exist*. For the six values of λ the Kurie plots are shown in Fig. 3. For each value of G_S/G_T other than 1, one of the two values of λ gives the straight Kurie plot while the other not. The latter are indicated by the mark \times in Table I. Among the former three, the one for $G_S/G_T = 1/2$ has an unreasonable point, namely, very large cancellation occurs in this case, and the corrected $f_c t$ -value for $Q_3(\beta\sigma \times \nu, \nu)/3!$ is of the order of 3×10^{14} and it seems too small for the the third forbidden coordinate type nuclear matrix element. Therefore, this is marked \triangle in Table I. The corrected $f_c t$ -values for $Q_3(\beta\sigma \times \nu, \nu)/3!$ in the other two cases are of the order of $3 \times 10^{15} \sim 6 \times 10^{15}$. These are marked \circ .

iii) Tl^{99}

According to the single particle shell model, this β^2 -decay takes place with the transition of a nucleon $d_{3/2} \rightarrow g_{9/2}$, and this is the second forbidden transition. The calculated ratio of $Q_2(\beta\nu, \nu)$ to $Q_2(\beta\sigma \times \nu, \nu)$ is $-i$ ($x = 1$), as in the case of Fe^{59} . We use the experimental data of Taimuty¹³⁾**. The values of λ which make the Kurie plot straight at the maximum electron energy $W_0 = 1.57mc^2$, $W = 1.415mc^2$ ($p = 1.0mc$) and $W = 1.22mc^2$ ($p = 0.7mc$) are shown in Table I, and for these values of λ the Kurie plots are given in Fig. 4. $(-1/2, 0.742)$ fits the experiment well down to the electron energy $W = 1.05mc^2$. The Kurie plots for $(1/2, -5.61)$, $(1, -7.31)$, $(-1/2, -2.22)$, $(1/2, 2.27)$, $(-1, -0.586)$ and $(1, 3.04)$ are straight down to the electron energy $W = 1.2mc^2$, and below this energy they deviate upward. These deviations seem a little too large to be attributed to the experimental error, but it is doubtful whether these deviations exclude these combinations completely. Therefore they are labeled \triangle . In the case

*) If we use the experimental data of Curran, Dixon and Wilson¹²⁾, real values do not exist not only for $G_S/G_T = 1$ but also for $G_S/G_T = 1/2$.

**) Besides this experiment there is an experiment of Feldman and Wu¹⁴⁾, but the uncorrected Kurie plot is not given there. If we calculate it using the graphs of the functions $D_+ - (1/3)c$, a and E is their paper, we obtain the similar uncorrected Kurie plot to that of Taimuty¹³⁾. However, these graphs of D_+ , a and E are in disagreement with those calculated by Nakamura et al¹⁵⁾. The origin of this disagreement has not yet been examined.

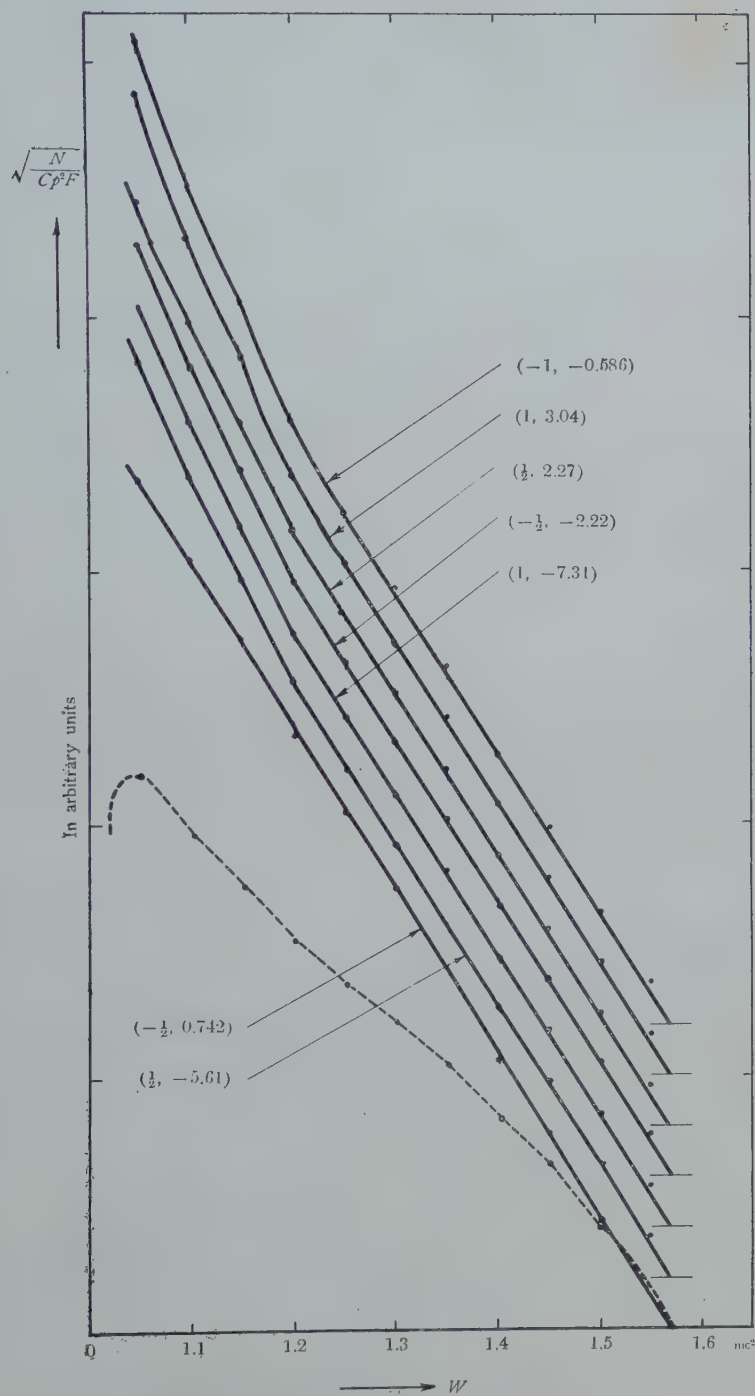


Fig. 4. Kurie plots for Tc^{99} . The sets of numbers in the figure are $(G_S/G_T, A)$. The dotted curve is the uncorrected Kurie plot.

of $(-1, 0.039)$ very large cancellation (about $1/500$) in the correction factor takes place and the tables of Rose *et al.*⁽⁶⁾ is insufficient for this case. However, such a large cancellation is very unlikely. If this large cancellation occurs, the corrected $f_c t$ -value for $Q_2(\beta\sigma \times r, r)/2!$ becomes of the order of 5×10^{10} , and this is smaller by about two orders of magnitude than the other $f_c t$ -values for second forbidden coordinate type nuclear matrix elements. For this reason $(-1, 0.039)$ is attached \times . The corrected $f_c t$ -values for $Q_2(\beta\sigma \times r, r)/2!$ of the other cases are of the order of $10^{12} \sim 2 \times 10^{15}$.

iv) Cs^{137}

According to the single particle shell model, the transition of the nucleon is $d_{5/2} \rightarrow g_{7/2}$, and this β -decay is second forbidden. The calculated ratio of $Q_2(\beta r, r)$ to $Q_2(\beta\sigma \times r, r)$

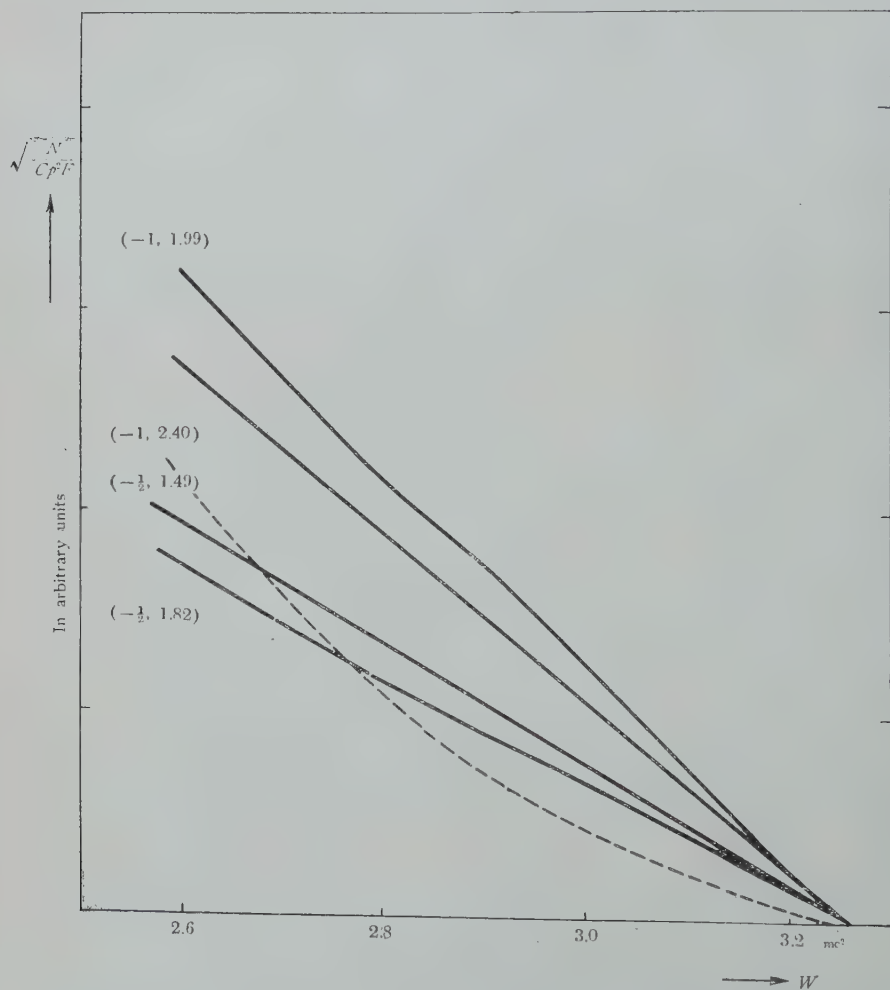


Fig. 5. Kurie plots for Cs^{137} . The sets of numbers in the figure are $(G_S/G_T, A)$. The dotted curve is the uncorrected Kurie plot.

is $i(x=-1)$, and the same as that of Rb^{87} . Using the experimental data of Langer and Moffat¹⁶⁾, the values of λ which make the Kurie plot straight at the maximum energy $W_0=3.26mc^2$, $W=2.97mc^2$ ($p=2.8mc$) and $W=2.6mc^2$ ($p=2.4mc$) are shown in Table I and the corresponding Kurie plots are given in Fig. 5. In this case no real value of λ exists for $G_s/G_T=1/2$ and 1. Even if we change the maximum energy by $0.1mc^2$, real value of λ does not exist for $G_s/G_T=1/2$ and 1. $(-1, 2.40)$ and $(-1/2, 1.49)$ fit the experiment very well. The Kurie plots for $(-1, 1.99)$ and $(-1/2, 1.82)$ deviate from straight line a little and they are labeled Δ . The corrected $f_c t$ -values for $Q_2(\beta\sigma \times \mathbf{r}, \mathbf{r})$ are of the order of 10^{13} , and suitable for the second forbidden coordinate type nuclear matrix element.

§ 3. Conclusion and discussion

To deduce the conclusion, it is better to look at Table I. When G_s/G_T is positive (1, and $1/2$), there are cases in which no real λ exists. On the other hand, when G_s/G_T is negative, real values of λ always exist. Especially in the case of $G_s/G_T=-1/2$ the values of λ , which fit the experiments very well, exist for all the elements. The fit of the case of $G_s/G_T=-1$ to the experiments is a little worse. Even if we take into account the roughness of our analysis, the minus sign of G_s/G_T is very probable, but it is impossible to exclude $G_s/G_T=-1$. Our conclusion is the following: "The sign of G_s/G_T is probably minus, and as to its absolute magnitude a somewhat smaller value than unity is favourable, but $G_s/G_T=-1$ cannot definitely be excluded".

Until now we have neglected the nuclear matrix element $Q_{n+1}(\beta\sigma, \mathbf{r})$. Even if we take into account this term, the conclusion will not be changed. Especially if we include this term in the case of Rb^{87} and Cs^{137} , the situation for the cases of positive values of G_s/G_T become worse, because this term has the completely opposite nature to the correction factors necessary for Rb^{87} and Cs^{137} .

In the calculation of section 2, we have decided the values of λ so as to fit the experiments. However, there is a theoretical value which has been calculated by one of the authors¹⁷⁾ using the method of Ahrens and Feenberg.¹⁸⁾ According to this theory λ takes the value near unity, but whatever the value G_s/G_T may be, λ cannot take the values near unity for all the elements. On the other hand, if we take the value of λ as unity at the beginning and then decide x so as to explain the experiments, x takes the value calculated with the shell model, and this value of x seems unlikely. Therefore, it is natural to regard the relations of Ahrens, Feenberg¹⁸⁾ and Yamada¹⁷⁾ as statistical ones which give only the order of magnitude in each case.

The authors are greatly indebted to Profs. S. Nakamura and T. Yamanouchi, Dr. M. Umezawa and Mr. H. Takebe for their kind guidances and valuable discussions. Two of the authors (M. M. and M. Y.) are also indebted to Yukawa Yomiuri Fellowship for the financial aid.

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On the β - γ Angular Correlation of Sb^{124} , II*

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A consistent explanation of both the β - γ angular correlation and the β -ray spectrum of Sb^{124} is obtained with the ST type interaction in the Fermi theory of β -decay. It is concluded that (i) the decay scheme is $3(-)-2(+)-0(+)$, (ii) the β -decay is a first forbidden transition which contains the reduced nuclear matrix elements $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta \alpha)$, $\mathfrak{M}(\beta \sigma \times r)$ and $\mathfrak{M}(B_{ij}^3)$, and (iii) the γ -decay is an electric quadrupole radiation. It is also noticed that $4(+)-2(+)-0(+)$ is ruled out.

§ 1. Introduction

The nucleus Sb^{124} changes into Te^{124} in two steps: first, Sb^{124} changes into the first excited state of Te^{124} (Te^{124*}) emitting an electron** ; and then, emitting a γ -ray quantum, Te^{124*} goes into the ground state of Te^{124} which has spin zero and even parity.

We abbreviate the five interaction types in the Fermi theory of β -decay as S, V, T, A and P hereafter. A linear combination of these types is written as ST, for example. The ratio of the coupling constants is not restricted to special values.

Recently the authors⁽²⁾ studied the β -decay of Sb^{124} with the single type among the five relativistic invariants in the Fermi theory (Part 1). It was concluded that a certain linear combination of the reduced nuclear matrix elements (RNME's) $\mathfrak{M}(\beta \alpha)$, $\mathfrak{M}(\beta \sigma \times r)$ and $\mathfrak{M}(B_{ij}^3)$ in T gives consistent explanation for both the β - γ angular correlation and β -ray spectrum, and the decay scheme, ground state of $\text{Sb}^{124} \rightarrow$ first excited state of $\text{Te}^{124} \rightarrow$ ground state of Te^{124} , $3(-)-2(+)-0(+)$.

After the part I was published several new informations on Sb^{124} have been obtained. Namely, Kloepper, Lennox and Wiedenbeck⁽³⁾ measured the angular and direction-polarization correlations which are in good accord with the previous experimental values measured by Darby and Opechowski⁽⁴⁾ and Stevenson and Deutsch⁽⁵⁾; Hutchinson and Wiedenbeck⁽⁶⁾ reinvestigated the β -ray spectrum and internal conversion coefficient of Sb^{124} which agreed with the results of Langer et al.⁽⁷⁾ and of Metzger.⁽⁸⁾

Moreover, Langer, Lazar and Moffat⁽¹⁾ refined their experiments and explained the β -ray spectrum by ST which contains RNME's $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta \alpha)$ and $\mathfrak{M}(\beta \sigma \times r)$. $\mathfrak{M}(B_{ij}^3)$ was discarded in their treatment because they thought that the magnitude of $\mathfrak{M}(B_{ij}^3)$ is very small.

* The part I of this article was published in Prog. Theor. Phys. 8 (1952), 449. Preliminary report of the part II was published in Prog. Theor. Phys. 10 (1953), 111.

** The β -ray of Sb^{124} is complex and we take an interest in the highest energy β -group which has the maximum energy 2.317 Mev. recently measured by Langer.⁽¹⁾

On the other hand Nakamura, Takebe and Umezawa⁹⁾ explained the β -ray spectrum with a linear combination of $\mathcal{M}(A_{ij}^3)$ and $\mathcal{M}(T_{ij}^3)$ terms in the second forbidden of T. Metzger¹⁰⁾ reported lately based on his experiments on the lower energy β -ray group of Sb^{124} and γ - γ angular correlations of Te^{124} , that the ground state of Sb^{124} will perhaps have spin 4 and even parity rather than spin 3 and odd parity.

In order to clarify the relations between these experimental and theoretical investigations on Sb^{124} , it seems important to investigate the β - γ angular correlation and β -ray spectrum with ST. Necessity of both S and T looks quite certain.

That the γ -ray is electric quadrupole is shown by many evidences.^{1, 2, 3, 8, 10)} Therefore, $\text{Te}^{124*} \rightarrow \text{Te}^{124}$ is $2(+)-0(+)$. An ambiguity left over whether the ground state of Sb^{124} is $3(-)$ or $4(+)$, i.e. whether the β -decay of Sb^{124} is the first or second forbidden. The other assignments to Sb^{124} will be ruled out because of the inadequacy for β - γ angular correlations,²⁾ even if S is taken into account.

We study the assignment $3(-)-2(+)-0(+)$ in section 2, where we shall show that some linear combinations of RNME's in ST can explain the β -ray spectrum and β - γ angular correlation simultaneously. The assignment $4(+)-2(+)-0(+)$ is investigated in section 3, where we shall show that good agreement with angular correlation is hard to obtain in this scheme. These results are summarized and discussed in section 4.

The numerical calculations are performed on an approximation $(a/\lambda)^2 \ll 1$ which may be a fairly good approximation in our case. The finite de Broglie wave length effect¹¹⁾ (FDBWLE) is taken into account in the special cases only, and the effect of finite nuclear size correction¹²⁾ (FNSC) is neglected, because these effects are very complicated, and moreover we expect they will not affect¹³⁾ the main results of this paper.

§ 2. $3(-)-2(+)-0(+)$

Under this assignment, the β -decay is the first forbidden transition in which $\mathcal{M}(\beta\mathbf{r})$, $\mathcal{M}(\beta\boldsymbol{\alpha})$, $\mathcal{M}(\beta\boldsymbol{\sigma} \times \mathbf{r})$ and $\mathcal{M}(B_{ij}^3)$ in ST are contained. We use the same notation as YM.

Substituting $a_{22}^{(2n)}$'s of the electric quadrupole radiation, eq. (24) of YM into eq. (20) of YM, we get the β - γ angular correlation function $\mathcal{U}(\theta)$,

$$\mathcal{U}(\theta) = 7a_{11}^{(0)} + 7a_{22}^{(0)} + (-a_{11}^{(2)} + \sqrt{2}a_{12}^{(2)} + 4a_{22}^{(2)})P_2(\cos\theta). \quad (1)$$

Common factors are dropped here. $a_{L_1 L_2}^{(2n)}$'s are introduced by eqs. (19), (20), (S1), (T1) of YM and (ST1) of M,

$$\begin{aligned} F_{11}^0(\theta) &= a_{11}^{(0)} - 2a_{11}^{(2)}P_2(\cos\theta) \\ &= G_S^2[\mathcal{M}(\beta\mathbf{r})]^2[\{(1/3)K^2L_0 + (2/3)KN_0 + 2L_1 + M_0\} \\ &\quad + \{(4/3)KL_{12} + 2L_1 + 4N_{12}\}P_2(\cos\theta)] \\ &\quad + G_T^2[\mathcal{M}(\beta\boldsymbol{\alpha})]^2L_0 + G_T^2[\mathcal{M}(\beta\boldsymbol{\sigma} \times \mathbf{r})]^2[\{(1/6)K^2L_0 - (2/3)KN_0 + (1/2)L_1 \\ &\quad + M_0\} + \{(2/3)KL_{12} + (1/2)L_1 - 2N_{12}\}P_2(\cos\theta)] \\ &\quad - G_T^2\{\mathcal{M}^*(\beta\boldsymbol{\alpha})\mathcal{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\}[\{(1/3)KL_0 - N_0\} + L_{12}P_2(\cos\theta)] \end{aligned}$$

$$\begin{aligned}
 & -G_S G_T \{i\mathfrak{M}^*(\beta\mathbf{r})\mathfrak{M}(\beta\mathbf{a}) + \text{c.c.}\} [\{N_0 + (1/3)KL_0\} + 2L_{12}P_2(\cos\theta)] \\
 & + G_S G_T \{i\mathfrak{M}^*(\beta\mathbf{r})\mathfrak{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) + \text{c.c.}\} [(L_1 - M_0) \\
 & + (KL_{12} + L_1 - N_{12})P_2(\cos\theta)], \\
 F_{12}^1(\theta) & = a_{12}^{(2)} P_2(\cos\theta) \\
 & = -G_T^2 \{i\mathfrak{M}^*(\beta\mathbf{a})\mathfrak{M}(B_{ij}^{\beta}) + \text{c.c.}\} \{3/(2\sqrt{2})\} L_{12} P_2(\cos\theta) \\
 & - G_T^2 \{i\mathfrak{M}^*(\beta\boldsymbol{\sigma} \times \mathbf{r})\mathfrak{M}(B_{ij}^{\beta}) + \text{c.c.}\} \\
 & \times (1/\sqrt{2}) \{-(1/2)KL_{12} + (3/4)L_1 + (3/2)N_{12}\} P_2(\cos\theta) \\
 & + G_S G_T \{\mathfrak{M}^*(\beta\mathbf{r})\mathfrak{M}(\beta_{ij}^{\beta}) + \text{c.c.}\} \\
 & \times (1/\sqrt{2}) \{-(1/2)KL_{12} + (3/2)L_1 - (3/2)N_{12}\} P_2(\cos\theta), \\
 F_{22}^0(\theta) & = a_{22}^{(0)} - 2a_{22}^{(2)} P_2(\cos\theta) + 6a_{22}^{(4)} P_4(\cos\theta) \\
 & = G_T^2 |\mathfrak{M}(B_{ij}^{\beta})|^2 [\{(1/12)K^2 L_0 + (3/4)L_1\} + (3/4)L_1 P_2(\cos\theta)]. \quad (2)
 \end{aligned}$$

The angular correlation coefficient $a(W)$ is defined by

$$a(W) = \{\mathcal{L}(\pi) - \mathcal{L}(\pi/2)\} / \mathcal{L}(\pi/2). \quad (3)$$

We use for the nuclear radius

$$\rho = 1.4 \times 10^{-13} A^{1/3} / (\hbar/mc), \quad (4)$$

and for the maximum energy of the electron

$$W_0 = 5.543 \text{ } mc^2. \quad (5)$$

We introduce real parameters⁹⁾ x , y and z^* as follows:

$$\begin{aligned}
 G_S \mathfrak{M}(\beta\mathbf{r}) / G_T \mathfrak{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) & = -ix, \quad \mathfrak{M}(\beta\mathbf{a}) / \mathfrak{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) = (uZ/2\rho) \cdot y \\
 \text{and } \mathfrak{M}(B_{ij}^{\beta}) / \mathfrak{M}(\beta\boldsymbol{\sigma} \times \mathbf{r}) & = iz. \quad (6)
 \end{aligned}$$

A linear combination of these four RNME's is symbolized by $(x, y, z)_1^*$. For example, Langer's case¹⁾ is written as $(1, 1, 0)_1$. We shall show the conclusion for various combinations of x, y, z in the following:

(a) Langer's case $(1, 1, 0)_1$

$(1, 1, 0)_1$ has an excellent correction factor. Unfortunately, its angular correlation is too small in absolute value, $a(5) \approx +0.02$, and has the opposite sign to that of the experimental value.

(b) $(x, y, 0)_1$

Whatever the ratios x and y may be the calculated angular correlation is not so large in absolute values compared to the experimental values. This excludes $(x, y, 0)_1$ definitely, and proves that we can not discard $\mathfrak{M}(B_{ij}^{\beta})$.

(c) $(1, 1, z)_1$

In order to remedy above defect, there is a possible way in which we use the linear

* The parameters x, y and z are introduced somewhat differently from those of part I. In the preliminary report of part II, z is written as r and $(x, r, 1)$ is equivalent to $(x, y, 0)_1$ in the present paper.

combination of $\mathfrak{M}(\beta_{ij}^{\beta})$ and $(1, 1, 0)_1$, i.e. $(1, 1, z)_1$. The interferences between them will cause an advantageous effect on $a(W)$. They do not exist in the correction factor. The suitable ratio z is $-6 > z > -9$ or $-50 > z > -150$. The graphs for $a(W)$ are shown in Figs. 1 and 2.

Since the relations between RNME's derived by Ahrens and Feenberg⁽⁴⁾ will have nothing more than statistical meaning,* we can not conclude $x=y=1$, therefore, we must examine various pairs of x and y .

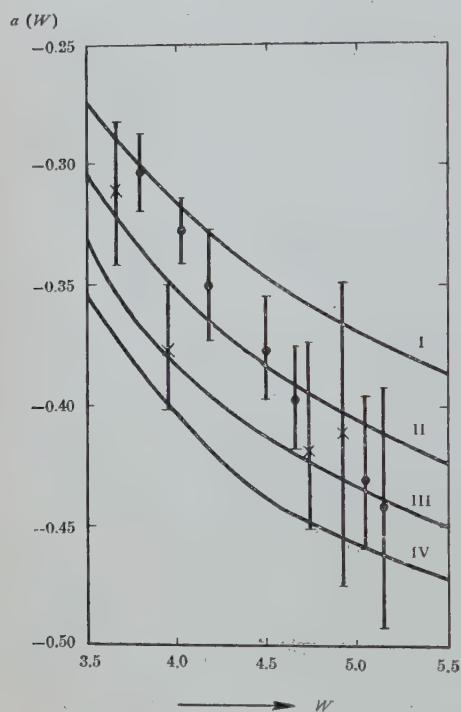


Fig. 1. Angular correlation coefficients for $3(-)-2(+)-0(+)$, in the cases of linear combinations of $\mathfrak{M}(\beta r)$, $\mathfrak{M}(\beta a)$, $\mathfrak{M}(\beta \sigma \times r)$ and $\mathfrak{M}(\beta_{ij}^{\beta})$.

- I. $(1, 1, -6)_1$.
- II. $(1, 1, -7)_1$.
- III. $(1, 1, -8)_1$.
- IV. $(1, 1, -9)_1$.

● Experimental values measured by Darby and Opechowski.⁽⁴⁾

× Experimental values measured by Stevenson and Deutsch.⁽⁵⁾

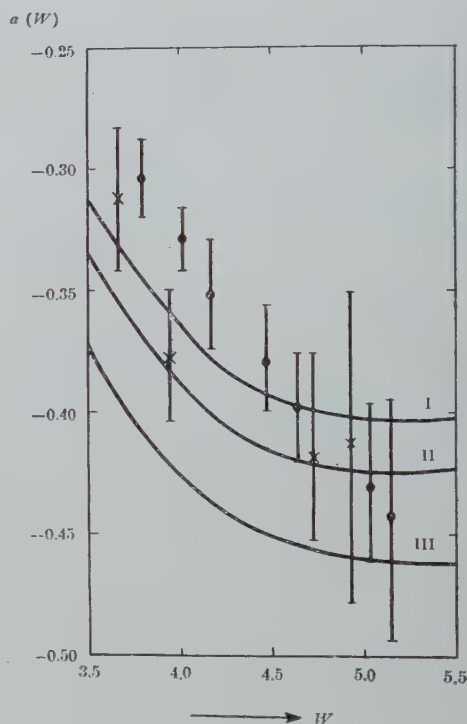


Fig. 2. Angular correlation coefficients for $3(-)-2(+)-0(+)$, in the same cases as Fig. 1.

- I. $(1, 1, -150)_1$.
- II. $(1, 1, -98)_1$.
- III. $(1, 1, -48)_1$.

* Because of the crude approximation.

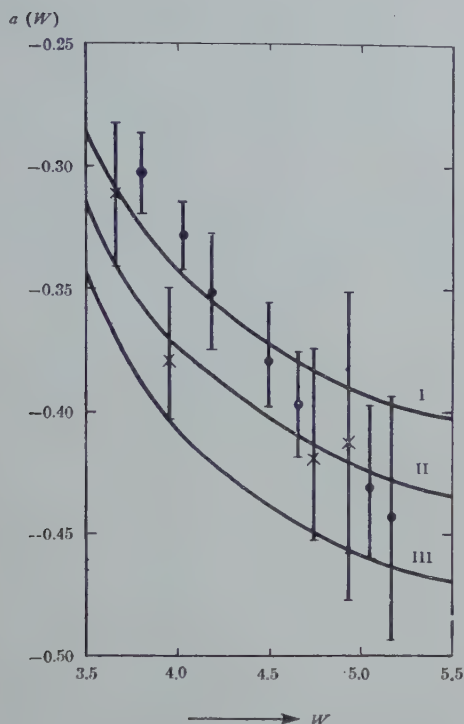


Fig. 3. Angular correlation coefficients for $3(-)-2(+)-0(+)$, in the same cases as Fig. 1.

- I. $(1, 1.54, -4.1)_1$.
- II. $(1, 1.54, -4.7)_1$.
- III. $(1, 1.54, -5.7)_1$.

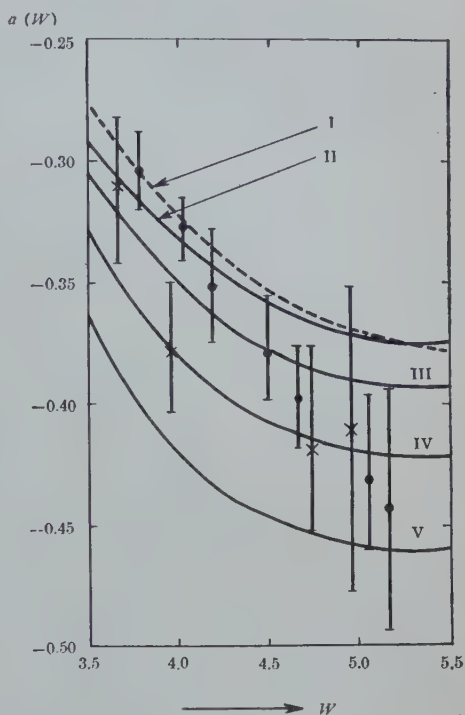


Fig. 4. Angular correlation coefficients for $3(-)-2(+)-0(+)$, in the same cases as Fig. 1.

- I. $(0, 0, \infty)_1$, corrected by the finite de Broglie wave length effect (dotted line).
- II. $(0, 0, \infty)_1$
- III. $(1, 1.54, -158)_1$.
- IV. $(1, 1.54, -58)_1$.
- V. $(1, 1.54, -28)_1$.

In order to perform this procedure in a simple manner we interpret $(1, 1, \infty)_1$ as follows. It should be noticed that $i(G_S/G_T) \cdot \mathcal{M}(\beta r)$ plays almost the same rôle as $\mathcal{M}(\beta \sigma \times r)$ in $F_{11}^0(\theta)$, $F_{12}^1(\theta)$ and the correction factor. Accordingly, we can make rough estimate for the case of ST using our previous work (Part I) with T only. According to Figs. 1 and 2. of I the following ratios

$$i\mathcal{M}(\beta a)/\mathcal{M}(B_{ij}^a) \sim -1 \quad \text{and} \quad i\mathcal{M}(\beta \sigma \times r)/\mathcal{M}(B_{ij}^a) \sim -0.2$$

give a good angular correlation in T only. If we put

$$G_S \mathcal{M}(\beta r)/G_T \mathcal{M}(\beta \sigma \times r) = -i,$$

$i(G_S/G_T) \cdot \mathcal{M}(\beta r)$ takes a half part of the rôle of $\mathcal{M}(\beta \sigma \times r)$ in T only and the ratios,

$$i\mathcal{M}(\beta a)/\mathcal{M}(B_{ij}^a) \sim -1, \quad i\mathcal{M}(\beta \sigma \times r)/\mathcal{M}(B_{ij}^a) \sim -0.1$$

$$\text{and} \quad G_S \mathcal{M}(\beta r)/G_T \mathcal{M}(B_{ij}^a) \sim +0.1,$$

i.e. $x=1$, $y \sim 10/(aZ/2\rho) \sim 1$ and $z \sim -10$, which agree approximately with $(1, 1, -6 > z > -9)_1$ in (c), will give the good angular correlation.

Generally Fig. 1 of I gives the order of the magnitude of $\mathfrak{M}(B_{ij}^\beta)$ in case of ST. Then, we must find its precise values to obtain the suitable angular correlation.

(d) $(x, y, z)_1$

From the above considerations we have found the combinations $(1, 1.54, z)_1$ where $z = -4.7$ or -58 . Both $(1, 1.54, 0)_1$ and $(0, 0, \infty)_1$ give good correction factors, so that $(1, 1.54, z)_1$ gives also a good correction factor. The graphs of $\alpha(W)$ are shown in Figs. 3 and 4.

In the cases of (c) and (d) the corrected ft-values for B_{ij}^β , i.e. $ft = (2\pi^3/G^2) \times \log 2 / \sum |B_{ij}^\beta|^2$, are

$$\approx 3 \times 10^{10} \quad \text{if } |z| \text{ is more than } 20,$$

$$\text{or} \quad \approx 9 \times 10^{10} \quad \text{if } |z| \text{ is } 1 \sim 10.$$

These values are considerably larger than the usual, but not inconsistent with the first forbidden transition.

In the above calculation FNSC⁽¹²⁾ and FDBWLE⁽¹¹⁾ are not considered. For the latter Rose et al.⁽¹¹⁾ calculated only L_i , M_i , N_i , L_i^- , M_i^- and N_i^- . In our cases, many other functions L_{ij} , M_{ij} and N_{ij} appear, and it is difficult to calculate FDBWLE for L_{ij} etc. with the same accuracy as Rose's tables.⁽¹¹⁾ For $\mathfrak{M}(B_{ij}^\beta)$ only, however, Rose's tables are sufficient, and $\alpha(W)$ taking into account FDBWLE is given in Fig. 4 (dotted line). It is almost the same as $\alpha(W)$ which neglects FDBWLE (full line). FDBWLE may have appreciable influence on the β -ray spectrum for which a large cancellation takes place, if the approximation $(aZ)^2 \ll 1$ is violated. However, as far as this approximation is valid, we can neglect FDBWLE.⁽¹²⁾ The situation of FNSC is almost similar as FDBWLE: it will have no appreciable effect, too.⁽¹³⁾

§ 3. $4(+)-2(+)-0(+)$

Under this assignment, the β -decay is the second forbidden transition which includes $\mathfrak{M}(R_{ij}^\beta)$, $\mathfrak{M}(A_{ij}^\beta)$, $\mathfrak{M}(T_{ij}^\beta)$ and $\mathfrak{M}(S_{ijk}^\beta)$ in ST.

Substituting $a_{22}^{(\beta n)}$ s for the electric quadrupole radiation, eq. (24) of YM, into eq. (31) of YM, we get the β - γ angular correlation function $\mathcal{U}(\theta)$,

$$\begin{aligned} \mathcal{U}(\theta) = & 126(a_{22}^{(0)} + a_{33}^{(0)}) \\ & + \{-36a_{22}^{(2)} + 135a_{33}^{(2)} + 45\sqrt{10}a_{23}^{(2)}\}P_2(\cos\theta) \\ & + \{-4a_{22}^{(4)} + 88a_{33}^{(4)} + 4\sqrt{10}a_{23}^{(4)}\}P_4(\cos\theta). \end{aligned} \quad (7)$$

Common factors are dropped here. $a_{L_1 L_1}^{(\beta n)}$ s are introduced by eqs. (20), (21), (S2), (T2) of YM and (ST2) of M as follows.

$$\begin{aligned}
 F_{22}^0(\theta) &= a_{22}^{(0)} - 2a_{22}^{(2)} P_2(\cos\theta) + 6a_{22}^{(4)} (\cos\theta) \\
 &= G_S^2 |\mathfrak{M}(R_{ij}^{\beta})|^2 [\{ (1/30) K^4 L_0 + (2/15) K^3 N_0 + (2/3) K^2 L_1 + (1/3) K^2 M_0 \\
 &\quad + 2KN_1 + (9/2) L_2 + 3M_1 \} + \{ (2/15) K^3 L_{12} + (1/3) K^2 L_1 + (2/3) K^2 N_{12} \\
 &\quad + (6/7) KL_{23} + 2KN_1 + (36/7) L_2 + 3M_1 + (18/7) N_{23} \} P_2(\cos\theta) \\
 &\quad + \{ (36/7) KL_{23} + (27/7) L_2 + (108/7) N_{23} \} P_4(\cos\theta)] \\
 &\quad + G_T^2 |\mathfrak{M}(A_{ij}^{\beta})|^2 [\{ (1/12) K^2 L_0 + (3/4) L_1 \} + (3/4) L_1 P_2(\cos\theta)] \\
 &\quad + G_T^2 |\mathfrak{M}(T_{ij}^{\beta})|^2 (1/12) [\{ (1/15) K^4 L_0 - (2/5) K^3 N_0 + K^2 L_1 + K^2 M_0 \\
 &\quad - 6KN_1 + 6L_2 + 9M_1 \} + \{ K^2 L_1 + (12/7) KL_{23} - 6KN_1 + (48/7) L_2 \\
 &\quad + 9M_1 - (36/7) N_{23} \} P_2(\cos\theta) \\
 &\quad + \{ (72/7) KL_{23} + (36/7) L_2 - (216/7) N_{23} \} P_4(\cos\theta)] \\
 &\quad - G_T^2 \{ \mathfrak{M}^*(A_{ij}^{\beta}) \mathfrak{M}(T_{ij}^{\beta}) + \text{c.c.} \} (1/4) [\{ (1/15) K^3 L_0 \\
 &\quad - (1/3) K^2 N_0 + KL_1 - 3N_1 \} \\
 &\quad + \{ KL_1 + (6/7) L_{23} - 3N_1 \} P_2(\cos\theta) + (36/7) L_{23} P_4(\cos\theta)] \\
 &\quad - G_S G_T \{ i \mathfrak{M}^*(R_{ij}^{\beta}) \mathfrak{M}(A_{ij}^{\beta}) + \text{c.c.} \} [\{ (1/30) K^3 L_0 + (1/6) K^2 N_0 \\
 &\quad + (1/2) KL_1 + (3/2) N_1 \} \\
 &\quad + \{ (1/6) K^2 L_{12} + (1/2) KL_1 + (9/14) L_{23} + (3/2) N_1 \} P_2(\cos\theta) \\
 &\quad + (27/7) L_{23} P_4(\cos\theta)] \\
 &\quad + G_S G_T \{ i \mathfrak{M}^*(R_{ij}^{\beta}) \mathfrak{M}(T_{ij}^{\beta}) + \text{c.c.} \} [\{ (1/6) (K^2 L_1 - K^2 M_0) \\
 &\quad + (3/2) (L_2 - M_1) \} \\
 &\quad + \{ (1/30) K^3 L_{12} + (1/6) (K^2 L_1 - K^2 N_{12}) + (5/14) KL_{23} \\
 &\quad + (12/7) L_2 - (3/2) M_1 - (3/14) N_{23} \} P_2(\cos\theta) \\
 &\quad + \{ (15/7) KL_{23} + (9/7) (L_2 - N_{23}) \} P_4(\cos\theta)], \\
 F_{23}^1(\theta) &= \sqrt{10} a_{23}^{(2)} P_2(\cos\theta) - \sqrt{10} a_{23}^{(4)} P_4(\cos\theta) \\
 &= -G_T^2 \{ i \mathfrak{M}^*(A_{ij}^{\beta}) \mathfrak{M}(S_{ijk}^{\beta}) + \text{c.c.} \} (\sqrt{10}/\sqrt{3}) [\{ (1/60) K^2 L_{12} \\
 &\quad + (3/28) L_{23} \} P_2(\cos\theta) + (15/56) L_{23} P_4(\cos\theta)] \\
 &\quad - G_T^2 \{ i \mathfrak{M}^*(T_{ij}^{\beta}) \mathfrak{M}(S_{ijk}^{\beta}) + \text{c.c.} \} (\sqrt{10}/\sqrt{3}) [\{ (1/300) (-K^3 L_{12} \\
 &\quad + 5K^2 N_{12}) + (1/28) (-KL_{23} + 2L_2 + 3N_{23}) \} P_2(\cos\theta) \\
 &\quad + (5/56) \{ -KL_{23} + 2L_2 + 3N_{23} \} P_4(\cos\theta)] \\
 &\quad + G_S G_T \{ \mathfrak{M}^*(R_{ij}^{\beta}) \mathfrak{M}(S_{ijk}^{\beta}) + \text{c.c.} \} (\sqrt{10}/\sqrt{3}) [(1/150) (-K^3 L_{12} \\
 &\quad + 5K^2 L_1 - 5K^2 N_{12}) + (1/14) (-KL_{23} + 3L_2 - 3N_{23}) \} P_2(\cos\theta) \\
 &\quad + (5/28) (-KL_{23} + 3L_2 - 3N_{23}) \} P_4(\cos\theta)], \\
 F_{33}^0(\theta) &= a_{33}^{(0)} - 4a_{33}^{(2)} P_2(\cos\theta) + 6a_{33}^{(4)} P_4(\cos\theta) - 20a_{33}^{(6)} P_6(\cos\theta) \\
 &= G_T^2 |\mathfrak{M}(S_{ijk}^{\beta})|^2 (1/72) [\{ (1/15) K^4 L_0 + 2K^2 L_1 + 15L_2 \} \\
 &\quad + \{ (8/5) K^2 L_1 + (120/7) L_2 \} P_2(\cos\theta) + (90/7) L_2 P_4(\cos\theta)].
 \end{aligned}$$

The analysis of $4(+)-2(+)-0(+)$ is carried out in the same way as in section 2. It should be noticed that $i(G_S/G_T) \cdot \mathfrak{M}(R_{ij}^0)$ plays nearly the same rôle as $\mathfrak{M}(T_{ij}^0)$ in $F_{22}^0(\theta)$, $F_{22}^1(\theta)$ and the correction factor. We introduce real parameters²⁾ x , y and z as follows,

$$\begin{aligned} G_S \mathfrak{M}(R_{ij}^0)/G_T \mathfrak{M}(T_{ij}^0) &= -ix/2, \quad \mathfrak{M}(A_{ij}^0)/\mathfrak{M}(T_{ij}^0) = uZy/4\rho \\ \text{and } \mathfrak{M}(S_{ijk}^0)/\mathfrak{M}(T_{ij}^0) &= iz. \end{aligned} \quad (9)$$

A linear combination of these four RNME's. is symbolized by $(x, y, z)_2$.

(a) $(x, y, 0)_2$

The good correction factor can be obtained with certain ratios x and y . But, whatever the ratios x and y may be, the calculated angular correlation is not so large in absolute values compared to the experimental value. This excludes $(x, y, 0)_2$ definitely, and proves that we can not discard $\mathfrak{M}(S_{ijk}^0)$.

(b) $(x, y, z)_2$

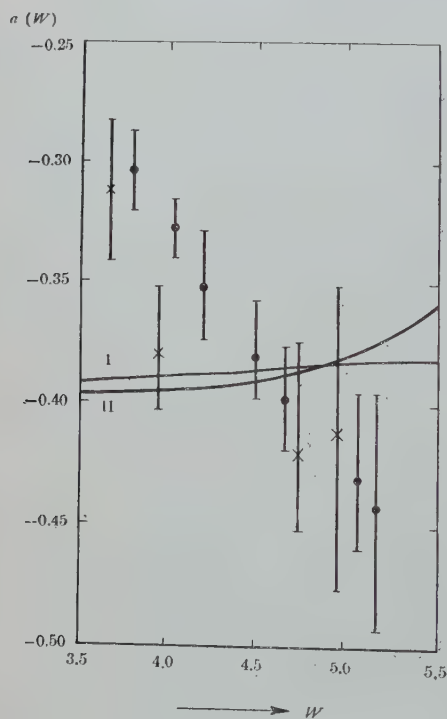


Fig. 5. Angular correlation coefficients for $4(+)-2(+)-0(+)$, in the cases of linear combinations of $\mathfrak{M}(R_{ij}^0)$, $\mathfrak{M}(A_{ij}^0)$, $\mathfrak{M}(T_{ij}^0)$ and $\mathfrak{M}(S_{ijk}^0)$.

- I. $(1, 1, -33.5)_2$.
II. $(1, 1, -57)_2$.

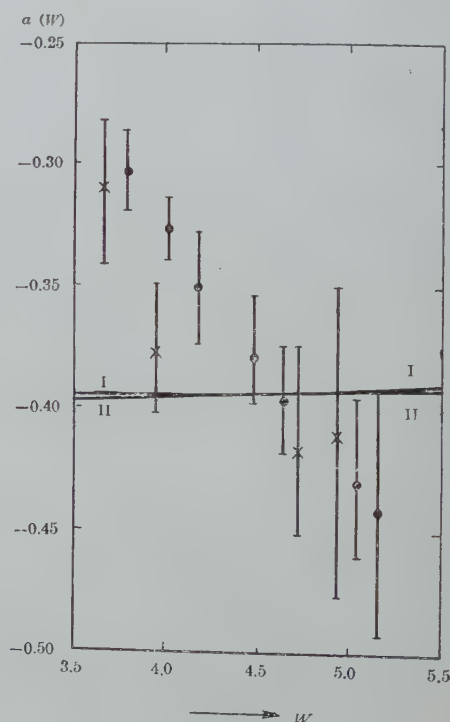


Fig. 6. Angular correlation coefficients for $4(+)-2(+)-0(+)$, in the same cases as Fig. 5.

- I. $(1, -1, -77.4)_2$.
II. $(1, -1, -134)_2$.

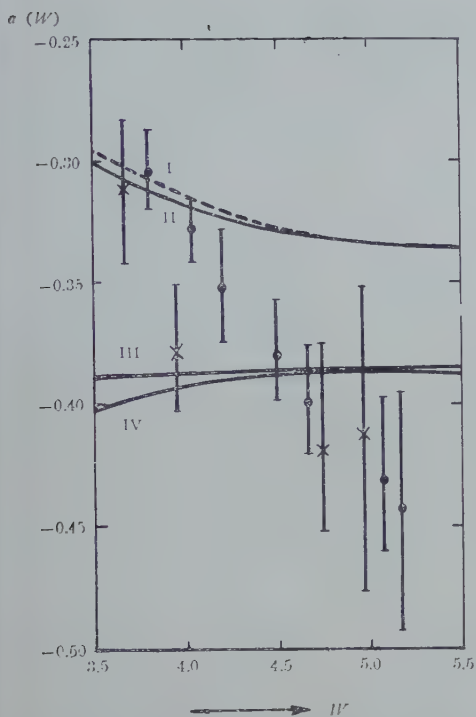


Fig. 7. Angular correlation coefficients for $4(+)-2(+)-0(+)$, in the same cases as

Fig. 5.

- I. $(0, 0, \infty)_2$, corrected by the finite de Broglie wave length effect (dotted line).
- II. $(0, 0, \infty)_2$.
- III. $(1, 1.3, -41)_2$.
- IV. $(1, 1.3, -28)_2$.

From the above results the following conclusion may be deduced: by the mixture of $\mathfrak{M}(R_{ij}^B)$, the situation becomes worse in ST than in T only. We can not help requiring the large magnitude of $\mathfrak{M}(S_{ijk}^B)$ which has $a(W)$ with only small energy dependence (See, I of Fig. 5 in part I). If we rely upon the experimental values of Darby and Opechowski which shows very large energy dependence, this is a destructive feature for the assignment $4(+)-2(+)-0(+)$ in ST.

(c) $(0, 0, \infty)_2$ taking into account FDBWLE

For $\mathfrak{M}(S_{ijk}^B)$ we calculated $a(W)$ with and without FDBWLE. The results are shown in Fig. 7.

For the cases (a) and (b) the Kurie plots are investigated, they have all similar properties to the case of $\mathfrak{M}(S_{ijk}^B)$ only (See, Fig. 6 in part I.) because of large z 's. They are convex downwards.

Corrected ft-values for S_{ijk}^B , $ft = (2\pi^3/G^2) \cdot \log 2 / \sum |S_{ijk}^B|^2$, are approximately 1×10^{10} in all cases.

We put simply $x = \pm 1$ and $y = \pm 1$ comparing with the Ahrens-Feenberg-Yamada¹⁴⁾ relations, eq. (9). To get large angular correlation at $W = 5mc^2$, z must be following values:

$(1, 1, z)_2$, where z is -33.5 or -57 .

(Fig. 5)

$(1, -1, z)_2$, where z is -134 or -77.4 .

(Fig. 6)

In these two cases we can not obtain the absolute value of $a(5)$ larger than 0.39 .

$(-1, 1, z)_2$, where z is 227 or 27.4 .

In the last case, $a(5) = -0.34$ and we can not obtain the absolute value of $a(5)$ larger than 0.35 . Moreover, in these three cases, $a(W)$'s have much smaller energy dependence than that of the data of Darby and Opechowski.⁴⁾

$i(G_s/G_T) \cdot \mathfrak{M}(R_{ij}^B)$ is similar to $\mathfrak{M}(T_{ij}^B)$.

Hence, in the same way as in section 2, we may find two examples $(1, 1.3, -28)_2$ and $(1, 1.3, -41)_2$ which make $a(5) = -0.385$. $a(W)$'s are shown in Fig. 7 which is almost the same as Figs. 5 and 6. $a(5)$'s of $(1, 1.3, z)_2$ can not have the absolute value larger than 0.39 .

§ 4. Discussion and summary

The energy levels in Sb^{124} and Te^{124} are shown in Fig. 8.

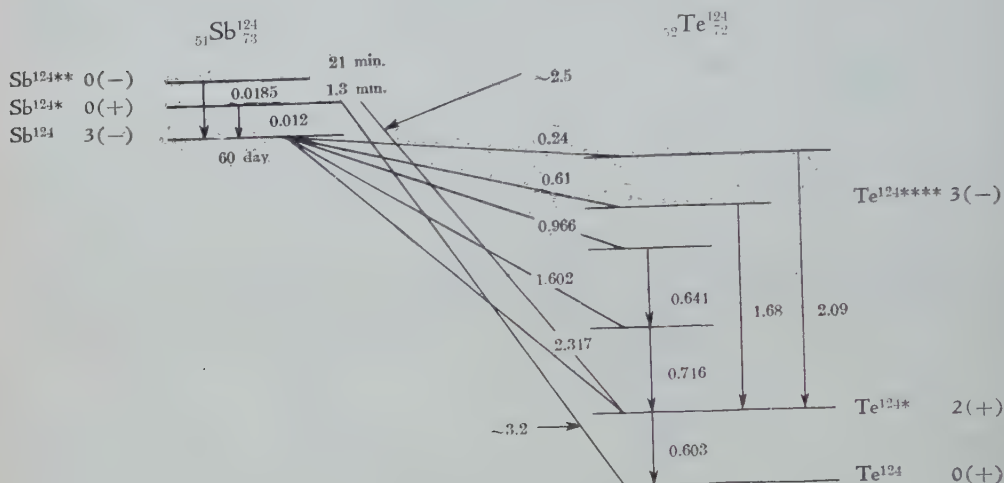


Fig. 8. Energy levels^{1, 10, 17)} of Sb^{124} and Te^{124} . Unit: Mev. Spins of Sb^{124} are newly assigned by the present authors.¹⁹⁾

Metzger measured the γ - γ (1.7–0.6 Mev.) angular correlation in Te^{124} which accorded with the assignment $3(-) \rightarrow 2(+)$ – $0(+)$, ($\text{Te}^{124****} \rightarrow \text{Te}^{124*} \rightarrow \text{Te}^{124}$). He insisted as follows: "If the ground state of Sb^{124} were $3(-)$ the β -transition to the 2.3 Mev. excited state of Te^{124} ($\text{Te}^{124****}$) with $3(-)$ would be allowed. But the $\log_{10} ft$ value for this transition is 7.7 and too large for the allowed transition, therefore, the assignment is $4(+)$ rather than $3(-)$ to the ground state of Sb^{124} ." However, the ft -values of the allowed transitions lie in very wide region¹⁷⁾ and $\log_{10} ft = 7.7$ does not seem inconsistent with the allowed transition.

If we consider the total decay scheme including the two isomers of Sb^{124} , we shall also find that the assumption of $3(-)$ for Sb^{124} is better than that of $4(+)$. The decay scheme is shown in Fig. 8. First, we consider the three energy levels of Sb^{124} . (We denote them by Sb^{124} , Sb^{124*} and Sb^{124**} as in Fig. 8) From the life-times of them both transitions $\text{Sb}^{124**} \rightarrow \text{Sb}^{124}$ and $\text{Sb}^{124*} \rightarrow \text{Sb}^{124}$ will be forbidden of degree higher than E2 and M2.^{16, 17)} Therefore, if either of the spin of Sb^{124*} or Sb^{124**} is not zero, we must take the difference between their spins six. The β -decay, $\text{Sb}^{124*} \rightarrow \text{Te}^{124}$, is allowed,¹⁷⁾ so that Sb^{124} has $0(+)$ or $1(+)$, and Sb^{124**} has either spin zero (Sb^{124*} has $0(+)$ in this case) or spin more than six. In the latter case, the β -decay $\text{Sb}^{124**} \rightarrow \text{Te}^{124*}$ is forbidden more highly than twice, and it will not be detected on account of the 21 minutes isomeric transition of Sb^{124**} . According to the experiments of E. der Mateosian et al.¹⁸⁾, weak β -rays of which maximum energy is 2.336 Mev. are observed, and this excludes the high spin of Sb^{124**} . Therefore, if we take the isomeric transition $\text{Sb}^{124**} \rightarrow \text{Sb}^{124}$ as E3 or M3,^{16, 17)} the spin of Sb^{124} must be three.

Except for the ft -values and energy dependences of $\alpha(W)$, $3(-) - 2(+)$ and $4(+)$ have much similarity. $\mathcal{M}(B_{ij}^3)$ corresponds to $\mathcal{M}(S_{ijk}^3)$ and they are indispensable to get a large angular correlation, and generally they must be taken much larger than the other RNME's. (Velocity type RNME's are compared dividing by $(uZ/2\rho)$. In the case of first forbidden transition, very large cancellation may occur, and in such cases other RNME's may be comparable with $\mathcal{M}(B_{ij}^3)$.) The $\log_{10} ft$ value of Sb^{121} is 10.5,¹⁾ which lies between first forbidden and second forbidden. Therefore, in the case of the first forbidden RNME's other than $\mathcal{M}(B_{ij}^3)$ must be very small, and in the case of the second forbidden $\mathcal{M}(S_{ijk}^3)$ must be extraordinary large. The smallness of the first forbidden RNME's are easily understood, according to the Mayer's shell model, which indicates that the i^2 -decay of Sb^{121} corresponds to the transition $h_{11/2} \rightarrow g_{7/2}$ of a nucleon. The first forbidden RNME's other than $\mathcal{M}(B_{ij}^3)$ vanish in the above model. On the contrary, the extraordinary largeness of $\mathcal{M}(S_{ijk}^3)$ is hard to understand. In this connection we should like to answer the criticism of Metzger¹⁰⁾ concerning the ft -value of Sb^{121} . In part I we excluded $4(+)$ and $2(+)$, and main reason was the small ft -value for $\mathcal{M}(S_{ijk}^3)$. Metzger insisted that the $\log_{10} ft$ value of Sb^{121} ($=10.5$) is not so different from that of Fe^{59} ($=10.9$) which belongs surely to second forbidden transition, and so our previous reason to exclude $4(+)$ and $2(+)$ is invalid. However, ft -values for the forbidden transitions should be compared in the corrected forms, e. g. the corrected ft -value for S_{ijk}^3 is $(2\pi^3/G^2) \log 2 / \sum |S_{ijk}^3|^2$. Only the corrected ft -values represent the magnitudes of RNME's. The corrected ft -value for S_{ijk}^3 of Sb^{121} must be of the order of 10^{10} ; on the other hand the corrected ft -values for the second forbidden coordinate type RNME's of Fe^{59} will be of the order of 10^{12} .²⁰⁾ There is a large difference between them. Therefore, we believe that our previous argument is not wrong.

In conclusion, it is much more likely that Sb^{121} has spin 3 and odd parity than the possibility that it has spin 4 and even parity. If we take the former assumption, we can explain the β -ray spectrum, β - γ angular correlation and other experimental data satisfactorily.

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- 19) Prof. R. E. Marshak, who came to Kyoto for the International Conference of Theoretical Physics in Sept. (1953), had the kindness to lend us the report of the Conference at Indiana University, Bloomington, in May (1953). This report contains the discussions of new experimental results of Sb¹²⁴ and Te¹²⁴ measured by Prof. F. R. Metzger, by Prof. L. M. Langer. and by Dr. Tomlinson, where some discrepancies are found. We take Langer's results here. Our conclusions do not change even if the other data are taken into account.
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Errata

On the β - γ Angular Correlation of Sb¹²⁴

M. Morita and M. Yamada, Prog. Theor. Phys. **8** (1952), 449.

In Figs. 4, 5 and 6 and Table II, the authors discarded the minus signs for all the numerical values of x and y . They should be read, for example, $x = -0.8$ and $y = -0.18$ instead of $x = 0.8$ and $y = 0.18$ in the case of Fig. 5. It is obvious that no essential changes are caused by these errors.

On the β -Ray Angular Correlations

M. Yamada and M. Morita, Prog. Theor. Phys. **8** (1952), 431.

	originally	should be read
Eq. (20),	$F_{12}^1(\theta) = -F_{12}^1(\theta),$	$F_{12}^1(\theta) = -F_{12}^{-1}(\theta).$
Eq. (28),	$g_{22} G_{11} = (1/6) (5d_{22} G_{11} - d_{22} D),$	$g_{22} G_{11} = (1/6) (5d_{22} G_{11} - d_{22} D_{11}).$
Eq. (29),	$(5/2) a^{(0)}_{22} a'^{(1)}_{22} \{ \},$	$(5/2) a^{(0)}_{22} a'^{(0)}_{22} \{ \},$
Tenth equation of eq. (30),	$d_{33} D_{22} = (1/5) (\quad),$	$d_{33} G_{22} = (1/5) (f_{33} G_{22} + 2f_{33} E_{22}).$

In Appendix, suffix 1 is missing in some of j_1 's. They must be replaced by j_1 's.

Convergence of Iterative Methods

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For various iterative methods in eigenvalue problem which are used to solve the bound states and scattering problems, we introduce the coefficient of convergency which indicates not only the convergency condition when the iteration is performed endlessly, but also the efficiency of successive approximation for any finite step. In this way the convergency properties of these iterative methods are investigated from a common stand-point, though the main new results stated in this paper concern the scattering problems. Comparing the convergence properties for Schwinger's iterative method and Born's successive approximation in scattering problem, we show the former is considerably superior in low energy domain, while in high energies the latter is more favourable by its simplicity. (Fig 2).

Then a more efficient iterative method (17), of which coefficient of convergency is expressed as (20), is presented, and its application to the variational technique is discussed. The method is shown to be convenient since no troublesome procedure solving the secular determinant is needed.

§ 1. Introduction

For the self-adjoint operators H_0 , H_1 , consider the following eigenvalue problem, (H_1 ; so to speak interaction Hamiltonian):

$$(H_0 - E)\psi = \lambda H_1 \psi,$$

which often appears in quantum mechanics. Recently, the iterative methods have been adopted to solve such problems, for example; as nucleon-nucleon scattering,^{1) 2)} states of deuteron³⁾ and lightest nuclei⁴⁾, estimation of errors in the Tomonaga's intermediate coupling theory,⁵⁾ etc. We shall discuss convergence-properties of these methods, for which few of the studies have been made theoretically for scattering problems in spite of the many arguments for bound state problems.⁶⁾

First, several classification seems to be necessary to show various iterative methods according to the ways the question is put forward.

(a). Let discrete eigenvalue E be known, and λ is to be found. (For example; deuteron problem). Consider

$$(H_0 - E)\psi_{n+1} = H_1 \psi_n.$$

Rewriting this into the following integral equation which contains some boundary conditions required

$$\psi_{n+1} = (H_0 - E)^{-1} H_1 \psi_n. \quad (1)$$

Since the normalization of wave functions has not essential meaning, we may drop λ which

is an unknown quantity. One may start with a suitable trial function ψ'_0 , and improve the wave function in usual iteration method.

If we find after n steps a sufficiently accurate ψ'_n , an approximate value of λ is obtained by calculating the following stationary quantity :

$$\lambda_{(n)} = (\psi_n (\mathbf{H}_0 - E) \psi'_n) / (\psi_n \mathbf{H}_1 \psi'_n).$$

(b). When λ is known, and discrete eigenvalue E is to be solved :

$$(\mathbf{H}_0 - \lambda \mathbf{H}_1) \psi_{n+1} = \psi_n, \quad (2)$$

$$E_{(n)} = (\psi_n (\mathbf{H}_0 - \lambda \mathbf{H}_1) \psi_n) / (\psi_n \psi_n).$$

(c). When continuous spectrum E is given, and the boundary condition at a large distance (for example ; phase shift) is also known. Then λ is found from

$$\psi_{n+1} = (\mathbf{H}_0 - E)^{-1} \mathbf{H}_1 \psi_n, \quad (3)$$

$$\lambda_{(n)} = (\psi_n (\mathbf{H}_0 - E) \psi_n) / (\psi_n \mathbf{H}_1 \psi_n).$$

(d). When continuous spectrum E and λ are known, and one wants to find the properties of ψ itself at a large distance (for example ; phase shift or scattering amplitude);*

$$(\mathbf{H}_0 - E) \psi_{n+1} = \mathbf{H}_1 \psi_n. \quad (4)$$

If one wants to calculate in practice, the iteration is performed by the integral equation as (1) or (3), therefore (2) or (4) should be rewritten in the integral equation, however, these integral equations are useless on account of presence of an unknown quantity in the kernel such as E or phase shift. In scattering problem, two methods which deal successfully with this difficulty are

(1). Born's iteration

$$\psi_{n+1} = f + \lambda \mathbf{G} \psi_n. \quad (5)$$

This is just the Born expansion when we take f as the initial trial function ψ'_0 , where f is a solution for $\lambda=0$. For example, in the problem of neutron-proton S-wave scattering²⁾

$$\mathbf{H}_0 - E = -d^2/dx^2 - k^2, \quad \mathbf{H}_1 = W(x).$$

The formula (5) therefore is written in the following form

$$\psi_{n+1}(x) = \sin(kx) + \lambda \int_0^\infty G(x, x') W(x') \psi_n(x') dx' \quad (5 \cdot a)$$

where

$$G(x, x') = k^{-1} \sin(kx_<) \cos(kx_>).$$

(2). Schwinger's iteration

$$\psi_{n+1} = (\mathbf{H}_0 - E)^{-1} \mathbf{H}_1 \psi_n$$

*) Iteration (4) is the approximate method which is suitable for small λ . There are many iterative formulae which seem to be suitable for various cases such as : the strong coupling case, high or low energy case, etc. However, these iterations are not applicable. The reason will be made clear in later arguments.

where $(\mathbf{H}_0 - E)^{-1}_n$ means that the unknown quantity in the kernel is replaced by the approximate quantity obtained from the Schwinger's variational method using ψ_n . If we take the same example in Born's iteration, the above formula is^{2)*}

$$\psi_{n+1}(x) = \int_0^\infty G_n^*(x, x') H(x') \psi_n(x') dx',$$

where

$$G_n^*(x, x') = (k \cot \delta_n) (\sin(kx)/k) (\sin(kx')/k) + G(x, x')$$

$$k \cot \delta_n = \frac{\int_0^\infty W(x) \psi_n^2(x) dx - \int_0^\infty dx \int_0^\infty dx' W(x) \psi_n(x) G(x, x') W(x') \psi_n(x')}{[k^{-1} \int_0^\infty W(x) \psi_n(x) \sin(kx) dx]^2}.$$

δ_p may of course be found from the other suitable formula.

Here we shall give brief explanations of necessity for the variational principle. Consider the following eigenvalue equation;

$$\mathbf{A}\psi = \lambda \mathbf{B}\psi, \quad (6)$$

where \mathbf{A} and \mathbf{B} are self-adjoint operators. Putting $c\phi = \psi$, (c ; c-number) we get

$$\lambda = (\phi \mathbf{A} \psi) / (\phi \mathbf{B} \psi).$$

Considering any (subject however to suitable boundary conditions) infinitesimal variations of ψ and ϕ independently,

$$\delta\{(\phi \mathbf{A} \psi) / (\phi \mathbf{B} \psi)\} = 0, \quad (6a) \quad \text{or} \quad \delta(\phi(\mathbf{A} - \lambda \mathbf{B})\psi) = 0, \quad (6b)$$

are equivalent to the equation (6). These are the well known variational principles. Schwinger's variational principle is also nothing but (6, a), though it looks different apparently.

Now let us estimate the error of λ which is due to any finite (not infinitesimal) variations of ψ and ϕ . For this purpose, defining

$$u = \psi + \Delta u, \quad v = \phi + \Delta v,$$

$$\lambda' = (v \mathbf{A} u) / (v \mathbf{B} u),$$

where $c\phi = \psi$, λ are the correct eigenfunction and the correct eigenvalue. Then it follows that

$$\lambda' = \lambda \{1 + (\Delta v (\mathbf{A} - \lambda \mathbf{B}) \Delta u) / (\phi \mathbf{A} \psi) + O(\Delta v \Delta u)^2\}.$$

If we take ψ_p, ψ_q , which are calculated by iteration (7), as u and v respectively, the following quantity $\lambda_{((p+q)/2)}$ will be defined,

$$\lambda_{((p+q)/2)} = (\psi_q \mathbf{A} \psi_p) / (\psi_q \mathbf{B} \psi_p),$$

since $\lambda_{((p+q)/2)} = \lambda_{((r+s)/2)}$ holds when $p+q=r+s$ is satisfied, and its error is accordingly

$$\Delta \lambda = \lambda_{((p+q)/2)} - \lambda = (\Delta \psi_q (\mathbf{A} - \lambda \mathbf{B}) \Delta \psi_p) / (\psi_q \mathbf{B} \psi_p) + O(\Delta \psi_p \Delta \psi_q)^2.$$

These are the errors of $\lambda_{((p+q)/2)}$ calculated from the iterative methods.

Next we shall consider variations of ψ and ϕ with finite freedom (not "any") as the practical calculations. Let a trial function ψ be expressed as follows;

$$\psi = \sum_{i=1}^n c_i f_i, \quad \delta \psi = \sum_{i=1}^n f_i \delta c_i,$$

where f_i are suitable functions which are restricted by the boundary conditions required. The equation (6.b) becomes

$$\sum_{j=1}^n (a_{ij} - \lambda_i b_{ij}) c_j = 0, \quad (i=1, \dots, n),$$

where

$$a_{ij} = (f_i \mathbf{A} f_j), \quad b_{ij} = (f_i \mathbf{B} f_j), \quad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji}.$$

The approximate λ_i is determined from the condition that n simultaneous homogeneity equations are compatible

*) For the three dimensional Schwinger iteration, see W. Kohn, Phys. Rev. **74** (1948), 1763.

with each other. Thus c_j can be solved. Multiplying $\sum_{i=1}^n c_i$ from left side, we get

$$\lambda_t = \sum_{ij} a_{ij} c_i c_j / \sum_{ij} b_{ij} c_i c_j = (\psi_t \mathbf{A} \psi_t) / (\psi_t \mathbf{B} \psi_t).$$

Therefore λ_t which is determined by the compatibility condition is stationary with respect to errors of ψ_t .

Let us consider following variational method; namely, putting g instead of ϕ in (6, b), we require the stationary property for only g ;

$$\delta(g(\mathbf{A} - \lambda_t' \mathbf{B})\psi_t) = (\delta g, (\mathbf{A} - \lambda_t' \mathbf{B})\psi_t) = 0, \quad \delta g = \sum_i g_i \delta d_i,$$

where δg need "not" always satisfy any boundary conditions. This method leads to the following simultaneous equations ;

$$\begin{aligned} \sum (c_{ij} - \lambda_t' b_{ij}) c_j &= 0, \\ c_{ij} &= (g_i \mathbf{A} f_j), \quad b_{ij} = (g_i \mathbf{B} f_j). \end{aligned}$$

Solving λ_t' from the compatibility condition, we get

$$\lambda_t' = (g_t \mathbf{A} \psi_t) / (g_t \mathbf{B} \psi_t)$$

where g_t which has to satisfy the boundary conditions required are constructed from s_i properly as follows ;

$$g_t = \sum_i d_i g_i,$$

however d_i are of course not determined uniquely. Thus, λ_t' found in this way contains a error which is order of the product of errors in ψ_t and in g_t . For this reason, the more accurate λ_t should be recalculated only by ψ_t , though the latter method is convenient for calculations of c_i by its simplicity. The similar method is also applicable for the approximate computation of initial-value-problem such as the equation (23) in the present paper.

Above discussions are independent of properties of \mathbf{A} , \mathbf{B} and the eigenfunction which we notice. If the eigenvalues are all positive, the following relation holds for the smallest eigenvalue. (For example ; for the ground state in (a) when \mathbf{H}_1 is positive definite).

$$\cdots \lambda_{(n)} > \lambda_{(n+1/2)} > \lambda_{(n+1)} \geq \cdots$$

for any n .

Thus we have seen that there are many iterative methods. In sec. 2, introducing the coefficient of convergency (=convergency coefficient) μ ; (12), it is derived that the necessary and sufficient condition of convergency in iteration methods is $\mu > 1$, and this condition is applied to physical problems. In case of (a), if \mathbf{H}_1 is positive definite, the above condition can be satisfied only for ground state, and in the case of (b), the procedure does not converge in general. Application of this condition to the scattering problems (c) and (d) are performed in sec. 3 and sec. 4. In sec. 3, the region of convergency of iteration (3) in (c) is shown. (Fig 1). Sec. 4 is concerned with (d). It is proved that in the region of divergence of (c), Schwinger's method also diverges, though in this region Born's iteration diverges by a more severe reason as discussed later. Next we show the convergency coefficient (μ_R) which shows definitely the efficiency of Born approximation in any step can be defined in the same way as in sec. 2, and comparing the region of convergency for Schwinger's iteration and Born expansion, we show the former is considerably superior in low energy region, however in high energy region the latter is favourable by its simplicity for calculation. (Fig 2). In sec. 5, it is shown that the iterative method of (17) has the great efficiency which may be seen from (21). In sec. 6, it is discussed that the formula (17) is convenient to apply the variational treatment due to the fact that usual treatment (22) contains the cumbersome process of solving the secular determinant, while

(23) method does not, so (23) is simpler for calculation. In appendix, several practical remarks are stated to make use of the method of sec. 6.

§ 2. Convergency of iterative methods

As we have seen in sec. 1, the equation we consider can be written in the following homogeneous form :

$$A\psi = \lambda B\psi, \quad (6)$$

with self-adjoint operators A and B , and the eigenvalue λ to be determined. The Born iteration is an exception for its inhomogeneous character, the arguments in connection with (5) will therefore be given in sec. 4, together with the arguments of Schwinger iteration which has slight differences with (6). To solve this equation by iteration, one starts with a suitable trial function ψ_0 , which satisfies given boundary conditions, and solve in turn

$$\left. \begin{aligned} A\psi_{n+1} &= B\psi_n, \\ \psi_{n+1} &= A^{-1}B\psi_n, \end{aligned} \right\} \quad (7)$$

where n and $n+1$ express the order of approximation of ψ .

We shall give two remarks with respect to "convergence of iteration" for the sake of definiteness.

(1). If and only if iteration (7) using any trial function converges to $\psi^{(0)}$ which is just one we want to obtain, (7) will be called "convergent". The other any cases are called divergent. Even if ψ_n approaches some definite function, in the case that this function is not one we want to find, we will therefore call it divergent.

(2). Even if the normalization makes ψ_n infinitely large or infinitesimally small when n is made larger and larger, it will be called "convergent", if the shape of the function itself approaches $\psi^{(0)}$ to arbitrary extent.

Solving the equation (6), eigenfunctions and eigenvalues are obtained as follows ; (Discrete eigenvalues are assumed, but generalization to continuous eigenvalue is easy.)

$$\begin{aligned} A\psi^{(i)} &= \lambda^{(i)} B\psi^{(i)}, \\ \psi^{(i)} &= \lambda^{(i)} A^{-1} B\psi^{(i)}. \end{aligned} \quad (i = \dots -1, 0, 1, 2, \dots)$$

These $\psi^{(i)}$ constitute a complete set and the superfix i indicates the number of the base of this set. As the solution we want to obtain is one of these $\psi^{(i)}$, $\lambda^{(i)}$, these will be written $\psi^{(0)}$, $\lambda^{(0)}$. Expanding ψ_n in terms of $\psi^{(i)*}$

$$\begin{aligned} \psi_0 &= \sum_i a_0^{(i)} \psi^{(i)}, \\ \psi_n &= \sum_i a_n^{(i)} \psi^{(i)} = \sum_i a_0^{(i)} / (\lambda^{(i)})^n \cdot \psi^{(i)} \end{aligned} \quad (8)$$

and defining "Norm" $|\psi^{(i)}|$ as*

*) This is in the sense that

$$(gB\psi_n) = \sum_i a_n^{(i)} (gB\psi^{(i)})$$

where g is any function having the same boundary conditions as ψ_n and $\psi^{(i)}$. The same is true for sec. 4.

$$|\psi^{(i)}| = \sqrt{|(\psi^{(i)} \mathbf{B} \psi^{(i)})|}, \quad (9)$$

normalization of $\psi^{(i)}$ is

$$|\psi^{(i)}| = 1 \quad (10)$$

and that of ψ_n, ψ_{n+1} are chosen as follows ;

$$|(\psi_n \mathbf{B} \psi^{(0)})| = |(\psi_{n+1} \mathbf{B} \psi^{(0)})| = 1 \quad (11)$$

being multiplied by suitable factors after the iteration (7). If we put

$$\Delta \psi_n = \psi_n - \psi^{(0)}, \quad \Delta \psi_{n+1} = \psi_{n+1} - \psi^{(0)},$$

(11) means

$$|\psi_n| = 1 + O(\Delta \psi_n)^2, \quad |\psi_{n+1}| = 1 + O(\Delta \psi_{n+1})^2,$$

and we get from (8) ~ (11)

$$|\Delta \psi_n| / |\Delta \psi_{n+1}| \geq |\lambda^{(i)}|_{\min} / |\lambda^{(0)}| = \mu, \quad (n=0, 1, 2, \dots), \quad (12)$$

where $|\lambda^{(i)}|_{\min}$ is the minimum value of $|\lambda^{(i)}|$ excepting $i=0$. Since $\Delta \psi_n$ and $\Delta \psi_{n+1}$ will be considered as errors of ψ_n and ψ_{n+1} , the formula (12) expresses the efficiency of iteration (7) performed once. We shall call μ is "convergent coefficient" or "coefficient of convergency". Since the equal sign in (12) is always necessary, the necessary and sufficient condition under which iteration (7) converges for any trial function with $(\psi_0 \mathbf{B} \psi^{(0)}) \neq 0$ is $\mu > 1$.

Now, we will proceed to consider this condition for the physical problems as follows :

(a). When binding energy E is known and interaction strength λ is to be solved ; in this case, if \mathbf{H}_1 is positive definite, it certainly converges for the ground state, and diverges for the excited state. Because the eigenvalues $\lambda^{(i)}$ are always positive since there is the boundary condition that ψ should vanish at large distance, and $\lambda^{(0)}$ which corresponds to the ground state is necessarily smaller than any other $\lambda^{(i)}$. (The ground states of the deuteron and the light nuclei were solved actually in this way. See the examples in the present paragraph.)

(b). When λ is known and binding energy $E (E < 0)$ is to be solved ; in this case, as there exist continuous eigenvalues in $E > 0$, therefore the discussions stated above must be modified in a simple manner as follows ;

$$\mathbf{A} \psi^{(i)} = E^{(i)} \mathbf{B} \psi^{(i)} \text{ and } \mathbf{A} \psi(E) = E \mathbf{B} \psi(E),$$

$$\psi_0 = \sum a_0^{(i)} \psi^{(i)} + \int a_0(E) \psi(E) dE,$$

**) Norm of arbitrary function ψ will be expressed by expanding ψ into $\psi^{(i)}$ as

$$|\psi| = \sqrt{\sum \frac{|(\psi^{(i)} \mathbf{B} \psi)|^2}{|(\psi^{(i)} \mathbf{B} \psi^{(i)})|}}.$$

The same is true for sec. 4.

$$\psi_n = \sum \frac{\alpha_0^{(i)}}{(E^{(i)})^n} \psi^{(i)} + \int \frac{a_0(E)}{E^n} \psi(E) dE.$$

Since continuous eigenvalues E spread usually from zero to ∞ , it follows that

$$|\lambda^{(i)}|_{M_{in}} = 0.$$

Thus it always diverges except for a special trial function ψ_0 .

(c). Though it is a scattering problem, the way the question is presented is unusual. The interaction strength $\lambda (= \lambda^{(0)})$ is to be determined from the phase shift. In this case, there are both domain of convergence and divergence, but generally speaking, as being presumable from (12), it converges when interaction strength is weak in a suitable way. (c. f. sec. 3. Fig 1.)

(d). It is the case E and λ are given and phase shift or scattering amplitude are to be solved. In this connection, region of convergency of Schwinger's method and of Born's method are investigated in sec. 4. (c. f. Fig 2.)

Let good ψ_n be obtained by (7), $\lambda_{(n)}$ may be calculated as stated in sec. 1.

Example

A neutron and a proton interact through the potential of square well type or Hulthén's type with the binding energy γ^2 . Let iteration (7) be applied to the ground state.

$$A = -d^2/dx^2 + \gamma^2, \quad \gamma > 0,$$

$$B = \begin{cases} 1 : x \leq 1 \\ 0 : x > 1 \end{cases} ; \text{ for square well,}$$

$$B = e^{-x}/(1 - e^{-x}) ; \text{ for Hulthén well.}$$

Boundary conditions :

$$\psi^{(i)}(0) = \psi^{(i)}(\infty) = 0. \quad (i=0, 1, 2, \dots).$$

Square well : eigenvalues

$$\lambda^{(i)} = (1/2 + i)^2 \pi^2 + 2\gamma + \{1 - 1/(1/2 + i)^2 \pi^2\} \gamma^2 + \dots,$$

coefficient of convergency

$$\mu = 9(1 - 64/9\pi^2 \cdot \gamma + \dots).$$

Hulthén well :

$$\lambda^{(i)} = (i+1)(i+1+2\gamma),$$

$$\psi^{(i)} = \xi \cdot (1-\xi)^{-2\gamma} \cdot d^{i+1}/dx^{i+1} [\xi^i \cdot (1-\xi)^{i+1+2\gamma}], \quad \xi = 1 - e^{-x},$$

$$\mu = (4+4\gamma)/(1+2\gamma) = 4(1-\gamma+\dots).$$

In these typical examples, when γ is small μ is so large that the convergency can be rapid. However if any tensor force potential is added to central force, μ may decrease and can be possibly order of one.³⁾

§ 3. Region of convergency in scattering problem

In this paragraph, we will take up the problem assigned to (c) class. Theoretically, as it needs only to apply the theorem in sec. 2, we had better show directly the region of convergence for a typical example.

Let us suppose neutron-proton scattering by a square well potential. Notations are the same in examples of sec. 2, except that γ^2 is replaced by $-k^2$. The correct S -wave phase shift δ is related as a function of relative wave number k and the potential strength

λ in the following way :

$$\sqrt{k^2 + \lambda} \cot \sqrt{k^2 + \lambda} = k \cot (k + \delta), \quad k^2 + \lambda > 0,$$

$$\sqrt{-k^2 - \lambda} \coth \sqrt{-k^2 - \lambda} = k \cot (k + \delta), \quad k^2 + \lambda < 0.$$

Having been given δ in this way, there are series of discrete eigenvalues $\lambda^{(i)}$ and eigenfunctions $\phi^{(i)}$ corresponding with $\lambda^{(i)}$ which are restricted by the following boundary conditions

$$\begin{aligned} \phi^{(i)}(0) &= 0, \\ \phi^{(i)} &\rightarrow \sin(kx + i\pi + \delta), \quad x \rightarrow \infty, \\ \dots \lambda^{(-1)} &< \lambda^{(0)} < \lambda^{(1)} < \lambda^{(2)} \dots \end{aligned} \quad (i = \dots -1, 0, 1, 2, \dots)$$

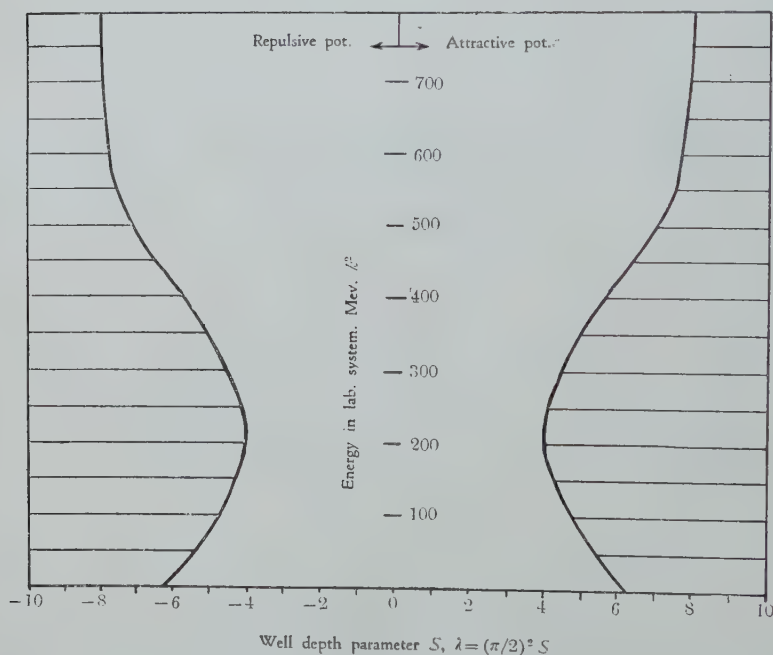


Fig 1. Region of convergence \square and divergence \square (horizontal lines) by iteration (c) for S-wave n - p scattering. The range of square well is 2.2×10^{-13} cm.

These $\phi^{(i)}$ constitute a complete set in the sense noticed in the foot note of sec. 2. The convergency condition which have been studied in sec. 2 is

$$\mu > 1,$$

or

$$|\lambda^{(i)}|/|\lambda^{(0)}| > 1, \quad \text{for any } i \ (i \neq 0).$$

Therefore the region of convergency is obtained as the function of k and λ . The results

are summarized in Fig 1.

For actual nuclear forces, λ are about 2.3 for spin singlet state and 3.5 for spin triplet state, (namely S are about 0.9 and 1.4), so that the convergency is good enough. Generally, for the potential having well-depth parameter²⁾ S , the coefficient of convergency in vanishing energy is about (if one neglects a quantity of order 0.001, the following relations are correct),

$$\mu \geq 25/|4S|, |S| \leq 25/4; \quad \mu < 25/|4S|, |S| > 25/4.$$

Above discussions are concerned with S-wave. For higher waves the region of convergency will be much increased. For example, at zero energy the bounds of convergence for S, P and D-waves are $S=4\lambda/\pi^2=6.25, =11.25, =17.30$ respectively.

§ 4. Schwinger's iteration method and Born expansion

Now let us proceed to study the problems in (d). First we have to consider Schwinger's iterative method:

$$\psi_{n+1} = A_n^{-1} B \psi_n,$$

where $A_n^{-1} B$ stands for that, for example, the phase shift δ contained in $A^{-1} B$ is replaced by the approximate δ_n which is obtained from Schwinger's stationary expression²⁾ of $k \cot \delta$ using ψ_n . We shall postulate, namely "In divergent region (excepting border lines) of $\psi_{n+1} = A^{-1} B \psi_n$ which is already discussed in sec. 3, Schwinger's iteration $\psi_{n+1} = A_n^{-1} B \psi_n$ also diverges".

(Proof)

Eigenvalues and eigenfunctions are

$$\psi^{(i)} = \lambda^{(i)} A^{-1} B \psi^{(i)}, \quad (i = \dots -1, 0, 1, 2, \dots), \quad (13)$$

$$\psi_n^{(i)} = \lambda_n^{(i)} A_n^{-1} B \psi_n^{(i)}, \quad (n=0, 1, 2, \dots), \quad (14)$$

where n indicates the order of approximation in Schwinger's iteration and i is the number of the bases in respective complete sets. Furthermore let the same number i of different sets be chosen correspondingly. Assuming that iteration

$$\psi_{n+1} = A^{-1} B \psi_n \quad (15)$$

diverges and moreover

$$\psi_{n+1} = A_n^{-1} B \psi_n \quad (16)$$

converges. We expand ψ_n in terms of $\psi_n^{(i)}$ and $\psi^{(i)}$

$$\psi_n = \sum_i a_n^{(i)} \psi_n^{(i)} = \sum_i a_n^{(i)'} \psi^{(i)}.$$

Iterating once by (16), we obtain

$$\psi_{n+1} = \sum_i a_n^{(i)} / \lambda_n^{(i)} \cdot \psi^{(i)} = \sum_i a_n^{(i)''} / \lambda^{(i)} \cdot \psi^{(i)}.$$

As the trial functions are arbitrary, it may be possible that $a_n^{(i)}$, $a_n^{(i)'}$, $a_n^{(i)''}$ are in general not zero. Since (16) method converges, $A_n^{-1} B$ approaches to correct $A^{-1} B$, namely $\psi_n^{(i)}$ converges to $\psi^{(i)}$ and $\lambda_n^{(i)}$ to $\lambda^{(i)}$, then there exist j for any m which are larger than j for which the following relations hold for any positive ϵ ,

$$1 + \epsilon > |a_m^{(i)''} / a_m^{(i)'}| > 1 / (1 + \epsilon).$$

However since (15) diverges, there are $j(j \neq 0)$ and Δ such as

$$|\lambda^{(0)}|/|\lambda^{(j)}| > 1 + \Delta, \quad \Delta > 0.$$

Putting

$$\begin{aligned} \varepsilon &= (1 + \Delta)^{1/3} - 1 > 0, \\ |a_{m+1}^{(j)'}|/|a_{m+1}^{(0)'}| &= |\lambda^{(0)}|/|\lambda^{(j)}| \cdot |\alpha_m^{(j)'}|/|\alpha_m^{(0)'}| > |\lambda^{(0)}|/|\lambda^{(j)}| \cdot |\alpha_m^{(j)'}|/|\alpha_m^{(0)'}| \cdot 1/(1 + \varepsilon)^2 > \\ &> |\alpha_m^{(j)'}|/|\alpha_m^{(0)'}| \cdot (1 + \Delta)/(1 + \varepsilon)^2 = |\alpha_m^{(j)'}|/|\alpha_m^{(0)'}| \cdot (1 + \varepsilon). \end{aligned}$$

As these relations hold for any m larger than j ,

$$|a_{m+n}^{(j)'}|/|a_{m+n}^{(0)'}| > (1 + \varepsilon)^n |\alpha_m^{(j)'}|/|\alpha_m^{(0)'}|.$$

Therefore

$$|\alpha_n^{(j)'}|/|\alpha_n^{(0)'}| \rightarrow \infty, \quad n \rightarrow \infty.$$

This contradicts the assumption that (16) converges.

Speaking of the example in sec. 3, above proof guarantees that Schwinger's iterative method diverges in the region of horizontal lines in Fig. 1. However it is difficult to prove definitely that Schwinger's iteration $\psi_{n-1} = A_n^{-1} B \psi_n$ is always convergent in convergent region of $\psi_{n+1} = A^{-1} B \psi_n$, though we can exemplify the convergency as example 2 of this paragraph. One may suppose however that the convergent regions of the two methods are equal, provided that the trial ψ_0 is chosen properly. This prediction is corroborated from the next discussions.

Let us consider the method that the phase shift δ_n in $A_n B$ is recalculated occasionally (not each time) by ψ_n which is less efficient than proper Schwinger's iteration. If the interval of recalculating the phase shift are large enough and B is positive definite, it can be proved that such Schwinger's method converges in the convergent region of $\psi_{n+1} = A^{-1} B \psi_n$ in sec. 3, provided the trial ψ_0 is f which is a solution for $\lambda=0$. The points of this proof are as follows:

Let $\lambda^{(0)} = \lambda$ be positive. Writing $A^{-1} B$ which is replaced by δ_m in place of correct δ as $A_m^{-1} B$, it follows that

$$\begin{aligned} \psi_m^{(i)} &= \lambda_m^{(i)} A_m^{-1} B \psi_m^{(i)}, \\ \psi_m^{(i)} &\rightarrow \cos kx + \cot \delta_m \sin kx, \quad (x \rightarrow \infty), \\ k \cot \delta_{m+1} &= k \cot \delta_m - (\lambda^{(0)} - \lambda_m^{(0)}) (\psi_m B \psi_m) \cdot \lambda_m^{(0)} / \lambda^{(0)}. \end{aligned}$$

Due to the property of Schwinger's variational method,^{7a)} as the relation

$$\lambda^{(0)} - \lambda_m^{(0)} > 0, \quad \text{for any } m$$

holds, if $\lambda_m^{(0)} > 0$ is satisfied for any m which is satisfied for $\psi_0 = f$, it follows that

$$k \cot \delta_{m+1} < k \cot \delta_m, \quad \text{for any } m.$$

Also from the properties of Schwinger's variational method, for the correct phase δ

$$k \cot \delta \leq k \cot \delta_m, \quad \text{for any } m.$$

If $k \cot \delta < k \cot \delta_\infty$ holds, as there exist Δ such as

$$\lambda^{(0)} > \lambda_m^{(0)} + \Delta, \quad \Delta > 0, \quad \text{for any } m$$

it follows that

$$k \cot \delta_\infty = -\infty.$$

This is self-contradiction, hence

$$k \cot \delta = k \cot \delta_\infty.$$

T. Kato^{7b)} showed that Schwinger's method could be improved and such modified method converged provided $|\delta| < \pi$.

Next we will take up the properties of Born expansion which is a perturbation method in scattering problem. The condition of convergency on this method when the iteration is performed infinitely was given by R. Jost and A. Pais^{(8)-(10)*} The equation of Born iteration is

$$\psi_{n+1} = \lambda A^{-1} B \psi_n = f + \lambda G \psi_n. \quad (5)$$

We assume that a trial ψ_0 and the correct $\psi^{(0)}$ can be expanded by the eigenfunctions of G as follows :

$$\begin{aligned} \varphi^{(i)} &= \nu^{(i)} G \varphi^{(i)} \text{ or } \varphi^{(i)} = \nu^{(i)m} G^m \varphi^{(i)}, \\ \psi_0 &= \sum a^{(i)} \varphi^{(i)} \text{ and } \psi^{(0)} = \sum b^{(i)} \varphi^{(i)}. \end{aligned}$$

Also from (5)

$$\begin{aligned} \psi^{(0)} &= f + \lambda G f + \dots + \lambda^{n-1} G^{n-1} f + \lambda^n G^n \psi^{(0)}, \\ \psi_n &= f + \lambda G f + \dots + \lambda^{n-1} G^{n-1} f + \lambda^n G^n \psi_0. \end{aligned}$$

Errors of ψ_n are

$$\delta\varphi_n = \psi_n - \psi^{(0)} = \lambda^n G^n (\psi_0 - \psi^{(0)}) = \sum_i (\lambda/\nu^{(i)})^n (a^{(i)} - b^{(i)}) \varphi^{(i)}.$$

Defining the "Norm"

$$\|\varphi^{(i)}\| = \sqrt{(\varphi^{(i)} B \varphi^{(i)})}$$

it follows that

$$\|\delta\varphi_n\|/\|\delta\varphi_{n+1}\| \geq |\nu^{(i)}|_{\min}/|\lambda| = \mu_B, \quad (n=0, 1, \dots)$$

where $|\nu^{(i)}|_{\min}$ means the minimum value of $|\nu^{(i)}|$. This μ_B indicates the efficiency of the Born approximation and is expressed in the same form as (12), which is a remarkable result because this expresses the relations of accuracy in each step for any "finite" n . Therefore μ_B can be called "coefficient of convergency." The necessary and sufficient condition of convergency of Born iteration for any trial function, as in sec. 2, is $\mu_B > 1$.

If we take f as trial ψ_0 (namely, usual Born approximation), it is possible to prove the following relation :

$$\begin{aligned} \delta\varphi_0 &= f - \psi^{(0)}, \\ \|\delta\varphi_0\|/\|\psi^{(0)}\| &\leq 1/\mu_B, \end{aligned}$$

hence

$$\|\delta\varphi_n\|/\|\psi^{(0)}\| \leq 1/(\mu_B)^{n+1}.$$

*) Convergence properties of the perturbation methods concerning discrete eigenvalues had been discussed in detail by T. Kato. (c. f. reference (10)).

However, since the purpose of this paper is a unified treatment of various iterative methods and comparison of them, the proof will be given elsewhere with more detailed study on the lowest Born approximation.

When B is a potential with well depth parameter^{2)*} S , as $|\nu^{(i)}|_{Min}$ at zero energy corresponds with $S=1$, the convergent coefficient for zero energy is

$$\mu_B = 1/S.$$

If we decompose the system into partial waves, for such one dimensional iteration, as (5·a) $\mu_B=1$ corresponds $|\delta|=\pi/2$. (δ : phase shift).

We now arrive at the problem of the comparison between Schwinger's iteration and Born iteration. Confining the subsequent discussions only for the one dimensional iteration, first we shall prove the following postulate. "Convergent region of iteration $\psi_{n+1}=A^{-1}B\psi_n$ (15) or (3) discussed in sec. 3 is in general more extensive than that of Born iteration for any energies, if B is positive definite." As convergent region of $\psi_{n+1}=A^{-1}B\psi_n$ in sec. 3 is inferred to agree with that of Schwinger's iteration, this result is of use for understanding the convergence properties of Schwinger iteration and Born iteration.

(Proof)

Since B is positive definite, there are $\lambda^{(i)}$ series as sec. 3,

$$\dots \lambda^{(-1)} < \lambda^{(0)} < \lambda^{(1)} < \lambda^{(2)} \dots$$

The condition that ψ_n converges to $\psi^{(0)}$ corresponding with $\lambda^{(0)}$ is

$$|\lambda^{(0)}|/|\lambda^{(i)}|_{Min} < 1, \quad (i \neq 0)$$

hence, at boundary of convergency, the following relation holds,

$$|\lambda^{(0)}| = |\lambda^{(i)}|_{Min} = \lambda_s, \quad (i \neq 0)$$

where λ_s is the upper bound of convergency for Schwinger's iteration. Now if $\lambda^{(0)} > 0$ ($\lambda^{(0)} < 0$), it follows that

$$\lambda_s = -\lambda^{(-1)} = \lambda^{(0)}, \quad (A)$$

$$(\lambda_s = \lambda^{(1)} = -\lambda^{(0)}, \quad (B)).$$

While in Born iteration, there exist $\nu^{(i)}$ series which correspond with $\delta = (2i+1)\pi/2$. From them choosing $\nu^{(+)}$ and $\nu^{(-)}$ which correspond with $\delta = \pi/2$ and $\delta = -\pi/2$ respectively, there are two cases as follows:

$$\lambda_B = \nu^{(+)}, \quad |\nu^{(i)}|_{Min} = \nu^{(+)}, \quad (a)$$

$$\lambda_B = |\nu^{(-)}|, \quad |\nu^{(i)}|_{Min} = |\nu^{(-)}|, \quad (b)$$

First we shall consider the combination (A) and (a). If

$$\nu^{(+)} \leq \lambda^{(0)}, \quad (\text{namely } \pi/2 \leq \delta)$$

it follows that

$$\dots < \nu^{(-)} \leq \lambda^{(-1)} < 0 < \nu^{(+)} \leq \lambda^{(0)} < \dots,$$

$$|\nu^{(-)}| \geq |\lambda^{(-1)}| = \lambda_s = \lambda^{(0)} \geq \nu^{(+)} = \lambda_B.$$

If

$$\nu^{(+)} > \lambda^{(0)}, \quad (\text{namely } \delta < \pi/2)$$

it follows that

*) When we consider the system into the partial waves separately the definition is such that, for any given angular momentum the depth of the potential which have just one bound state of zero binding energy is expressed as $S=1$.

$$\dots < \lambda^{(-1)} < \nu^{(-)} < 0 < \lambda^{(0)} < \nu^{(+)} < \dots,$$

$$|\nu^{(-)}| < |\lambda^{(-1)}| = \lambda_s = \lambda^{(0)} < \nu^{(+)}.$$

As this contradicts the assumption (a), it is not the case. Therefore for (A) and (a),

$$|\nu^{(-)}| \geq \lambda_s \geq \nu^{(+)} = \lambda_n.$$

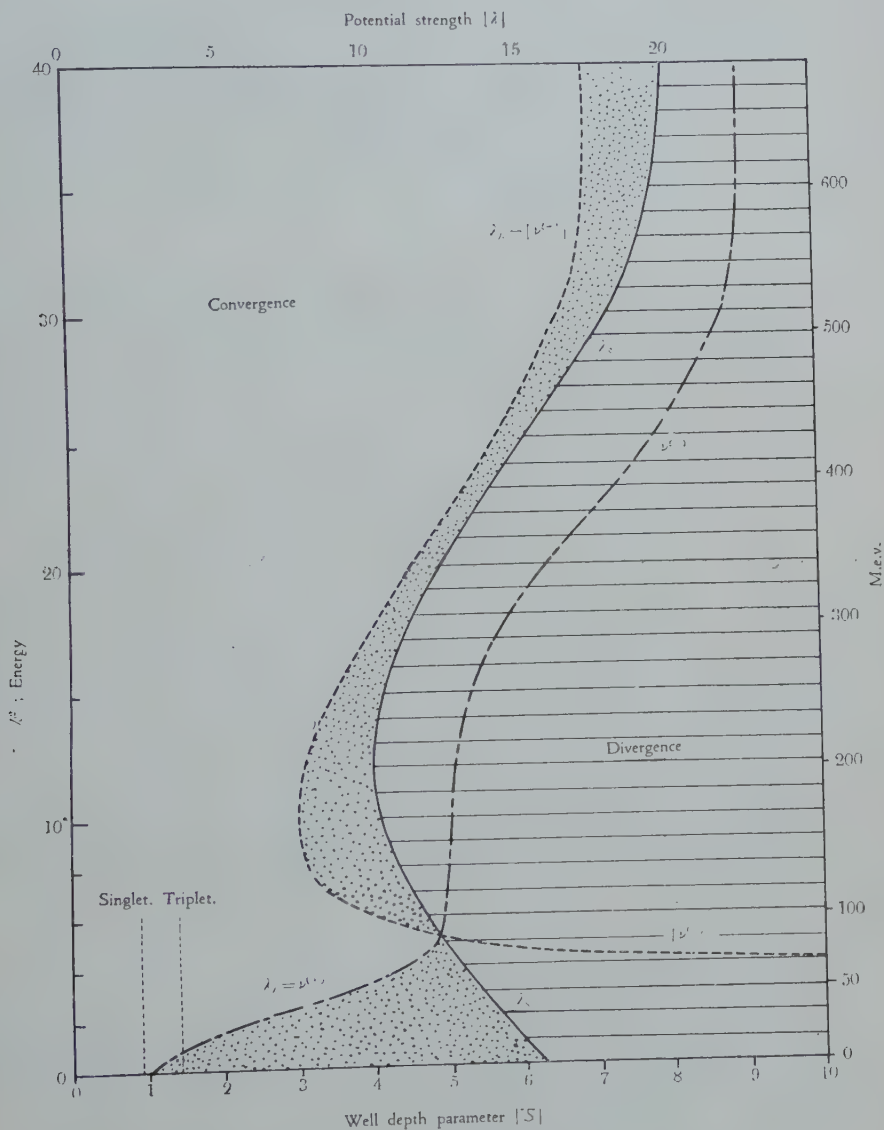




Fig 2.  Region of divergency for only Born expansion.
 Region of divergency for both Schwinger's method and Born expansion for neutron-proton
 S-wave scattering by square well potential. The range of square well is 2.2×10^{-13} cm.

From the combination (A) and (B)

$$\nu^{(+)} \geq \lambda_S \geq |\nu^{(-)}| = \lambda_B$$

holds in the same manner as above. Writing into a unified form

$$|\nu^{(\pm)}|_{>} \geq \lambda_S \geq \lambda_B = |\nu^{(\pm)}|_{<},$$

where $|\nu^{(\pm)}|_{>}$ stands for the larger one of $|\nu^{(-)}|$, $\nu^{(+)}$, and $|\nu^{(\pm)}|_{<}$ stands for the smaller one of $|\nu^{(-)}|$, $\nu^{(+)}$. The same is true for (B). The essence of this proof is that the phase shift in the kernel for Born iteration is restricted by $\cot \delta = 0$, while the one for Schwinger's is free from such restriction.

In the course of above proof, an upper bound for λ_S is found. It has been also confirmed that the convergence region of Schwinger's iteration is larger than that of Born iteration for all the energy range. Then we shall explain these situations by a few examples.

Example 1.

The same example in sec. 3, namely neutron-proton S-wave scattering by a square well potential. Region of divergency on two methods are showed in Fig. 2.

The boundary lines of divergent region run through the following points.

For Born iteration :

$$\begin{aligned} k^2 &= f^2 \pi^2 / 4, \quad \lambda_B = (2f - 1) \pi^2 / 4, \quad f \geq 2, \\ f &: \text{odd}, \quad d(k^2) / d\lambda_B = \infty, \quad f : \text{even}, \quad d(k^2) / d\lambda_B = f^2 / (2f - 1), \\ \text{two exceptions; } &k^2 = 0, \quad \lambda_B = \pi^2 / 4, \quad d(k^2) / d\lambda_B = 1, \\ &k^2 = \pi^2 / 4, \quad \lambda_B = 3\pi^2 / 4, \quad d(k^2) / d\lambda_B = 1/3; \end{aligned}$$

For Schwinger iteration :

$$\begin{aligned} k^2 &= (f^2 + 1) \pi^2 / 4, \quad \lambda_S = f^2 \pi^2 / 2, \quad f \geq 2, \\ f &: \text{odd}, \quad d(k^2) / d\lambda_S = (f^2 + 1) / 2f, \quad f : \text{even}, \quad d(k^2) / d\lambda_S = \infty. \end{aligned}$$

As can be seen from Fig. 2, Schwinger's method is much favourable in low energy region. For example, for zero energy, the coefficients of convergency μ of two methods are about

$$\mu_S \geq 25\mu_B / 4, \quad |S| < 25/4.$$

Since the nuclear force correspond to $\lambda \sim 3.5$ for spin triplet state, from Fig. 2 we can see that divergent region of Born iteration is about 0~15 Mev.

Example 2.

Neutron-proton S-wave scattering by a δ -function type potential.

$$B = \lambda \delta(r - r_0).$$

In this case both eigenfunctions of $A^{-1}B$ and G don't constitute the complete sets, hence preceeding arguments by expanding into complete sets lose their validity. Nevertheless the following results are derived easily.

(1) Schwinger's iteration converges in the whole $\lambda - k^2$ plane by any trial function which is continuous and not zero only at $r = r_0$. Of course, since convergent region of (c) are just whole plane, this fact does not conflict the theorem in the present paragraph.

(2) The border line of convergency in Born iteration is*

$$\lambda_B = k^2 \cot kr_0 + \tan kr_0.$$

*) The border line of convergency for S-wave in three dimensional Born iteration is

$$\lambda_B = k^2 \operatorname{cosec} kr_0.$$

Thus for such a singular potential Schwinger's iteration is also more favourable than Born expansion for all the energy range.

§ 5. An effective method of iteration

In the preceding paragraphs we have seen the convergent properties of several iterative methods. Now in this and next paragraphs we shall investigate a more effective method of iteration and its application to variational technique.

For solving the equation (6), consider the following iterative method:

$$\text{or} \quad (\mathbf{A} - \lambda_{(n)} \mathbf{B}) \phi_{n+1} = \mathbf{B} \phi_n, \quad (17)$$

$$\phi_{n+1} = (\mathbf{A} - \lambda_{(n)} \mathbf{B})^{-1} \mathbf{B} \phi_n,$$

$$\text{where} \quad \lambda_{(n)} = (\phi_n \mathbf{A} \phi_n) / (\phi_n \mathbf{B} \phi_n). \quad (18)$$

Convergent coefficient of iteration (17) is

$$\mu = |\lambda^{(i)} - \lambda_{(n)}|_{Min} / |\lambda^{(0)} - \lambda_{(n)}|, \quad (i \neq 0).$$

After normalizing as (10) and (11), we define

$$\Delta \phi_n = \phi_n - \phi^{(0)}.$$

From (18) we obtain

$$|\lambda^{(0)} - \lambda_{(n)}| = |(\Delta \phi_n (\mathbf{A} - \lambda^{(0)} \mathbf{B}) \Delta \phi_n)| + O(\Delta \phi_n)^4 \quad (19)$$

and

$$|\lambda^{(i)} - \lambda_{(n)}|_{Min} = |\lambda^{(i)} - \lambda^{(0)}|_{Min} + O(\Delta \phi_n)^2, \quad (i \neq 0).$$

Then

$$\mu = |\lambda^{(i)} - \lambda^{(0)}|_{Min} / |(\Delta \phi_n (\mathbf{A} - \lambda^{(0)} \mathbf{B}) \Delta \phi_n)| \cdot \{1 + O(\Delta \phi_n)^2\}.$$

If we assume that the error of ϕ_n arises from the component to the eigenfunction corresponding to the eigenvalue nearest to $\lambda^{(0)}$, μ is reduced into

$$\mu = \{1 + O(\Delta \phi_n)^2\} / |\Delta \phi_n|^2. \quad (20)$$

This means

$$|\Delta \phi_{n+1}| = |\Delta \phi_n|^3 \cdot \{1 + O(\Delta \phi_n)^2\}.$$

Rewriting this without specification of the normalization on ϕ_n and ϕ_{n+1} , it is written as

$$|\Delta \phi_{n+1}| / |\phi_{n+1}| = |\Delta \phi_n|^3 / |\phi_n|^3 \cdot \{1 + O(\Delta \phi_n)^2\}. \quad (21)$$

Thus the efficiency of iteration (17) is very good and is furthermore independent whether ϕ_n corresponds to the ground state or not. It is enough for usual purposes to iterate only once using a crude trial function ϕ_0 . If one wants to solve (6) numerically, since $\lambda^{(0)} = \lambda$ is unknown, $\lambda^{(0)}$ must be sought many times by the trial and error method, while when one solves numerically the equation (17) there is no longer such trouble. After having obtained ϕ_{n+1} in this way, $\lambda_{(n+1)}$ is calculated as

$$\lambda_{(n+1)} = (\psi_{n+1} \mathbf{A} \psi_{n+1}) / (\psi_{n+1} \mathbf{B} \psi_{n+1}).$$

This $\lambda_{(n+1)}$ contains the error which is order of only $(\Delta \psi_{n+1})^2$. Finally it may be remarked that if we can simply estimate an approximate value of $\lambda^{(0)}$, $\lambda_{(n)}$ in (17) need not be always calculated from (18). In this case the relation (19) are subject to some alteration.

Example

To find ψ and λ when a neutron and a proton interact through the square well potential which just make the system a bound state with zero binding energy. Namely

$$\mathbf{A} = -d^2/dx^2, \quad \mathbf{B} = 1, \quad 1 > x; \quad = 0, \quad 1 < x, \\ \psi(0) = d\psi(1)/dx = 0.$$

The correct eigenfunctions are

$$\psi^{(m)} = \sin(m+1/2)\pi x, \quad x < 1; \quad = (-1)^m, \quad x > 1,$$

and eigenvalues are

$$\lambda^{(m)} = (m+1/2)^2 \pi^2.$$

We will take the following $\psi_0^{(0)}$ and $\psi_0^{(1)}$ as the trial function ψ_0 in the formula (17) for $m=0$ and 1.

$$\psi_0^{(0)} = 2.1 - x^2, \quad x < 1, \\ \psi_0^{(1)} = 4x - 2x^2 - 12x^3 + 9x^4.$$

These contain the errors of several percents compared to the correct solutions:

$$\psi^{(0)} = \sin(\pi x/2), \quad \psi^{(1)} = \sin(3\pi x/2).$$

In fact

$$\lambda_0^{(0)} = (\psi_0^{(0)} \mathbf{A} \psi_0^{(0)}) / (\psi_0^{(0)} \mathbf{B} \psi_0^{(0)}) = 5/2, \quad \therefore |\Delta \psi_0^{(0)}| / |\psi_0^{(0)}| \sim 0.04, \\ \lambda_0^{(1)} = 424/19, \quad \therefore |\Delta \psi_0^{(1)}| / |\psi_0^{(1)}| \sim 0.07.$$

Iterating $\psi_0^{(0)}$ and $\psi_0^{(1)}$ once by (17), we get

$$\psi_1^{(0)} = (2/5)^2 \sec \sqrt{5/2} \cdot \{4/5 \cdot \cos[\sqrt{5/2} \cdot (1-x)] + (x^2 - 2x - 4/5) \cos \sqrt{5/2}\}, \\ \psi_1^{(1)} = (19/424)^2 \sec \sqrt{424/19} \cdot \{725/53 \cdot \cos[\sqrt{424/19} (1-x)] - 144 \sqrt{19/424} \sin \sqrt{424/19} x \\ + (-3816x^4/19 + 5088x^3/19 + 2900x^2/19 - 3064x/19 - 725/53) \cos \sqrt{424/19}\}.$$

From table 1, we see the correctness of the relation (21).

	correct $\psi^{(0)}$	$\psi_1^{(0)}$	$\psi_0^{(0)}$	correct $\psi^{(1)}$	$\psi_1^{(1)}$	$\psi_0^{(1)}$
$\psi(1/4)$	0.38268	0.38260	0.4375	0.92388	0.92360	0.7227
$\psi(1/2)$	0.70711	0.70702	0.7500	0.70711	0.70654	0.5625
$\psi(3/4)$	0.92388	0.92384	0.9375	-0.38268	-0.38296	-0.3398
$\psi'(0)$	1.57080	1.57035	2.0000	4.71239	4.71123	4.0000

Table 1. Two examples of the iteration (17). $\psi_0^{(0)}$ are trial functions, and $\psi_1^{(0)}$ are obtained from first iteration (17). The values of functions are normalized as $\psi^{(0)}(1) = 1$, $\psi^{(1)}(1) = -1$.

§ 6. Application to variational treatment

Let us consider the variational method in order to solve the equation (6) approximately. This consist in solving the equation (22) for all the possible variations of trial ψ which has n adjustable parameters linearly and satisfies suitable boundary conditions.

$$\delta(\psi, (A - \lambda B)\psi) = 0. \quad (22)$$

Since λ is unknown, first we must solve the troublesome secular determinant of n -th order in order to determine the approximate λ from the condition that n simultaneous homogeneity equations are compatible with each other.* On the other hand using the equation:

$$\delta(\psi, (A - \lambda_{(0)} B)\psi - 2B\psi_0) = 0, \quad (23)$$

$$\lambda_{(0)} = (\psi_0 A \psi_0) / (\psi_0 B \psi_0),$$

the calculations are made simple in consequence of the fact that it does not need to compute such a secular equation. Let us consider the condition under which (23) can be used with the same accuracy instead of (22). This becomes remembering (21),

$$|\Delta\psi|/|\psi| > |\Delta\psi_0|^3/|\psi_0|^3. \quad (24)$$

where $\Delta\psi$ means the error of ψ when parameters in ψ are adjusted so as to be most suitable. If the trial function ψ contains so many parameters that it can approach to an arbitrary function by taking suitable values of parameters, ψ in (22) and (23) become to be true for ψ and ψ_1 in (17) respectively. So if the sign of inequality in (24) is reverse, the accuracy in (22) is in general superior to the one in (23) provided that trial functions ψ in (22) and (23) have the same form. However this condition is satisfied easily for almost cases because the accuracy $|\Delta\psi|/|\psi|$ we want to obtain is more rough than $|\Delta\psi_0|^3/|\psi_0|^3$ in general. Two examples concerning to the bound state problems are given to confirm above arguments. This method also can be used conveniently for such scattering problems that the potential strength λ is to be found when the phase shift is known.

Example 1

The same example as in sec. 5. Only the ground state ($m=0$) is solved by both (22) and (23). For ψ_0 in (23)

$$\psi_0 = 2x - x^2 \quad (25)$$

is taken, and for a trial ψ for both (22) and (23) is assumed to be

$$\psi = c_1 x + c_2 x^3 + c_3 x^5, \quad (x < 1), \quad (26)$$

where c_i are adjustable parameters to be determined by (22) or (23). The results are summarized in Table 2.

	true $\psi^{(0)}$	(22) method	(23) method	ψ_0
$\psi(1/4)$	0.38268	0.38258	0.38251	0.4375
$\psi(1/2)$	0.70711	0.70721	0.70712	0.7500
$\psi(3/4)$	0.92388	0.92412	0.92408	0.9375
$\psi'(0)$	1.57080	1.57007	1.56975	2.0000
λ	2.4674011	2.4674023	2.4674024	2.5000

Table 2. The values of wave functions which are solved by (22) and (23) using the ψ_0 of (25) and ψ of (26). $\psi(1)=1$ for all wave functions.

*) Recently Hulthén and Lauriainen devised a method of solving this determinant, which was however still troublesome (c. f. reference (11)).

From the above table we can see that the accuracies of (22) and (23) are comparable. Thus in this case the usefulness of (23) has been confirmed. The error of λ in the (22) $\Delta\lambda$ is

$$\Delta\lambda/|\lambda^{(0)} - \lambda^{(4)}|_{M_{878}} \sim 6/10^8 = 1/(4000)^2,$$

hence

$$|\Delta\psi|/|\psi| \sim 1/4000.$$

From the example in sec. 5

$$|\Delta\psi_0|^{1/3}/|\psi_0|^{1/3} \sim (0.04)^3 = 1/17000.$$

Thus the inequality of (24) is satisfied. Comparing λ obtained by (22) and (23), it is noticeable that λ in (23) is very close to λ in (22) due to the stationary property of λ when the inequality (24) is properly satisfied as above, though λ in (22) is always better than λ in (23) on account of the fact that λ obtained by (22) is minimum and the best value for this trial function. Comparison between Table 1 and 2 is also instructive. One may see the following relation:

$$\psi(23) - \psi(22) \doteq \psi_1^{(0)} - \psi^{(0)},$$

which seems to be plausible.

Example 2.

Let the potential strength be given and the binding energy of a neutron and a proton system is to be found. Namely situations of energy and potential strength are opposite in the example 1 and in the present example. Here we take up Yukawa type with the depth 5/2.

$$A = -d^2/dx^2 - 5e^{-x}/2x, \quad B = 1,$$

$$\psi(0) = 0, \quad \int_0^\infty |\psi(x)|^2 dx = \text{finite}.$$

The correct eigenfunction and the eigenvalue (binding energy) are⁽¹⁾

$$\psi = e^{-0.374013x} \cdot (1 - e^{-x}) (1 + ae^{-x} + be^{-2x} + \dots),$$

$$\lambda = -0.139886.$$

Taking a very crude ψ_0 on purpose,

$$\psi_0 = e^{-0.6x} (1 - e^{-x}),$$

then

$$\lambda_{(0)} = -0.113.$$

For trial ψ , we take

$$\psi = e^{-0.4x} \cdot (1 - e^{-x}) (c_1 + c_2 e^{-x}).$$

The results are

$$\lambda = -0.139242, \quad \sigma_0 = 1.016, \quad \sigma_\infty = 0.956, \quad \text{for (22) method,}$$

$$\lambda = -0.139235, \quad \sigma_0 = 1.018, \quad \sigma_\infty = 0.952, \quad \text{for (23) method.}$$

where σ_0 and σ_∞ are the quantities⁽¹⁾ defined by Hulthén in order to estimate the errors of ψ in the vicinity of the origine and at large distance respectively. For correct ψ , σ_0 and σ_∞ are to be one. The circumstances which are confirmed in example 1 also hold good for this example.

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Appendix

Several remarks will be stated with regard to practical use of the (23) method. First we note that the method using (23) formula requires ψ_0 which does not appear in (22).

However since this ψ_0 is only restricted by (24), it is sufficient to take in general for ψ_0 a very crude function which is derived by simple variational calculations with one or a few adjustable parameters or is determined with one's eye, in fact the latter is good enough for many cases, therefore the additional complications may be trivial. Thus it is possible to conclude in almost cases that (23) method is more simple than (22) method because it needs no cumbersome calculation of solving the secular equation.

Secondly discussions with regards to solving the simultaneous linear equations

$$\sum_j^n a_{ij} c_j = a_i, \quad (i=1, \dots, n) \quad (\text{A} \cdot 1)$$

will be given. Since A and B are Hermitian operators, a_{ij} are symmetric. Then it is convenient to extend the method given by H. M. James and A. S. Coolidge.⁽¹²⁾ For this purpose, we constitute c_{ij} , and e_i by the following recurrence formulae:

$$e_{ij} = a_{ij} - \sum_{k=1}^{i-1} e_{ki} c_{kj} / e_{kk},$$

$$e_i = a_i - \sum_{k=1}^{i-1} e_{ki} c_i / e_{kk}$$

and

$$\sum_j^n e_{ij} c_j = e_i, \quad (i=1, \dots, n). \quad (\text{A} \cdot 2)$$

The equations (A·1) and (A·2) are equivalent. Since $e_{ij}=0$ ($i>j$), (A·2) are reduced to

$$\begin{array}{l} e_{11}c_1 + e_{12}c_2 + \dots + e_{1n}c_n = e_1, \\ \quad \vdots \\ e_{i1}c_1 + \dots + e_{in}c_n = e_i, \\ \quad \vdots \\ e_{nn}c_n = e_n. \end{array}$$

Thus it is possible to obtain c_i in turn from the bottom to above. It can be shown that $\sum e_{ij} + e_i$ are obtained by the same recurrence formulae, which provides a method of test.

Thirdly, we would like to mention the errors which arise from solving the equations (A·1) or (A·2). It may be inferred that since $\lambda_{(0)}$ in (23) is very close to λ in (22), the value of the determinant of the coefficients in (A·1) is a small quantity, namely the available figures are diminished by cancellation. Putting

$$\Delta = |c_{ij}|,$$

c_i 's are determined by the determinant

$$c_i = \begin{vmatrix} \dots & a_{1, i-1} & a_1 & a_{1, i+1} & \dots \\ & \vdots & \vdots & \vdots & \\ \dots & a_{n, i-1} & a_n & a_{n, i+1} & \dots \end{vmatrix} / \Delta.$$

However we must notice that it is enough for our purpose to obtain the shape of the function itself. In the other words, only the ratios of c_i are our concern, so that we

need not find the value of Δ . When we solve (A.1) using (A.2), if small quantities appear in the course of calculation, they can be handled as if they had the same available figures as a_{ij} . Slight considerations for using (A.2) will be helpful to explain this situation. Let only e_{nn} be small quantity ϵ , and other c_{ij} and c_i be order of one. This does not lose generality, because the above assumption is valid if we take the base φ_n of trial function ψ corresponding to n a function near to the correct function while the other bases $\varphi_i (i \neq n)$ of trial function are different to ψ_n and each other. ($\psi = \sum c_i \varphi_i$). Since

$$c_n = e_n / e_{nn},$$

$$c_{n-1} = e_{n-1} / e_{n-1, n-1} - e_{n-1, n} e_n / e_{n-1, n-1} e_{n, n},$$

it follows that

$$c_{n-1} = e_{n-1} / e_{n-1, n-1} e_n / e_{n-1, n-1} e_{n, n}.$$

Thus if all e_{ij} and e_i have the rounding errors of order of δ ,

$$\Delta e_{nn} / e_{nn} \sim O(\delta / \epsilon),$$

$$\text{other } \Delta e / e \sim O(\delta).$$

$$\Delta c_n / c_n \sim \Delta c_{n-1} / c_{n-1} \sim O(\delta / \epsilon),$$

however

$$\Delta(c_{n-1} / c_n) / (c_{n-1} / c_n) \sim O(\delta),$$

so that in general

$$\Delta(c_i / c_n) / (c_i / c_n) \sim O(\delta).$$

Thus the mentioned difficulty is excluded, though there exist accumulated errors as in usual calculations.

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Pseudoscalar Interaction in the Theory of Beta-decay and RaE^{*)}

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The correction factors for the linear combinations of pseudoscalar and tensor interaction, and of pseudoscalar and axial vector interaction for the first forbidden $0 \rightarrow 0$ (yes) transition are deduced by taking the following into consideration; i) the effect of the finite size of nucleus, ii) contribution of a term which contains a derivative of lepton wave function in the case of pseudoscalar interaction. The beta-ray spectrum of RaE is examined by the linear combination of pseudoscalar and tensor interaction.

§ 1. Introduction

Of the Fermi theory of beta-decay the following seem to be currently accepted as having been established that the scalar (S) or the vector (V) interaction is necessary to explain the decay schemes of C^{10} and O^{14} ¹⁾, from the necessity of Gamow-Teller selection rule to account for the short life transitions with a unit spin change the axial vector (A) or the tensor (T) interaction must be included, and the T interaction is more favoured than the A interaction by the experiments on the correlation between electron and recoil nucleus in He^6 decay²⁾. By these arguments and the so-called Fierz interference³⁾ the combination $(S, T)^{4)5)}$ or $(V, T)^{6)}$ seems to fit in with the experimental data. On the other hand, as to the pseudoscalar (P) interaction, we have no evidence for its necessity except one case, namely, the explanation of beta-ray spectrum of RaE by Petschek and Marshak⁶⁾ with (T, P) combination.

To explain the beta-ray spectrum of RaE, Petschek and Marshak have at first calculated the correction factor, assuming that the nuclear charge is concentrated at the centre of nucleus, and then the effect of the finite size of nucleus, calculated by Rose and Holmes⁷⁾, taken into account. This procedure seems to be incomplete because of the following three reasons. First, although we must take 84 for the decay of RaE as the atomic number Z , the effect of the finite size of nucleus calculated by Rose and Holmes is that for $Z=83$. Secondly, Petschek and Marshak have taken the effect of the so-called finite deBroglie wave length⁸⁾ into consideration in their calculation of the correction factor without the effect of the finite size of nucleus, but Rose and Holmes have not considered the former effect. So it may become meaningless to combine these two calculations. Thirdly, if the pseudoscalar nuclear matrix element $\int \beta \gamma_5$ is small⁹⁾, the procedure to evaluate the radial part

*) The contents of this paper were read at the Kyoto meeting of the Physical Society of Japan on the 7th May, 1953.

of the lepton wave function at the boundary of the nucleus becomes incomplete. Then we must proceed to the next order of approximation, and should consider the term which contains a derivative of lepton wave function. Ahrens et al.^{10,11)} have pointed out this fact, too. According to Ruderman's¹²⁾ calculation, however, the matrix element $\int \beta \gamma_5$ is not so small owing to the so-called pair term, if the nuclear force arises from pseudoscalar mesons with pseudoscalar coupling. But Brueckner et al.¹³⁾ have shown, although their calculations is incomplete as they have considered only the simplest case, that there is a possibility to reduce this term. Even if the nuclear matrix element $\int \beta \gamma_5$ is not small, the term containing the derivatives of lepton wave function may contribute to the correction factor, for, in order to explain the decay of RaE by the linear combination (T, P) the main terms should cancel each other.

By the above mentioned reasons it seems worth while to reexamine the pseudoscalar interaction and the beta-ray spectrum of RaE. Therefore, we have deduced the matrix element for pseudoscalar interaction in § 2, considered the deviation of the lepton wave function due to the effect of the finite size of nucleus in § 3, deduced the correction factors for the P , (T, P) and (A, P) interaction in § 4, and lastly in § 5 tried to explain the beta-ray spectrum of RaE using these results.

§ 2. Pseudoscalar interaction

The matrix element for the pseudoscalar interaction may be written as:

$$H = \lambda_p \int d\tau (\Psi_f^* Q_k \beta \gamma_5 \Phi_i) (\psi^* \beta \gamma_5 \phi). \quad (1)$$

In this expression λ_p is the coupling constant of the nuclear particle with the lepton field, Φ_i and Ψ_f are the wave functions of the initial and final nuclei, Q_k is an operator which replaces Φ_i with a wave function describing a nucleus in which the k -th neutron is replaced by a proton. The quantities ψ and ϕ are the quantized wave functions of the electron and anti-neutrino, respectively, normalized to one particle in a sphere of unit radius.

To use only the positive energy solutions of the Dirac equation¹⁴⁾, we consider an operator

$$C = \beta u_y \quad (2)$$

then

$$\begin{aligned} (\psi^*(+) \beta \gamma_5 \phi(-)) &= (\psi(+) C \beta \gamma_5 \phi(+))^* \\ &= i(-\psi_1 \phi_2 + \psi_2 \phi_1 - \psi_3 \phi_4 + \psi_4 \phi_3)^* \\ &= \sum_{k=1}^4 R_k(r) \Theta_k(\theta, \varphi). \end{aligned} \quad (3)$$

ψ_i and ϕ_i are the components of $\psi(+)$ and $\phi(+)$, respectively, where $\psi(+)$ and $\phi(-)$ represent the positive and negative energy state solutions of Dirac equation. The ψ_i' for

the case of pure Coulomb potential are calculated by Rose¹⁵⁾. For ϕ_i , see reference 14). $R_k(r)$ and $\theta_k(\theta, \varphi)$ are defined as follows:

$$\left. \begin{aligned} R_1(r) \theta_1(\theta, \varphi) &= -i(\phi_1 \phi_2)^*, \\ R_2(r) \theta_2(\theta, \varphi) &= i(\phi_2 \phi_1)^*, \\ R_3(r) \theta_3(\theta, \varphi) &= -i(\phi_3 \phi_4)^*, \\ R_4(r) \theta_4(\theta, \varphi) &= i(\phi_4 \phi_3)^*, \end{aligned} \right\} \quad (4)$$

$$\theta_k(\theta, \varphi) = r^2 Y_{lm}^*(\theta, \varphi) Y_{00}. \quad (5)$$

If we expand $R_k(r)$ at the nuclear radius ρ , we get for eq. (1):

$$H = \lambda_p \sum_{k=1}^4 \left[R_k(\rho) \int Q_{ki} \gamma_5 \theta_k(\theta, \varphi) + \sum_{n=1}^{\infty} \frac{R_k^{(n)}(\rho)}{n!} \int Q_{ki} \gamma_5 (r-\rho)^n \theta_k(\theta, \varphi) \right] \quad (6)$$

For the sake of simplicity, we have omitted $d\tau$, Ψ_f^* and ϕ_i in the expression (6). $R_k^{(n)}(r)$ is the n -th derivative of $R_k(r)$ with respect to r . The first term is the customarily used transition matrix element for the P interaction.¹⁶⁾¹⁷⁾

The effect of nuclear force affects the results only if the nuclear force arises from pseudoscalar mesons with pseudoscalar coupling. So we assume that nucleon satisfies the following Dirac equation:

$$(E + \boldsymbol{\alpha} \cdot \mathbf{p} + \beta M - i\beta \gamma_5 U) \Psi = 0. \quad (7)$$

E , p and M stand for energy, momentum and mass of nucleon; $-i\beta \gamma_5 U$ is the potential for the nucleon.

If we consider only the first two terms in (6), then

$$\begin{aligned} H &= \lambda_p \sum_{k=1}^4 \left[R_k(\rho) \int Q \gamma_5 \theta(\theta, \varphi) \right. \\ &\quad \left. + \frac{i}{2M} R'_k(\rho) \int Q_k \cdot (\boldsymbol{\sigma} \cdot \nabla + 2U) (r-\rho) \theta_k(\theta, \varphi) \right]. \end{aligned} \quad (8)$$

Hereafter, we confine ourselves to the transition $0 \rightarrow 0$ (yes). In this case

$$\theta_k(\theta, \varphi) = \frac{1}{4\pi}, \quad k=1, 2, 3, 4, \quad (9)$$

and if we put

$$R(r) = \sum_{k=1}^4 R_k(r), \quad (10)$$

eq. (8) becomes as follows:

$$H = \frac{\lambda_p \lambda'}{4\pi} \frac{i}{2M\rho} [IR(\rho) + R'(\rho)] \int \boldsymbol{\sigma} \cdot \mathbf{r}, \quad (11)$$

where

$$\left. \begin{aligned} \lambda' &= \frac{\int \sigma \cdot r + \rho \int 2(r-\rho) U}{\int \sigma \cdot r}, \\ \frac{1}{\lambda'} \int \beta \gamma_5 &= \frac{i}{2M} \frac{I'}{\rho} \int \sigma \cdot r. \end{aligned} \right\} \quad (12)$$

According to the calculation of Ahrens et al.⁹⁾¹⁰⁾,

$$I/\rho = A^2 (uZ/2\rho)^2, \quad A = 1 + (W_0 - 2.5) A^{1/3}/Z. \quad (13)$$

This relation may be correct as an average over a range of A, Z values, but it seems somewhat inadequate to apply this to individual elements. Hence we regarded I/ρ as a parameter of which value should be determined from experimental data.

The second term in the numerator of λ' is of the next order to the first as to ρ , and if we neglect that term, λ' becomes 1. But as we do not know to what extent U contributes, we will write λ' instead of 1. The fact that Ruderman could make $\int \beta \gamma_5$ large is due to the effect of U . We therefore may not be able to take λ' as 1, that is, if $\int \beta \gamma_5$ is really large. And, as it may be seen in the following chapters, to take λ' as 1 or not is only equivalent to multiplying λ_p and I/ρ by constant factors or not.

§ 3. The effect of the finite size of the nucleus

To reexamine the beta-ray spectrum of RaE, we must make eq. (13) include the effect of the finite size of the nucleus. This effect has been calculated by Rose and Holmes⁷⁾ and Malcolm¹⁸⁾. As it has been mentioned in the introduction, Rose and Holmes have computed for $Z=83$, and as they have not considered the effect of the finite deBroglie wave length, we cannot use their result in our case. Malcolm has calculated the effect for $Z=84$, but his work came to our notice after the present investigation was finished and we could not use his results. The correction of the second term of eq. (13) by the finite nucleus has been calculated by neither Rose and Holmes nor Malcolm.

For these reasons, we had to calculate the effect of the finite size of nucleus for $Z=84$. We have done it according to the procedure given by Rose and Holmes. We use their notation in the following discussion.*)

*) $x = \pm(j+1/2)$; $\mathfrak{F}_x(r) = rf_x(r)$, $\mathfrak{G}_x(r) = rg_x(r)$, where f_x and g_x are regular radial wave functions in a Coulomb field. When we consider the effect of the finite nucleus, $\mathfrak{F}_x(r)$ and $\mathfrak{G}_x(r)$ are replaced by $\Phi_x(r)$ and $\Gamma_x(r)$.

$$\mu'_x = \left(\frac{\mathfrak{F}/\mathfrak{G} - \mathfrak{F}^{(i)}/\mathfrak{G}^{(i)}}{\mathfrak{F}^{(i)}/\mathfrak{G}^{(i)} - \mathfrak{F}/\mathfrak{G}} \right)_{x, \text{at } p}, \quad \mu_x = \left(\frac{\mathfrak{F}/\mathfrak{G}}{\mathfrak{F}/\mathfrak{G}} \right)_{x, \text{at } p} \cdot \mu'_x$$

\mathfrak{F} and \mathfrak{G} are irregular solutions for pure Coulomb field. $\mathfrak{F}^{(i)}$ and $\mathfrak{G}^{(i)}$ are regular solutions for the potential $V_i(r)$ which is determined by the charge distribution inside the nucleus. The x which is outside the parentheses is common for all the quantities inside.

Rose and Holmes have calculated this effect for

$$L_{\kappa} = (1/2\rho^2 F) \cdot (\mathfrak{G}_{-\kappa}^2 + \mathfrak{F}_{\kappa}^2) / \rho^{2\kappa} \quad \text{etc.,}$$

but we have calculated the following quantities for the convenience of later calculations :

$$A_{\kappa} = \Phi_{\kappa}(\rho) / \mathfrak{F}_{\kappa}(\rho), \quad B_{\kappa} = \Gamma_{\kappa}(\rho) / \mathfrak{G}_{\kappa}(\rho), \quad (14)$$

$$C_{\kappa} = \Phi'_{\kappa}(\rho) / \mathfrak{F}'_{\kappa}(\rho), \quad D_{\kappa} = \Gamma'_{\kappa}(\rho) / \mathfrak{G}'_{\kappa}(\rho). \quad (15)$$

The primes attached to Φ, \mathfrak{F} etc. denote the derivatives with respect to ρ .

These quantities are :

$$A_{\kappa} = 1 + \mu_{\kappa}, \quad B_{\kappa} = 1 + \mu'_{\kappa} \quad (16)$$

$$C_{\kappa} = \left\{ 1 + \frac{\mathfrak{F}'(\rho)}{\mathfrak{F}(\rho)} \cdot \frac{\mathfrak{F}(\rho)}{\mathfrak{F}'(\rho)} \cdot \mu \right\}_{\kappa}, \quad D_{\kappa} = \left\{ 1 + \frac{\mathfrak{G}'(\rho)}{\mathfrak{G}(\rho)} \cdot \frac{\mathfrak{G}(\rho)}{\mathfrak{G}'(\rho)} \cdot \mu' \right\}_{\kappa}. \quad (17)$$

The results of numerical calculations are shown in Figs. 1, 2, 3 and 4 for the uniform nuclear charge distribution.

As it is unnecessary in the following discussions to consider the cases $|x| \geq 3$, we have not computed these cases. The greater the value of $|x|$ the more the wave function is pushed away, and the effect of the finite nucleus becomes smaller and smaller. Then this effect may be neglected for these cases except the very special cases e.g. the case where the correction factors cancel each other almost entirely when some types of interaction are combined.

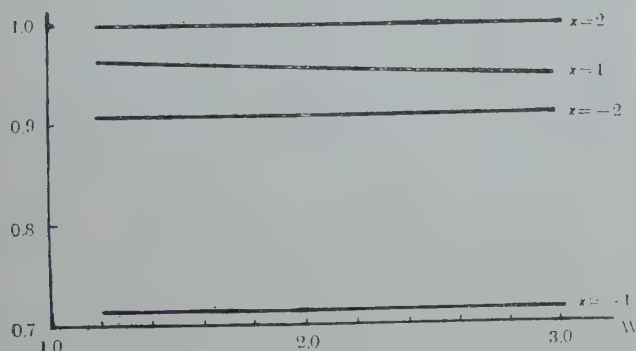


Fig. 1, A_{κ}

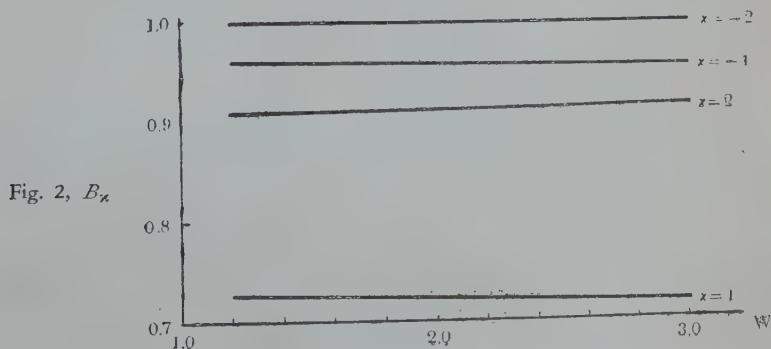


Fig. 2, B_{κ}

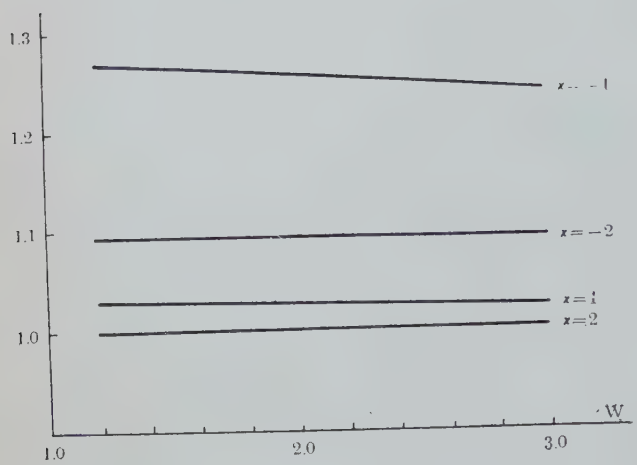


Fig. 3, C_x

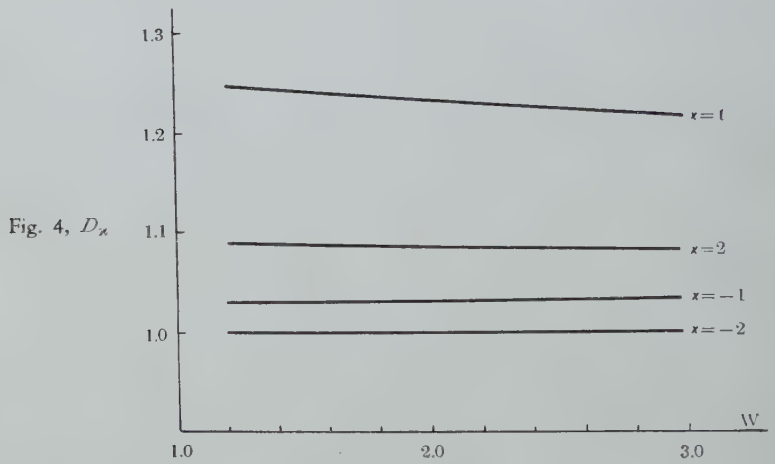


Fig. 4, D_x

§ 4. Correction factors

The correction factor C_p of pseudoscalar interaction for $0 \rightarrow 0$ (yes) transition may be written :

$$C_p = \frac{1}{4F(Z, W) p^0 q^0} \frac{\lambda'^2}{(2M\rho)^2} \left| \int \boldsymbol{\sigma} \cdot \mathbf{r} \right|^2 \sum_{(j_1, l_1, m_1)} \sum_{(j_2, l_2, m_2)} |R'(\rho) + IR(\rho)|^2 \quad (18)$$

in which W is the electron energy, $p = (W^2 - 1)^{1/2}$ is its momentum, q is the neutrino momentum, and j_1, l_1, m_1 and j_2, l_2, m_2 are the total angular momentum quantum number, azimuthal quantum number and magnetic quantum number of electron and neutrino, respectively. And

$$F(Z, W) = 4(2p\rho)^{2\gamma-2} e^{\pi\alpha ZW/p} |\Gamma(\gamma + iaZW/p)|^2 / \Gamma^2(2\gamma + 1), \quad (19)$$

where

$$\gamma = \sqrt{1 - \alpha^2 Z^2}.$$

In this case, we can consider the linear combinations (T, P) and (A, P) . And we have calculated not only C_p , but also the cross correction factors for these two combinations. In such cases where $R'(\rho)$ does not exist, these are calculated by Smith⁽¹⁹⁾ and Pursey⁽²⁰⁾, and for neutron decay by Kotani et al.⁽²¹⁾.

The results are :

$$C_p = \frac{\lambda'^2}{(2M)^2} \left| \int \boldsymbol{\sigma} \cdot \mathbf{r} \right| \left[l_{0p} + 2l_{1p} + 2l'_{2p} + l'_{2p} + 2l_{3p} + l_{4p} \right. \\ \left. + m_{0p} + 2m_{1p} + m_{2p} + 2n_{0p} + 2n_{1p} + 2n'_{1p} + 2n_{2p} + 2n_{3p} \right. \\ \left. + \frac{2p^2 q^2}{(3!)^2} \frac{F_1(Z, W)}{F(Z, W)} l_{0p}^* \right], \quad (20)$$

$$C_{PT} = \lambda'/2M \cdot \left| \int \boldsymbol{\sigma} \cdot \mathbf{r} \right|^2 \left[l_{0PT} + l_{1PT} + l_{2PT} - m_{0PT} - m_{1PT} - m_{2PT} \right. \\ \left. - n_{0PT} - n_{1PT} - n'_{1PT} - n_{2PT} - n'_{2PT} - n_{3PT} \right], \quad (21)$$

$$C_{AP} = \lambda'/2M \cdot \left| \int \boldsymbol{\sigma} \cdot \mathbf{r} \right|^2 \left[l_{0AP} + l_{1AP} + l_{2AP} + m_{0AP} + m_{1AP} + m_{2AP} \right. \\ \left. + n_{0AP} + n_{1AP} + n'_{1AP} + n_{2AP} + n'_{2AP} + n_{3AP} \right]. \quad (22)$$

In this calculation, we have assumed

$$\int \beta \boldsymbol{\sigma} \cdot \mathbf{r} = - \int \boldsymbol{\sigma} \cdot \mathbf{r}. \quad (23)$$

Above are used the abbreviations :

$$F_1(Z, W) = (4!)^2 (2p\rho)^{2\gamma_1-4} e^{\pi\alpha_Z W/p} \left| \Gamma(\gamma_1 + i\alpha_Z W/p) \right|^2 / \Gamma^2(2\gamma_1 + 1), \\ \gamma_1 = \sqrt{4 - \alpha_Z^2 Z^2},$$

and

$$\left. \begin{aligned} l_{i\lambda} &= l_i^L \frac{x_{i\lambda} + y_{i\lambda}}{2} + l_i^S \frac{x_{i\lambda} - y_{i\lambda}}{2}, \\ m_{i\lambda} &= m_i^L \frac{x_{i\lambda} + y_{i\lambda}}{2} + m_i^S \frac{x_{i\lambda} - y_{i\lambda}}{2}, \\ n_{i\lambda} &= n_i^L \frac{x_{i\lambda} + y_{i\lambda}}{2} + n_i^S \frac{x_{i\lambda} - y_{i\lambda}}{2}, \end{aligned} \right\} \quad (24)$$

$i=0, 1, 2, \dots$, $\lambda=P, TP$, and AP .

$l_i^L, l_i^S, m_i, m_i^S, n_i^L$ and n_i^S are given as follows :

$$l_0^L = (1 + \gamma)/2,$$

$$l_0^S = -\frac{1+\gamma-a^2Z^2}{2},$$

$$l_1^L = \frac{-1}{2\gamma+1} \left\{ \frac{\gamma}{W} + (2\gamma+3)W \right\} \cdot \frac{aZ}{2\rho},$$

$$l_1^S = (\gamma+1) \cdot aZ/2\rho,$$

$$l_2^L = \frac{1}{2(2\gamma+1)(2\gamma+2)} [-(\gamma+1)^2 p^2 + 2a^2 Z^2 \{\gamma + (\gamma+2)W^2\}],$$

$$l_2^S = \frac{1}{2(2\gamma+1)(2\gamma+2)} \left\{ \gamma(\gamma+1)^2 \frac{p^2}{W} - 2a^2 Z^2 (\gamma+1)^2 W \right\},$$

$$l_2^{*L} = \frac{2}{(2\gamma+1)^2} \left[\frac{1-\gamma}{a^2 Z^2} p^2 + 2\{1+2\gamma+2(\gamma+2)W^2\} \right] \cdot \left(\frac{aZ}{2\rho} \right)^2,$$

$$l_2^{*S} = \frac{-2}{(2\gamma+1)^2} \left[\frac{1-\gamma}{a^2 Z^2} \frac{p^2}{W} + \left\{ \frac{1}{W} + (8\gamma+9)W - 4a^2 Z^2 W \right\} \right] \cdot \left(\frac{aZ}{2\rho} \right)^2,$$

$$l_3^L = \frac{1}{(2\gamma+1)^2(2\gamma+2)} \left[(\gamma+1)(\gamma+W)^2 \frac{p^2}{W} + 2\gamma(\gamma+3)p^2 W \right. \\ \left. - 2a^2 Z^2 W \{3(\gamma+W)^2 + 2(\gamma+1)W^2 + 2\} \right] \cdot \frac{aZ}{2\rho},$$

$$l_3^S = \frac{-1}{(2\gamma+1)^2(2\gamma+2)} \left[\{(\gamma+1)^2 + 2\gamma(\gamma+3)\} p^2 \right. \\ \left. - 2a^2 Z^2 \{(3\gamma+4)W^2 - \gamma\} + 4a^4 Z^4 W^2 \right] \cdot \frac{aZ}{2\rho},$$

$$l_4^L = \frac{1}{2(2\gamma+1)^2(2\gamma+2)^2} [(\gamma+1)^3 p^4 - 4a^2 Z^2 p^2 \{\gamma(\gamma+1) + (\gamma^2+4\gamma+1)W^2\} \\ + 4a^4 Z^4 W^2 \{2(\gamma+1) + (\gamma+3)W^2\}],$$

$$l_4^S = \frac{-1}{2(2\gamma+1)^2(2\gamma+2)^2} \left[(\gamma+1)^3 \frac{p^4}{W} + a^2 Z^2 p^2 \left\{ \frac{(\gamma+1)^2}{W} \right. \right. \\ \left. \left. - (9\gamma^2+22\gamma+5)W \right\} + 4a^4 Z^4 \{-\gamma + (4\gamma+5)W^2\} - 4a^6 Z^6 W^3 \right],$$

$$l_0^{*L} = 2 + \gamma,$$

$$l_0^{*S} = -(2 + \gamma_1 - a^2 Z^2/2),$$

$$m_0^L = 2 \cdot \frac{1-\gamma}{a^2 Z^2} \cdot \left(\frac{aZ}{2\rho} \right)^2,$$

$$m_0^S = 2 \cdot (1-\gamma-a^2 Z^2)/a^2 Z^2 \cdot (aZ/2\rho)^2,$$

$$m_1^L = \frac{-1}{2\gamma+1} \left\{ \frac{\gamma}{W} + (1-2\gamma W) \right\} \cdot \frac{aZ}{2\rho},$$

$$m_1^S = (1-\gamma) \cdot aZ/2\rho,$$

$$m_2^L = \frac{1}{2(2\gamma+1)^2} \{ (1+\gamma)p^2 + 2a^2Z^2(1-2\gamma p^2) \},$$

$$m_2^S = \frac{1}{2(2\gamma+1)^2} \left\{ (1+\gamma) \frac{p^2}{W} + a^2Z^2 \left(\frac{1}{W} + W \right) - 4a^4Z^4W \right\},$$

$$m_2'^L = (\gamma-1)p^2/4,$$

$$m_2'^S = \frac{\gamma-1}{4} \cdot \frac{-1}{2\gamma+1} \left(\gamma \frac{p^2}{W} - 2a^2Z^2W \right),$$

$$n_0^L = aZ/2\rho,$$

$$n_0^S = -\gamma/W \cdot aZ/2\rho,$$

$$n_1^L = \frac{-2}{2\gamma+1} \left(\frac{1-\gamma}{a^2Z^2} \frac{p^2}{W} + \frac{1}{W} + 2W \right) \cdot \left(\frac{aZ}{2\rho} \right)^2,$$

$$n_1^S = 2(aZ/2\rho)^2,$$

$$n_1'^L = \frac{1}{2(2\gamma+1)} \left\{ (1+\gamma) \frac{p^2}{W} + a^2Z^2 \left(\frac{1}{W} - 2W \right) \right\},$$

$$n_1'^S = a^2Z^2/2,$$

$$n_2^L = \frac{-1}{(2\gamma+1)(2\gamma+2)} \{ (3\gamma-1)p^2 - 2a^2Z^2(1+W^2) \} \cdot \frac{aZ}{2\rho},$$

$$n_2^S = \frac{-1}{2(2\gamma+1)} \left\{ -\gamma \frac{p^2}{W} + 2a^2Z^2W \right\},$$

$$n_2'^L = \frac{-1}{(2\gamma+1)^2} \{ (1+4\gamma)p^2 - 4a^2Z^2W^2 \} \cdot \frac{aZ}{2\rho},$$

$$n_2'^S = \frac{1}{(2\gamma+1)^2} \left\{ \gamma \frac{p^2}{W} - 4a^2Z^2(1+\gamma)W \right\} \cdot \frac{aZ}{2\rho},$$

$$n_3^L = \frac{-1}{2(2\gamma+1)^2(2\gamma+2)} \left[(\gamma+1)^2 \frac{p^4}{W} + a^2Z^2 p^2 \left\{ \frac{1+\gamma}{W} - 2W(1+4\gamma) \right\} \right. \\ \left. + 2a^4Z^4W(1+2W^2) \right],$$

$$n_3^S = \frac{-a^2Z^2}{2(2\gamma+1)^2(2\gamma+2)} \left[(1+5\gamma+2\gamma^2)p^2 - 2a^2Z^2(3+2\gamma)W^2 \right].$$

$x_{i\lambda}$ and $y_{i\lambda}$ are shown in Table I. In this table, the following abbreviations are used ;

Table Ia.

P			(P, T)	
	x	y	x	y
l_0	$(\theta_0 + \theta_0' q^2)$	$(\beta_0 + \beta_0' q^2)$	$q(\theta_0 + \theta_0' q^2) \sigma_1/3$	$q(\beta_0 + \beta_0' q^2) \tau_{-1}/3$
l_1	$(\theta_0 + \theta_0' q^2) \theta_1$	$(\beta_0 + \beta_0' q^2) \beta_1$	$q \theta_1 \sigma_1/3$	$q \beta_1 \tau_{-1}/3$
l_2	$(\theta_0 + \theta_0' q^2) \theta_2$	$(\beta_0 + \beta_0' q^2) \beta_2$	$q \theta_2 \sigma_1/3$	$q \beta_2 \tau_{-1}/3$
l_2'	θ_1^2	β_1^2		
l_3	$\theta_1 \theta_2$	$\beta_1 \beta_2$		
l_4	θ_2^2	β_2^2		
l_0^*	ζ_0^2	η_0^2		
m_0	$x_0^2 q^2$	$a_0^2 q^2$	$q x_0 \tau_1$	$q a_0 \sigma_{-1}$
m_1	$x_0 x_1 q^2$	$a_0 a_1 q^2$	$q(x_0 + x_1) \tau_1$	$q(a_0 + a_1) \sigma_{-1}$
m_2	$x_1^2 q^2$	$a_1^2 q^2$	$q x_1 \tau_1$	$q a_1 \sigma_{-1}$
n_0	$(\theta_0 + \theta_0' q^2) x_0 q$	$(\beta_0 + \beta_0' q^2) a_0 q$	$(\theta_0 + \theta_0' q^2) \tau_1 - q x_0 \sigma_1/3$	$(\beta_0 + \beta_0' q^2) \sigma_{-1} - q a_0 \tau_{-1}/3$
n_1	$\theta_1 x_0 q$	$\beta_1 a_0 q$	$\theta_1 \tau_1$	$\beta_1 \sigma_{-1}$
n_1'	$(\theta_0 + \theta_0' q^2) x_1 q$	$(\beta_0 + \beta_0' q^2) a_1 q$	$(\theta_0 + \theta_0' q^2) \tau_1 - q^2 x_1 \sigma_1/3$	$(\beta_0 + \beta_0' q^2) \sigma_{-1} - q^2 a_1 \tau_{-1}/3$
n_2	$\theta_2 x_0 q$	$\beta_2 a_0 q$	$\theta_2 \tau_1$	$\beta_2 \sigma_{-1}$
n_2'	$\theta_1 x_1 q$	$\beta_1 a_1 q$	$\theta_1 \tau_1$	$\beta_1 \sigma_{-1}$
n_3	$\theta_2 x_1 q$	$\beta_2 a_1 q$	$\theta_2 \tau_1$	$\beta_2 \sigma_{-1}$

Table Ib. $I = -i\Gamma_1 = -i \int \gamma_5 / \int \sigma \cdot r$.

(A, P)		
	x	y
l_0	$(q/3 + I) (\theta_0 + \theta_0' q^2) \sigma_1$	$(q/3 + I) (\beta_0 + \beta_0' q^2) \tau_{-1}$
l_1	$(q/3 + I) \theta_1 \sigma_1$	$(q/3 + I) \beta_1 \tau_{-1}$
l_2	$(q/3 + I) \theta_2 \sigma_2$	$(q/3 + I) \beta_2 \tau_{-1}$
m_0	$q x_0 \tau_1$	$q a_0 \sigma_{-1}$
m_1	$q(x_0 + x_1) \tau_1$	$q(a_0 + a_1) \sigma_{-1}$
m_2	$q x_1 \tau_1$	$q a_1 \sigma_{-1}$
n_0	$(\theta_0 + \theta_0' q^2) \tau_1 + q(q/3 + I) x_0 \sigma_1$	$(\beta_0 + \beta_0' q^2) \sigma_{-1} + q(q/3 + I) a_0 \tau_{-1}$
n_1	$\theta_1 \tau_1$	$\beta_1 \sigma_{-1}$
n_1'	$(\theta_0 + \theta_0' q^2) \tau_1 + q(q/3 + I) x_1 \sigma_1$	$(\beta_0 + \beta_0' q^2) \sigma_{-1} + q(q/3 + I) a_1 \tau_{-1}$
n_2	$\theta_2 \tau_1$	$\beta_2 \sigma_{-1}$
n_2'	$\theta_1 \tau_1$	$\beta_1 \sigma_{-1}$
n_3	$\theta_2 \tau_1$	$\beta_2 \sigma_{-1}$

$$\beta_0 = (\gamma D_{-1} - B_{-1}) \rho^{-2} + B_{-1} \Gamma \rho^{-1},$$
$$\beta_0' = -(\gamma D_{-1} + B_{-1})/12,$$
$$\beta_1 = (\gamma + 1) D_{-1} - B_{-1} + \rho \Gamma B_{-1},$$
$$\beta_2 = (\gamma + 2) D_{-1} - B_{-1} + \rho \Gamma B_{-1},$$

$$\theta_0 = (\gamma C_1 - A_1) \rho^{-2} + A_1 \Gamma \rho^{-1},$$
$$\theta_0' = -(\gamma C_1 + A_1)/12,$$
$$\theta_1 = (\gamma + 1) C_1 - A_1 + \sigma \Gamma A_1,$$
$$\theta_2 = (\gamma + 2) C_1 - A_1 + \rho \Gamma A_1,$$

$$\begin{aligned}
 a_0 &= (\gamma C_{-1} + \rho \Gamma A_{-1})/3, & x_0 &= x_0 (\gamma D_1 + \rho \Gamma B_1)/3, \\
 a_1 &= \{(\gamma + 1) C_{-1} + \rho \Gamma A_{-1}\}/3, & x_1 &= \{(\gamma + 1) D_1 + \rho \Gamma B_1\}/3, \\
 \zeta_0 &= (\gamma_1 C_2 + \rho \Gamma A_2)/3, & \eta_0 &= (\gamma_1 D_{-2} + \rho \Gamma B_{-2})/3.
 \end{aligned}$$

If we do not take the effect of the finite nucleus into account, the second terms in eq. (24) vanish. l_i^L , m_i^L and n_i^L are the constituents of L , M , and N given by Konopinski and Uhlenbeck⁽¹⁶⁾. For reference, we re-write the Konopinski and Uhlenbeck's correction factors for $0 \rightarrow 0$ (yes) transition by our notation with the finite size correction :

$$\begin{aligned}
 C_T &= \left| \int \sigma \cdot r \right|^2 [l_{0T} + m_{0T} + 2m_{1T} + m_{2T} + 2m'_{2T} - n_{0T} - n_{1T} - n'_{1T}], \\
 C_A &= \left| \int \sigma \cdot r \right|^2 [l_{0A} + m_{0A} + 2m_{1A} + m_{2A} + 2m'_{2A} + n_{0A} + n_{1A} + n'_{1A}]
 \end{aligned}$$

Table II

	T		A
	x	$-y$	
l_0	$g^2 \sigma_1^2/9$	$g^2 \tau_{-1}^2/9$	$l_{0A} = (1 + 3I/q)^2 l_{0T}$
m_0	τ_1^2	σ_{-1}^2	$m_{0A} = m_{0T}$
m_1	τ_1^2	σ_{-1}^2	$m_{1A} = m_{1T}$
m_2	τ_1^2	σ_{-1}^2	$m_{2A} = m_{2T}$
m'_2	τ_1^2	σ_{-1}^2	$m_{2A}' = m_{2T}'$
n_0	$2q\sigma_1\tau_1/3$	$2q\sigma_{-1}\tau_{-1}/3$	$n_{0A} = (1 + 3I/q)n_{0T}$
n_1	$2q\rho^2\sigma_1\tau_1/3$	$2q\rho^2\sigma_{-1}\tau_{-1}/3$	$n_{1A} = (1 + 3I/q)n_{1T}$
n'_1	$2q\sigma_1\tau_1/3$	$2q\sigma_{-1}\tau_{-1}/3$	$n_{1A}' = (1 + 3I/q)n_{1T}'$

§ 5. Beta-ray spectrum of RaE

The beta-ray spectrum of RaE was explained by Konopinski and Uhlenbeck⁽¹⁶⁾ with the T interaction, and by Smith⁽²²⁾ with the linear combinations (V, T) and (S, A) . All of them had regarded this transition as second forbidden. But from the prediction of the shell model and the ft -value, this transition seems to be first forbidden⁽²³⁾. Nakamura, Umezawa and the author have shown that this spectrum can not be explained by the first forbidden transition of the tensor interaction⁽²³⁾. Radiative correction⁽²⁴⁾⁽²⁵⁾ and mesonic correction⁽²⁶⁾ may be unable to explain this singular spectrum.

Recently, Petschek and Marshak⁽⁶⁾ have succeeded in the explanation of this spectrum by the linear combination (T, P) . But, as was mentioned in the introduction, it seems necessary to reexamine their calculation.

Using the above obtained correction factor, we have attempted to know whether the

RaE beta-spectrum can be explained by (T, P) or not*. We have not considered the linear combination (A, P) , for the A interaction may not exist²⁾. The RaE beta-spectrum has been measured by many authors²⁶⁾. We have used the data of Langer given in the Annual Review of Nuclear Science²⁶⁾.

We have tried to explain the RaE beta-spectrum by adjusting the two parameters $\lambda = \lambda_T / (\lambda_P \lambda' / 2M)$ and Γ/ρ .

The condition that the three points on the Kurie plot for electron energy W_0 (=maximum energy), W_1 and W_2 should lie on a straight line is represented by a quadratic for λ and Γ/ρ . The graphs of these quadratics for various values of W_1 and W_2 are shown in Fig. 5. These are hyperbolas. They coincide practically with each other except near the apices, and we may be able to regard them as one hyperbola. In the explanation of the RaE beta-spectrum, λ and Γ/ρ must be based on this hyperbola.

From Fig. 5, we can conclude $\lambda \cdot \Gamma/\rho < 0$. By the estimation (14) given by Ahrens et al.^{9,10)} the value of Γ/ρ is about 200, and we cannot explain the spectrum by this value for any value of λ .

But not all of the points on this hyperbola can explain the RaE beta-spectrum, for, although many hyperbolas seem to coincide with each other, they do not necessarily coincide at every point.

Therefore we have plotted the Kurie plots of RaE for various points on the hyperbola. And we found that when

Fig. 5, The relation between $\lambda = \lambda_T / (\lambda_P \lambda' / 2M)$ and Γ/ρ . All of the upper branches of the hyperbolas nearly coincide.

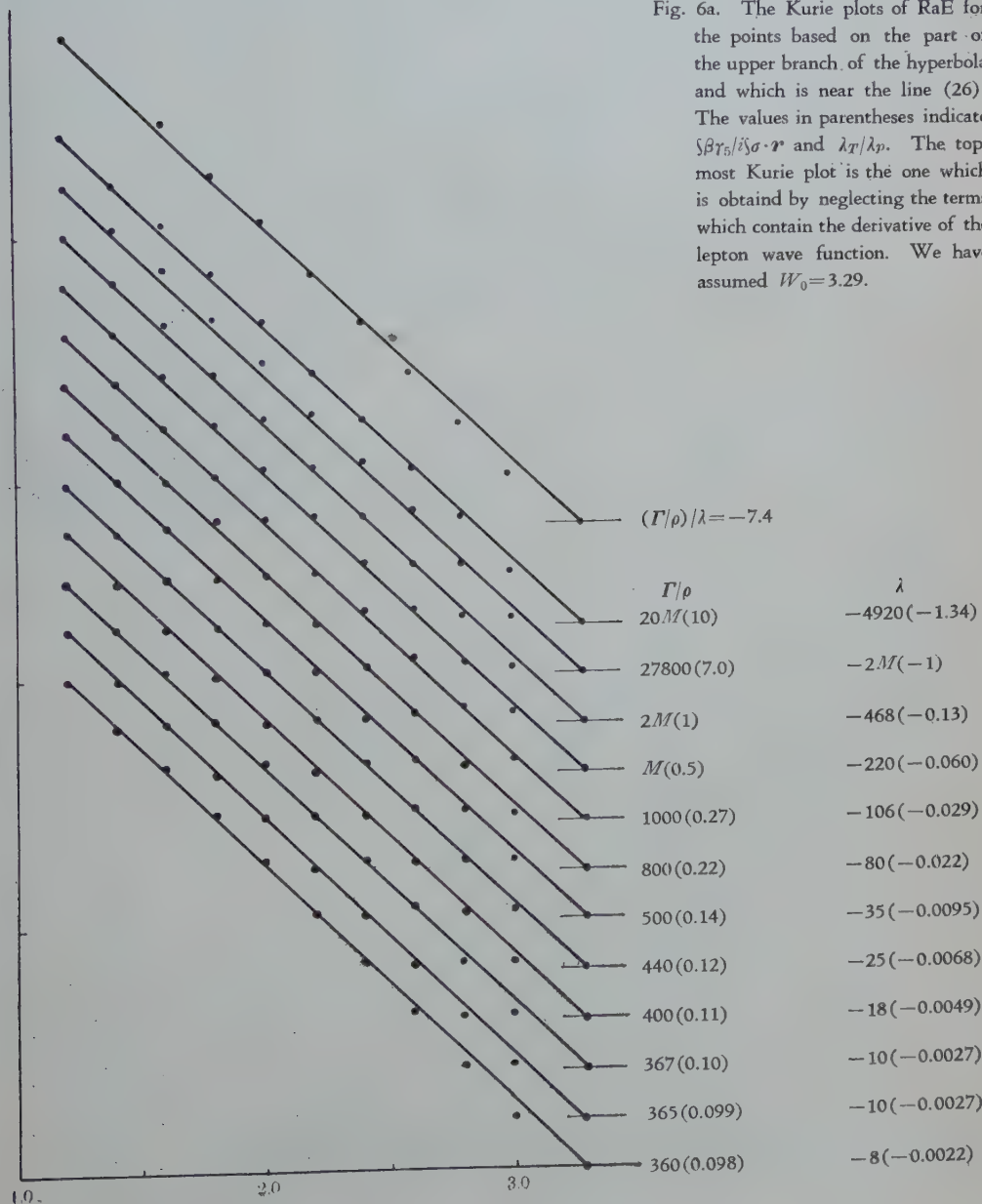


*) Very recently, Yamada has shown that this spectrum can be explained by (S, T) , assuming the transition to be first forbidden²⁷⁾.

$$360 > \Gamma/\rho > -1500,$$

or putting $\lambda' = 1$, when

$$0.098 > \int \beta \gamma_5 / i \int \sigma \cdot r > -0.41,$$



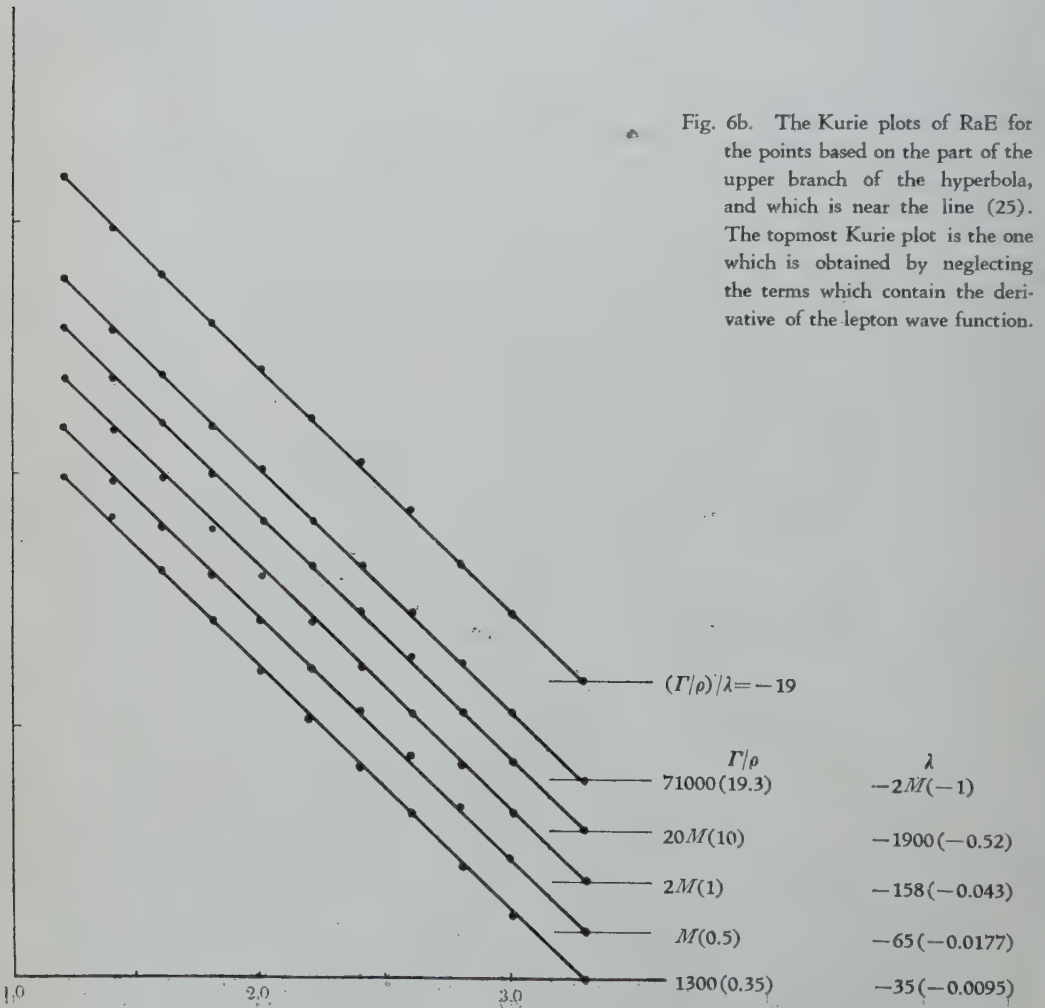
we can not explain the RaE beta-spectrum for any ratio of the coupling constants of the T and P interaction.

If the magnitude of $\int \beta_{T_5}$ or Γ/ρ is so large as was shown by Ruderman¹²⁾, the RaE beta-spectrum can be undoubtedly explained. And even if it reduces to about one tenth of Ruderman's value by virtue of the damping effect calculated by Brueckner et al.¹³⁾, we can explain the RaE beta-spectrum by the linear combination (T, P) . But when we introduce more complicated processes into the calculation of Brueckner et al., if the magnitude of $\int \beta_{T_5}$ more reduced, we cannot explain the RaE beta-spectrum by the linear combination (T, P) .

As for λ , if the RaE decay is $0 \rightarrow 0$ (yes), the region

$$120 > \lambda > -10,$$

or assuming $\lambda' = 1$



$$0.033 > \lambda_T/\lambda_P > -0.0027$$

is forbidden^{*)}.

The asymptotes of this hyperbola are :

$$\Gamma/\rho = -19\lambda + 680, \quad (25)$$

$$\Gamma/\rho = -7.4\lambda + 160. \quad (26)$$

These are the relations between Γ/ρ and λ , when the term $R'(\rho)$ does not appear in eq. (13).

For the large values of $|\Gamma/\rho|$ the part of the hyperbola which is close by the line (25) seems to fit in better with the experimental data, and for the small values of $|\Gamma/\rho|$, the part which is near the line (26) seem to be better.

The best Kurie plots for various values of Γ/ρ and for $\lambda = \pm 2M$ are shown in Fig. 6. These seem to explain the RaE beta-spectrum fairly well.

If a cancellation occurs between the terms which do not contain the derivatives of lepton wave function, it occurs also between the terms which contain the derivatives, and

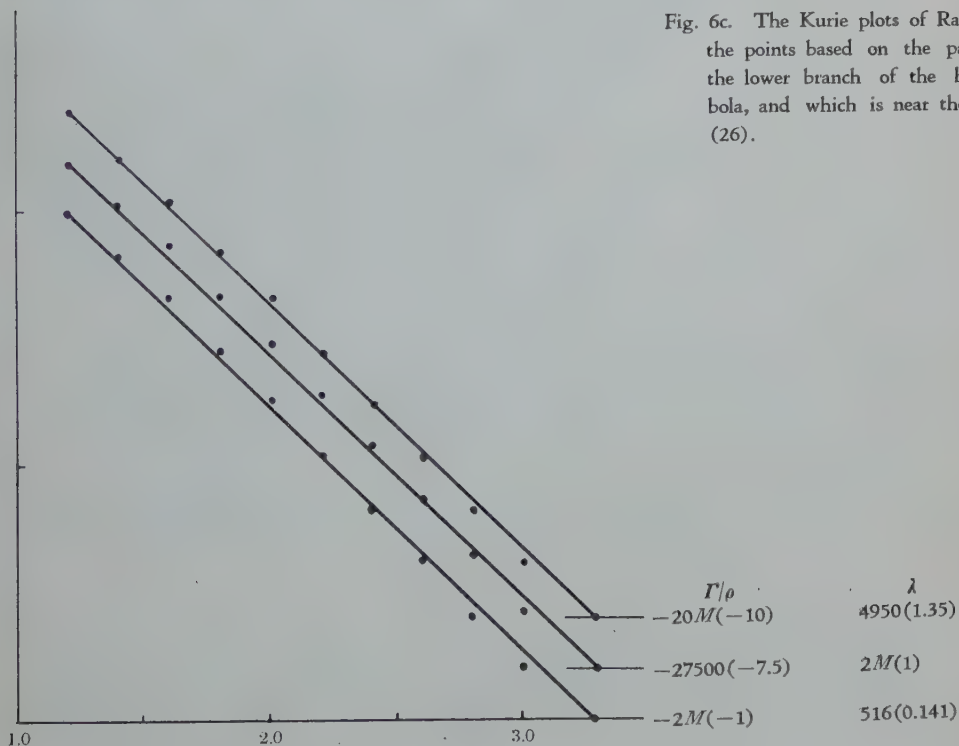


Fig. 6c. The Kurie plots of RaE for the points based on the part of the lower branch of the hyperbola, and which is near the line (26).

*) Fujita and Yamada³⁰⁾ and Kofoed-Hansen and Winther³¹⁾ have obtained the extent to which the ratio of λ_P and λ_T can exist from the analysis of the beta-ray spectra of He^6 and B^{12} .

the term $R'(\rho)$ in eq. (13) hardly affects the results except near the apex of the hyperbola.

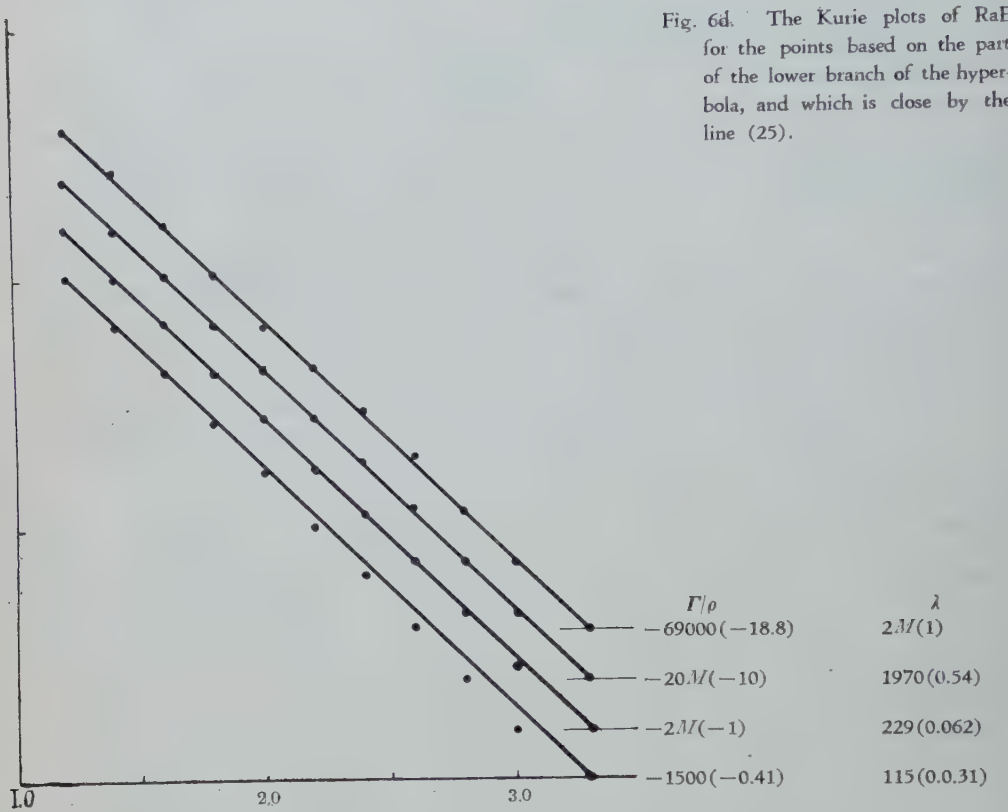


Table III. The ratios of the nuclear matrix elements for $\lambda_T/\lambda_P=\pm 1$

λ_T/λ_P	-1	+1
$\S\beta\gamma_5/i\Sigma\sigma\mathbf{r}$		
Takebe	19.3 or 7.03	-18.8 or -7.49
Petschek & Marshak	13	-13

Acknowledgments

The author wishes to express his cordial thanks to Professor R. E. Marshak for his valuable discussions at Kyoto. In addition, the author wishes to thank Professor T. Yamanouchi, Professor M. Taketani, Professor S. Nakamura, Dr. M. Umezawa, Dr. M. Morita, Dr. M Yamada and Dr. S. Okubo for their kind interest in this work. Miss M. Hayashi, Miss T. Nakano, Mr. T. Hibino, Mr. Y. Nakai, Mr. T. Takebe and Mr. M. Watanabe were very kind and helpful in the numerical calculations. The author is indebted to the "Yukawa Yomiuri Fellowship" for the financial aid.

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Letters to the Editor

Note on the Neutral Vector Meson Theory

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In a recent paper¹⁾ we have discussed a formulation of the neutral vector meson theory which exhibits a gauge invariance analogous to that of quantum electrodynamics. The treatment employs a vector field A_ν and a scalar field B , both solutions of the Klein-Gordon equation, together with state vectors subject to the supplementary condition $(\partial_\nu A_\nu + \mu^{-1}B)\Psi=0$. It was noted that following a contact transformation the interaction Hamiltonian reduces to the simple form $\mathcal{H}=j_\nu A_\nu$, so that computations with the theory are closely similar to those of electrodynamics. Some differences occur, of course, in problems involving renormalization of the meson mass. The meson mass occurs in both the A_ν and B fields and undergoes the same renormalization in each. In practice this may be recognized in any self-energy calculation with the simplified interaction Hamiltonian by using the supplementary condition to express occurrences of the divergence of A_ν in terms of B .

In a paper appearing elsewhere in this issue²⁾, O. Hara and H. Okonogi consider the self-energy problem for the field A_ν in the Heisenberg representation. In the absence of real nucleons their computation of the current vector in terms of A_ν reduces the equation of motion $(\square - \mu^2)A_\nu = j_\nu$ to a homogeneous equation for A_ν . They show that in the second order of perturbation theory j_ν has the gauge invariant form $C(A_\nu - \mu^{-2}\partial_\nu\partial_\lambda A_\lambda)$, in which C is an infinite constant³⁾. The presence of the derivative terms in this expression might appear, as the authors indicate to prevent the interpretation of C as an increment of μ^2 . This would indeed be the case if the field B were not implicitly present. That the renormalization procedure is actually unobstructed may be seen by employing a variant of the device of letting μ be the observed mass and subtracting appropriate

counter-terms from the Lagrangian. Instead of choosing the familiar counter-terms $1/2 \cdot \delta(\mu^2)A_\nu A_\nu$ and $1/2 \cdot \delta(\mu^2)B^2$ we replace the latter one by $1/(2\mu^2) \cdot \delta(\mu^2)(\partial A_\lambda/\partial x_\lambda)^2$, an expression whose expectation value, the supplementary condition assures us will be the same for all admissible state vectors. The equation of motion for A derived with the altered Lagrangian then contains counter terms which, with the identification $\delta(\mu^2)=C$, precisely cancel the expression for j_ν .

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- 2) O. Hara and H. Okonogi, *Prog. Theor. Phys.* **10** (1953), 191. We are indebted to the authors for a copy of their manuscript.
- 3) This form holds in the higher orders as well. The constant C contains a factor of μ^2 and therefore vanishes in the electromagnetic limit.

On the lower energy levels of Li^6

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November 19, 1953

From various evidences about two-nucleon data¹⁾, the existence of the tensor interactions between two nucleons seems to be almost decisive. We have tried the calculations of the energy levels for several light nuclei, taking the tensor forces into account, provided the independent particle shell model is valid, and have obtained some results for lower levels of Li^6 , which are reported below.

The interaction potential between two nucleons is assumed to be of the form, according to the neutral theory^{1), 2), 3)}

$$V_{12} = \{ (1-g) + gP_{\sigma} \} V^C(r_{12}) + S_{12} \cdot V^T(r_{12})$$

with

$$S_{12} = 3(\sigma_1 r_{12})(\sigma_2 r_{12})r_{12}^{-2} - (\sigma_1 \sigma_2),$$

where P_{σ} is the spin exchange operator, g is a numerical constant, and $V^C(r_{12})$ and $V^T(r_{12})$ are the radial dependencies of the scalar and the tensor interactions respectively. For both $V^C(r_{12})$ and $V^T(r_{12})$ we adopt the Yukawa well shapes with different ranges r_C and r_t respectively:

$$V^C(r_{12}) = -V_0 \cdot \exp(-r_{12}/r_C) / (r_{12}/r_C),$$

$$V^T(r_{12}) = -\gamma V_0 \cdot \exp(-r_{12}/r_t) / (r_{12}/r_t).$$

For the radial part of the wave functions of the two nucleons outside the closed shell the harmonic oscillator function⁴⁾ is employed, which is written as, corresponding to the assumed configuration p^2 for Li^6 ,

$$R_p(r) = N \cdot r \exp(-\nu r^2/2)$$

with normalization

$$\int_0^{\infty} R_p^2(r) r^2 dr = 1.$$

Several sets of numerical values of i) the potential depth V_0 , ii) the ratio of two force ranges $\rho = r_t/r_C$ and iii) the ratio of the depth of tensor to that of central force γ , were selected from the Feshbach-Schwinger's results¹⁾ in order to fit the calculated values to the experimental data^{5), 6)} of the following three quantities: (a) the magnetic moment of the ground state μ , (b) the interval between the $1S_0$ and the ground ($3S_1$ main) state, and (c) the interval between the $3D_2$ and the ground state. The exchange force constant g was adjusted also to fit to the singlet scattering result $V_0(1-2g) = 46.48$ Mev as much as possible. Complete reconciliation between the two-nucleon- and the Li^6 -data was not possible, but by mere alterations of the potential depth V_0 (larger values than those of Feshbach-Schwinger) fairly good results were obtained as shown in Table I and Fig. 1. The results of the calculations do not depend upon the absolute value of nuclear radius, but only on its relative ratios to force ranges $\lambda_C = 1/(\sqrt{2\nu}r_C)$ and $\lambda_t = 1/(\sqrt{2\nu}r_t)$. For $r_C = 1.185 \times 10^{-13} \text{ cm}^{1), 3)}$ $\lambda_C = 1$ gives the "radius" $\sqrt{2/\nu} = 2r_C = 2.37 \times 10^{-13} \text{ cm}$.

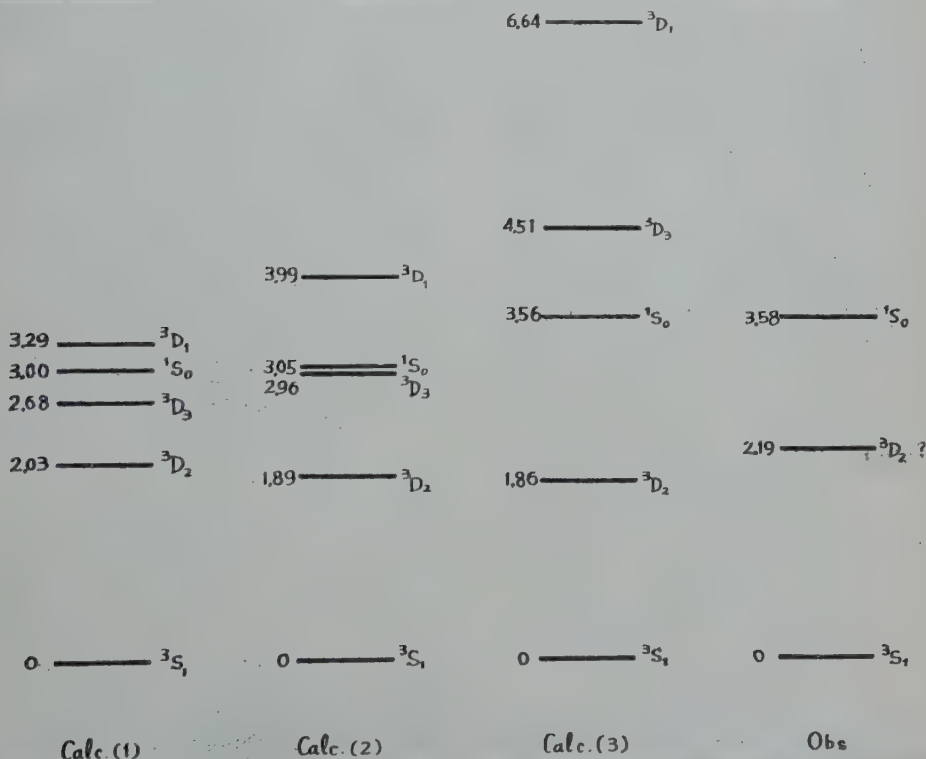


Table I.

	Calc. (1)	Calc. (2)	Calc. (3)
λ_C	1.5	1.5	1.2
λ_t	1.3	1.2	1.0
$\rho=r_t/r_C$	1.15	1.25	1.20
V_0 (Mev)	85	85	80
r	1.00	1.20	1.56
$2g$	0.58	0.55	0.40
$V_0(1-2g)$	35.7	38.3	48.0
μ^*	0.83839	0.81860	0.82206

*) $\mu_{\text{exp}}=0.82189$

Komoda and Sasaki²⁾ obtained a good result for the magnetic moment of Li^6 also taking into account the tensor interactions, but as they did not consider the spin-exchange interactions the position of the 1S_0 level could not be predicted correctly. Their assumption that the first excited level should be the 3D_1 level also seems incompatible with observation.

In conclusion we wish to express our indebtedness and thanks to Mr. H. Horie, Mr. S. Yanagawa and Mr. M. Sato for their valuable discussions and suggestions on this work.

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Non-perturbation Approach Method by Edwards

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In the case of the interaction between the scalar meson and scalar photon fields, it is shown that

$\Delta_{F'}(\not{p})$ and $\Gamma(\not{p}, \not{p})$ are obtained by the same method as the Edwards¹⁾ and the mass subtraction is done in the closed form.

The interaction Hamiltonian is given by

$$H = g\phi^*\phi A - \delta(\mu^2)\phi^*\phi. \quad (1)$$

Here ϕ and A represent the scalar meson and photon respectively, and $\delta(\mu^2)\phi^*\phi$ is the mass subtraction term. We consider only the all graphs of the form in Fig. 1. This corresponds to the approximation done by Edwards¹⁾.

Then the propagation function $\Delta_{F'}$ satisfies the next integral equation:

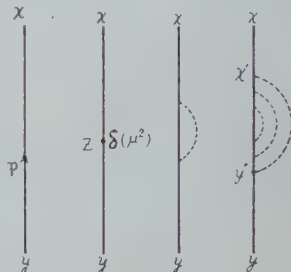


Fig. 1

$$\begin{aligned} \Delta_{F'}(x-y) &= \Delta_F(x-y) + i(\delta\mu^2) \int \Delta_{F'}(x-z) \\ &\times \Delta_F(z-y) dz - g^2 \int \Delta_{F'}(x-x') \Delta_{F'}(x'-y') \\ &\times \Delta_F(y'-y) \Delta_F(x'-y') dx' dy'. \end{aligned} \quad (2)$$

By the fourier transformation, this reduces to

$$\begin{aligned} \Delta_{F'}(\not{p}) &= -i \frac{1}{\not{p}^2 + \mu^2} - i \frac{\delta(\mu^2)}{(\not{p}^2 + \mu^2)^2} \\ &- i \frac{g^2}{(2\pi)^4} \frac{1}{(\not{p}^2 + \mu^2)^2} \int \frac{\Delta_{F'}(\not{p}')}{(\not{p} - \not{p}')^2} d^4 p' \end{aligned} \quad (3)$$

or

$$\begin{aligned} (\not{p}^2 + \mu^2)^2 \Delta_{F'}(\not{p}) &= -i(\not{p}^2 + \mu^2) - i\delta(\mu^2) \\ &- i \frac{g^2}{(2\pi)^4} \int \frac{\Delta_{F'}(\not{p}')}{(\not{p} - \not{p}')^2} d^4 p'. \end{aligned} \quad (3')$$

Following Edwards¹⁾, we put

$$\Delta_{F'}(\not{p}) = \int_0^\infty \frac{F(t)}{(\not{p}^2 + \mu^2 + t)^3} dt \quad (4)$$

and notice that

$$\int \frac{1}{(\not{p} - \not{p}')^2 (\not{p}'^2 + \lambda)^3} d^4 p' = \frac{i\pi^2}{2} \frac{1}{\lambda(\not{p}^2 + \lambda)}. \quad (5)$$

Here, \not{p}^2 in the denominator of (4) has the vanishingly small negative imaginary part. (5) is shown by the direct calculation of the usual method. Then

$$\int_0^\infty dt F(t) \left(1 + 2t \frac{d}{dt} + \frac{t^2}{2} \frac{d^2}{dt^2} \right) \frac{1}{(p^2 + \mu^2 + t)} \\ = -i(p^2 + \mu^2) - i\delta(\mu^2) \\ + \frac{g^2}{32\pi^2} \int_0^\infty \frac{F(t)}{(t + \mu^2)(p^2 + \mu^2 + t)} dt. \quad (3)''$$

Integrating partially the left hand of (3'') and equating the corresponding terms in both side, we get

$$\left. \begin{aligned} t^2 \frac{d^2 F}{dt^2} &= (g^2/16\pi^2) \cdot F/(t + \mu^2), \\ (tF)'_{t=0} &= (t^2 F')_{t=0} = 0, \end{aligned} \right\} \quad (6)$$

$$\left[\frac{2tF}{p^2 + \mu^2 + t} \right]_{t=\infty} - \frac{1}{2} \left[\frac{t^2 F}{(p^2 + \mu^2 + t)^2} \right]_{t=\infty} \\ - \frac{1}{2} \left[\frac{t^2 F' + 2tF}{p^2 + \mu^2 + t} \right]_{t=\infty} = -i(p^2 + \mu^2) - i\delta(\mu^2). \quad (7)$$

(7) is equivalent to the next

$$\left. \begin{aligned} F(\infty) - (tF')_\infty &= -2i\delta(\mu^2), \\ F'(\infty) &= -2i, \\ (F/t^2)_\infty &= (F'/t)_\infty = 0. \end{aligned} \right\} \quad (7)'$$

Put

$$\left. \begin{aligned} y &= -\mu^2/t, \\ \lambda &= g^2/16\pi^2\mu^2, \end{aligned} \right\} \quad (8)$$

then (6) and (7') become to

$$y(y-1) \frac{d^2 F}{dy^2} + 2(y-1) \frac{dF}{dy} - \lambda F = 0, \quad (9)$$

$$(F(y)/y)_{y=-\infty} = (dF/dy)_{y=-\infty} = 0, \quad (9)$$

$$\left. \begin{aligned} (y^2 \frac{dF}{dy})_{y=0} &= -2i\mu^2, \\ \delta(\mu^2) &= -\frac{1}{2i} \left[y \frac{dF}{dy} - F \right]_{y=0}. \end{aligned} \right\} \quad (9)'$$

The solutions of the first equation of (9) are given by

$$\left. \begin{aligned} F_1 &= F(a, \beta, 2, y) = 1 + \sum_{n=1}^\infty \frac{1}{n!(n+1)!} a(a+1) \\ &\quad \dots (a+n-1)\beta(\beta+1)(\beta+1) \dots \\ &\quad \dots (\beta+n-1)y^n, \\ F_2 &= F_1(y) \int_{-\infty}^y \frac{dy}{y^2 F_1^2(y)}. \end{aligned} \right\} \quad (10)$$

Here

$$a = (1 + \sqrt{1+4\lambda})/2, \quad \beta = (1 - \sqrt{1+4\lambda})/2 \quad (10)'$$

and $F(a, \beta, 2, y)$ represents a hypergeometric function. Now λ is positive, and so a is positive and β is negative. We suppose that the coupling constant

g is not large and so λ is smaller than two. Then

$$1 + \beta > 0 \quad (11)$$

and

$$F_1(y) \sim c \cdot 1/y^3, \quad (y \rightarrow -\infty).$$

Therefore, the second equation of (9) is satisfied by both F_1 and F_2 , but (9') can be satisfied only by F_2 .

$$\left. \begin{aligned} F(y) &= -2i\mu^2 F_2(y) \\ &= -2i\mu^2 F_1(y) \int_{-\infty}^y \frac{dy}{y^2 F_1^2(y)}, \\ \delta(\mu^2) &= -\frac{1}{2i} \left[y \frac{dF}{dy} - F \right]_{y=0}. \end{aligned} \right\} \quad (12)$$

Thus, the inhomogeneous equation (3) has the solution, different from Edwards' case. From this and (4),

$$\Delta F'(p) = -2i\mu^4 \int_{-\infty}^0 \frac{dy}{y} \frac{y F_2(y)}{[y(p^2 + \mu^2) - \mu^2]^3}. \quad (13)$$

It is interesting that $\delta(\mu^2)$ is determined uniquely in the contrast with the perturbation method, where

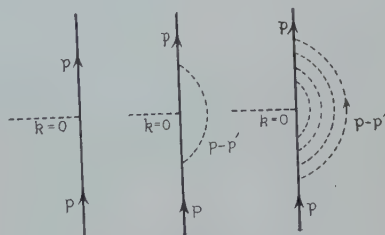


Fig. 2

the Dyson's²⁾ prescription is necessitated in order to make $\delta(\mu^2)$ unique.

The same method can be applied for the sum of such all graphs in Fig. 2 as Edwards¹⁾ considered originally. The integral equation is given by

$$\Gamma(p, p') = 1 - \frac{ig^2}{(2\pi)^4} \int \frac{1}{(p-p')^2} \frac{1}{(p'^2 + \mu^2)^2} \\ \times \Gamma(p', p') d^4 p'.$$

In the same way, this solution is given by

$$\Gamma(p, p') = (p^2 + \mu^2)^2 \int_0^\infty \frac{2}{(p'^2 + \mu^2 + t)^3} \\ F(a, \beta, 2, -\frac{\mu^2}{t}) dt \\ = 1 + \lambda \int_0^\infty \frac{F(a, \beta, 2, -y)}{(1+y)(1+y + (p^2/\mu^2)y)} dy. \quad (14)$$

α and β are the same as before and the condition (11) is postulated. The author represents his thanks to the Yukawa Yomiuri Fellowship for the financial aids.

- 1) S. F. Edwards, Phys. Rev. **90** (1953), 284.
- 2) F. J. Dyson, Phys. Rev. **75** (1949), 1736.

Fourth Order Calculations
of Meson-Proton Scattering in
the Symmetrical Ps(Ps) Theory

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November 28, 1953

Previously* we had calculated meson-proton scattering cross sections up to the fourth order in the symmetrical ps(ps) theory by the covariant perturbation method, in order to partly supplement the work of Ashkin et al¹⁾, and particularly to test whether the curious energy dependence just then observed²⁾, including signs of the scattering amplitudes³⁾, could be reproducible to any extent.

Since then, many experimental⁴⁾ and theoretical⁵⁾ researches have been carried out. Although a com-

* Read at Kyoto Meeting of Physical Society of Japan, May 4, 1953.

pletely satisfactory theory is yet unavailable, it was proved that the perturbation method is not good compared with the Tamm-Dancoff's⁶⁾. One reason for this is that the core term remains still much effective in the perturbational treatment, as shown Drell and Henley⁷⁾ and in a more consistent way by Sawada et al⁸⁾. The symmetrical ps(ps) theory seems to be promising, because it yields, besides, the equivalent pv-coupling term, a term which gives rise to a charge dependent s-wave scattering required by the experimental data. We shall briefly report the perturbational results for comparison and reference.

The Feynman-Dyson's graphs for meson-proton scatterings up to the fourth order are given in Fig. 1; the corresponding matrix elements,

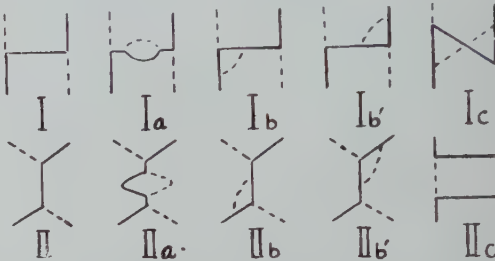


Fig. 1. Feynman-Dyson's graphs for meson-proton scatterings up to the fourth order.

except for the second order I and II, involve (after renormalization) many multiple integrals depending on two invariant parameters⁹⁾ $x_1 = -\lambda + 2\varepsilon'/MC^2$, $x_2 = -\lambda - 2\varepsilon/MC^2$ where $\lambda = (\mu/M)^2$ and $\varepsilon, \varepsilon'$ are the energies of the incident and scattered mesons in lab. system. Of these integrals, those occurring for

Table 1. Total cross sections $\sigma(\text{mb})$

$\sigma = Aa^2 + Ba^3 + Ca^4, \quad a \equiv f^2/4\pi\hbar c$

energy (Mev)	$\sigma(+, +)$			$\sigma(-, -)$			$\sigma(-, 0)$		
	A	B	C	A	B	C	A	B	C
0**	4.8	-4.3	1.0	3.7	-3.0	0.6	0.05	-0.06	0.02
60	4.6	-4.1	1.0	3.2	-2.2	0.4	0.07	-0.12	0.05
110	4.4	-3.9	1.0	2.8 ₅	-1.9	0.4	0.10	-0.15	0.08
135	4.3	-3.9	1.0	2.7	-1.8	0.3 ₅	0.11	-0.17	0.09
180	4.2	-4.0	1.1 ₅	2.5	-1.6 ₅	0.3 ₅	0.14	-0.21	0.11

** There is a discrepancy of about 40% between the results of Ashkin et al. and ours.

the crossed graph Ic and especially for the ladder graph IIc are much complicated; each of them is, however, by suitable transformations reduced to at most double integrals, most of them being able to be evaluated analytically with errors of a few% for $\epsilon \leq 200$ Mev.

Numerical results were obtained for five incident meson energies and five scattering angles (0, 45, 90, 135 and 180 degrees in lab. system), with about 10% errors. The meson-proton scattering process proves however to be almost isotropic for the energy range considered. The total cross sections are given in Tab. 1 and Fig. 2. For the latter, the conventional coupling constant $a \equiv f^2/4\pi\tilde{\kappa}c$ was chosen to be 3.4 so as to fit the experimental data for positive meson scattering at 60 Mev.¹⁰⁾ Total cross sections

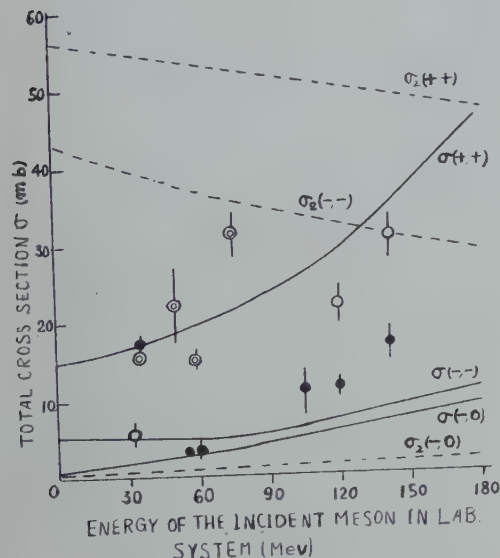


Fig. 2. Total cross sections of meson-proton scatterings for $a \equiv f^2/4\pi\tilde{\kappa}c = 3.4$. Full lines and dotted lines are total cross sections up to the 4-th and 2-nd order respectively. \odot , \bullet and \circ are experimental data for $\sigma(+, +)$, $\sigma(-, -)$ and $\sigma(-, 0)$, cf. reference 4).

thus obtained really increase as meson energy, but in much less extent compared with the experiments, even for the charge-exchange scattering, to which the core term does not contribute in the second order process. For this increase are the large fourth order matrix elements, with sign opposite to the second order ones, responsible, thus showing at the

same time invalidity of the perturbational approximation. Due to this situation, the coupling constant had to be taken about four times smaller than that usually accepted, though consistent with the value deduced from the fourth order radiative corrections to the anomalous magnetic moment of nucleon.¹¹⁾

The authors wish to thank Messrs. Ichikawa and Ishii for laborious numerical integrations.

- 1) Ashkin, Simon and Marshak, *Prog. Theor. Phys.* **5** (1950), 634. (The 4-th order numerical calculations are only for Thomson limit.)
- 2) Anderson, Fermi et al., *Phys. Rev.* **85** (1952), 934(L), 935(L), 936(L).
- 3) Bethe and Wilson, *Phys. Rev.* **83** (1951), 690; Kaplon, *Phys. Rev.* **85** (1952), 1059.
- 4) Most recent reports are Anderson, Fermi Martin and Nagle, *Phys. Rev.* **91** (1953), 155; Fowler, Fowler, Shutt, Thorndike and Whittemore, ditto 135; Roberts and Tinlot, *Phys. Rev.* **90** (1953), 951(L); Angell and Perry, ditto 724(L); Bodansky, Sachs and Steinberger, ditto 996(L), 997(L).
- 5) Most of them are for the symmetrical ps(pv) theory. Cf. also Proc. 3rd Annual Rochester Conference.
- 6) Dyson et al., *Phys. Rev.* **90** (1953), 372 (A); Fubini, *Nuovo Cimento* **10** (1953), 564; 851(L); Chiba and Yamazaki, private communication.
- 7) Drell and Henley, *Phys. Rev.* **88** (1952), 1053; cf. also Wentzel, *Phys. Rev.* **86** (1952) 802; Brueckner, Gell-Mann and Goldberger, *Phys. Rev.* **90** (1953), 476; Hasegawa and Azuma, *Prog. Theor. Phys.* **10** (1953), 240(L); Karplus, Kivelson and Martin, *Phys. Rev.* **90** (1953), 1072.
- 8) Sawada and Tani, read at Internat. Conf. of Theor. Phys., Kyoto, Sept. 21, 1953.
- 9) Brown and Feynman, *Phys. Rev.* **85** (1952), 231.
- 10) The value 20 mb was adopted; cf. reference 2) and Fowler, Fowler, Shutt, Thorndike and Whittemore, *Phys. Rev.* **86** (1952), 1053. Bodansky et al. recently reported somewhat smaller value, 15.3 mb.
- 11) Nakabayasi and Sato, *Prog. Theor. Phys.* **6** (1951), 242(L); *Sci. Rep. Tohoku Univ.* 1 ser. **34** (1950), 169. There are some unfortunate mistakes concerning numerical evaluations. We hope to publish the corrected results in the near future.

Fourth Order Phase Shifts for Meson-Proton Scattering in the Symmetrical Ps(Ps) Theory

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November 28, 1953

Recently the phase shift analysis of meson-proton scattering has been carried out by several authors¹⁾. For comparison, we also do it for our covariant fourth order perturbational calculations reported in the preceding letter.

For our case of the symmetrical ps(ps) theory, the scattering matrix element R defined by

$$\begin{aligned} \langle f', k' | S - 1 | f, k \rangle &= \delta^4(f' + k' - f - k) \\ \langle f', k' | R | f, k \rangle &= \sqrt{f_0' k_0'} / \sqrt{f_0 k_0} \end{aligned} \quad (1)$$

can be written as follows (in matrix notation for charge state)²⁾:

$$\begin{aligned} R &= R^{(1)} \mathbf{I} + R^{(2)} (\boldsymbol{\tau} \cdot \boldsymbol{\omega}) = (2 + \boldsymbol{\tau} \cdot \boldsymbol{\omega}) / 3 \cdot R^{(3/2)} \\ &+ (1 - \boldsymbol{\tau} \cdot \boldsymbol{\omega}) / 3 \cdot R^{(1/2)}, \end{aligned} \quad (2)$$

where $\boldsymbol{\tau}$, $\boldsymbol{\omega}$ are the isotopic spin of nucleon and meson respectively and

$$\begin{aligned} R^{(1)} &= a(R_I + R_{II}) + a^2[(3R_{Ia} - 2R_{Ib}) \\ &+ (3R_{IIa} - 2R_{IIb}) + 3(R_{Ic} + R_{IIc})], \\ R^{(2)} &= a(R_I - R_{II}) + a^2[(3R_{Ia} - 2R_{Ib}) \\ &- (3R_{IIa} - 2R_{IIb}) + (-R_{Ic} + R_{IIc})], \end{aligned} \quad (3)$$

$$R_{Ib} = R_{IIb}, \quad R_{IIb} = R_{IIb}; \quad a \equiv f^2 / 4\pi \hbar c;$$

indices I, II, Ia, etc., referring to the corresponding graphs in Fig. 1 in the preceding letter. Then the total cross sections are written, neglecting small f -wave contributions, as

$$\begin{aligned} \sigma(+, +) &= 4\pi^2 / (f_0 + k_0)^2 \cdot 4\pi |R^{(3/2)}|^2, \\ \sigma(+, -) &= 4\pi^2 / (f_0 + k_0)^2 \cdot 4\pi \cdot 9 \cdot |R^{(3/2)} + 2R^{(1/2)}|^2, \\ \sigma(-, 0) &= 4\pi^2 / (f_0 + k_0)^2 \cdot 4\pi / 9 \cdot |R^{(3/2)} - R^{(1/2)}|^2, \end{aligned} \quad (4)$$

where f_0 , k_0 are proton and meson energies in C.M. system respectively. From the above, we obtain the

phase shifts of s -waves with total isotopic spin $I=3/2$ and $1/2$ by the well known method,³⁾ assuming $e^{i\delta} \approx 1 + i\delta$ consistently in our approximation.

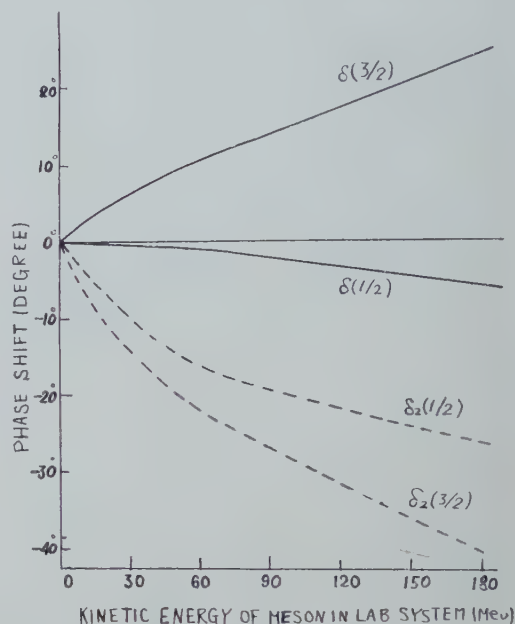


Fig. 1. S -wave phase shifts $\delta(3/2)$ and $\delta(1/2)$ for states with total isotopic spin $I=3/2$ and $1/2$. Full lines and dotted lines are those up to the 4-th and 2-nd order respectively.

In Fig. 1, the s -wave phase shifts $\delta(3/2)$ and $\delta(1/2)$, thus obtained; as well as the ones in the second order, are plotted for $a=3.4$. We observe that the positive fourth order corrections for $I=3/2$ largely overwhelm the negative phase shift in the second order, while $\delta(1/2)$ is small and remains still negative. This is completely in contradiction with the experiment and again shows the crudeness of our approximation, mainly due to the large core term effect. The next higher order corrections are expected to bring to a correct direction. In facts, Chiba et al⁴⁾ showed that Tamm-Dancoff's approximation allowing two mesons and one nucleon-pair in the intermediate states, taking into account mass and charge renormalizations,⁵⁾ reproduces the phase shifts with right signs qualitatively.

In Table. 1, we tabulate the matrix elements in C.M. system for five incident meson energies. It is noted that among the fourth order matrix elements the ladder type is dominant, though all are of the same order.

- 1) Cf. reference 2) in the preceding letter.
- 2) Otuka and Yamaguchi, *Soryusiron-Kenkyu* (mimeographed circular in Japanese), **4** (1952), No. 7, 61. R 's in (2) are the matrices such that the free nucleon spinors are inserted in $(\not{p}'\not{k}'|R|\not{p},\not{k})$ to get spin and momentum dependencies; the result of this shows indeed smallness of f -wave contributions.
- 3) Møller, D. Kgl. Danske Vidensk. Selskab. Mat-fys. Medd. XXIII (1945), Nr. 1; Lippmann and Schwinger, *Phys. Rev.* **79** (1950), 469; Otuka and Yamaguchi, reference 2).
- 4) Chiba and Yamazaki, private communication; cf. also Fubini, l.c. in the preceding letter.
- 5) Cf. Cini, *Nuovo Cimento* **10** (1953), 526, 614.

On the Relation between Non-Local Urmaterie Field and Irreducible Local Field

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November, 30, 1953

It was verified by many authors¹⁾²⁾ that Yukawa's non-local field³⁾ was a superposition of irreducible local fields with various spin. In their investigations Yukawa's third equation played an important role.

Recently O. Hara et al.⁴⁾⁵⁾⁶⁾ proposed a non-local "Urmaterie" field in order to describe elementary particles in unified way. They started from the following two fundamental equations*

$$\begin{aligned} (\partial^2/\partial X_\mu \partial X_\mu - \mathbf{M}^2/\lambda^2) U(X_\mu, r_\mu) &= 0, \\ (r_\mu r_\mu - \lambda^2) U(X_\mu, r_\mu) &= 0. \end{aligned} \quad (1)$$

In this case, \mathbf{M}^2 is not commutable with $r_\mu \partial/\partial X_\mu$, and Yukawa's third equation can not be adopted. It is shown in this note, however, that the free scalar "Urmaterie" field can be regarded also as a superposition of local fields with various spin and rest mass which satisfy Fierz's⁷⁾ equations.

Neglecting normalization constants, the eigenfunction of the internal motion which belongs to the eigenvalue $S(S+1)$ and m^2 of \mathbf{S}^2 and \mathbf{M}^2 is given by

* As for notations see Reference (4).

$$\begin{aligned} \phi_{s,m,n} &= \frac{1}{\{r_\mu r_\mu - (k_\mu r_\mu)^2/k_\mu k_\mu\}^{1/2}} \\ &\times \int_{-\pi}^{\pi} du (a_{0\mu} r_\mu + i a_{1\mu} r_\mu \cos u + i a_{2\mu} r_\mu \sin u) e^{i n u}, \end{aligned} \quad (2)$$

where $f_{s,m}$ is determined to be a product of Jacobi's polynome and some algebraic functions as an eigenfunction of \mathbf{M}^2 in the rest system of the center of mass, k_μ the energy-momentum vector of the external motion, and $a_{\mu\nu}$ are coefficients of Lorentz-transformation which transform k_μ into rest;

$$a_{\mu\nu} k_\nu = \begin{cases} 0 & \text{for } \mu=1, 2, 3, \\ i(m/\lambda) & \text{for } \mu=4. \end{cases} \quad (3)$$

Of course $a_{\mu\nu}$ satisfy

$$a_{\mu\lambda} a_{\nu\lambda} = \delta_{\mu\nu}. \quad (4)$$

In (2) Lorentz-invariant parameter n is introduced other than s and m to distinguish $(2s+1)$ independent eigenfunction belonging to the spin eigenvalue s , and the last factor is a four dimensional generalization of spherical harmonics which, in the rest system of the center of mass, is reduced to

$$P_s^{(n)}(\cos \theta) e^{i n \varphi} = \frac{\text{const.}}{r^s} \int_{-\pi}^{\pi} du (r_3 + i r_1 \cos u + i r_2 \sin u) e^{i n u}.$$

Expanding Fourier coefficients of the Urmaterie field by these eigenfunctions,

$$\begin{aligned} U(X_\mu, r_\mu) &= \int u(k_\mu, r_\mu) e^{i k_\mu X_\mu} (dk), \\ u(k_\mu, r_\mu) &= \sum A(k_\mu; s, m, n) \delta(r_\mu r_\mu - \lambda^2) / \{r_\mu r_\mu - (k_\mu r_\mu)^2/k_\mu k_\mu\}^{1/2} f_{s,m}(-k_\mu r_\mu/\lambda(-k_\mu k_\mu)^{1/2}) \\ &\times \int_{-\pi}^{\pi} du (a_{0\mu} r_\mu + i a_{1\mu} r_\mu \cos u + i a_{2\mu} r_\mu \sin u) e^{i n u}, \end{aligned} \quad (5)$$

and picking up the coefficient of $r_\lambda r_\mu r_\nu \dots$ from

$$\begin{aligned} \sum_{|n| \leq s} A(k_\mu; s, m, n) \int_{-\pi}^{\pi} du (a_{0\mu} r_\mu + i a_{1\mu} r_\mu \cos u \\ + i a_{2\mu} r_\mu \sin u) e^{i n u}, \end{aligned} \quad (6)$$

we see that it behaves as a symmetrical tensor contragradient to $r_\lambda r_\mu r_\nu \dots$, since $u(k_\mu; s, m)$ is scalar. Denoting it as $A_{\lambda\mu\nu\dots}(k_\mu; s, m)$, it is given by

$$A_{\lambda\mu\nu\dots}(k_\mu; s, m) = \sum_{|n| \leq s} \int_{-\pi}^{\pi} du e^{i n u} A(k_\mu; s, m, n)$$

$$\begin{aligned}
& \times (a_{11} + ia_{11} \cos u + ia_{21} \sin u)^\alpha \\
& \times (a_{32} + ia_{12} \cos u + ia_{22} \sin u)^\beta \\
& \times (a_{33} + ia_{13} \cos u + ia_{23} \sin u)^\gamma \\
& \times (a_{34} + ia_{14} \cos u + ia_{24} \sin u)^\delta, \quad (7)
\end{aligned}$$

where α , β , γ and δ mean numbers of 1, 2, 3, and 4 appearing in suffixes of $A_{\lambda\mu\nu\dots}(k_\mu; s, m)$ respectively. Using (1), (4) and (7), we get after some calculations

$$\begin{aligned}
& (k_\mu k_\mu^2 + (m/\lambda)^2) A_{\lambda\mu\nu\dots}(k_\mu; s, m) = 0, \\
& k_\lambda A_{\lambda\mu\nu\dots}(k_\mu; s, m) = 0, \quad (8) \\
& A_{\lambda\lambda\nu\dots}(k_\mu; s, m) = 0.
\end{aligned}$$

These are nothing but the equations of motion and the supplementary conditions for particles with spin s and rest mass (m/λ) as given by Fierz⁷⁾. It is clear from above deduction that (8) is a direct consequence of the fact that the eigenfunction of \mathbf{S}^2 is essentially spherical harmonics, and therefore is closely related to the rotation of a rigid sphere. Main features of Yukawa's original non-local theory are thus maintained in our theory of the Urmaterie field.

Equations (5) and (8) show that the Urmaterie field is equivalent to a superposition of various local fields with definite spin and rest mass, the value of which are determined by the internal motion described by eigenfunctions accompanying to each of them. Such a structure of the Urmaterie field suggests a way for its quantization, since the quantization is a procedure to reproduce particle aspect from that of the wave, and particles that appear in our observation seem to have definite spin and rest mass at least in the case of no interaction. Thus, it would be reasonable to expect that the quantization of the (free) Urmaterie field can be achieved by quantizing those part of it which corresponds to local fields. Eigenfunctions of the internal motion will, on the other hand, be responsible to the law of interaction between them. Although the concrete form of this law is not known at the present stage, it is easy to foresee, for example by substituting (5) into the S -matrix given by Yukawa⁸⁾, that it leads to a non-local interaction with form factors which are composed from these eigenfunctions. Thus, in the theory of the Urmaterie field, the form of form factors is determined uniquely by the structure of participating particles. As emphasized by Yukawa⁹⁾, this is one of the great advantages of the theory of this type over the

conventional theory of non-local interaction, where we can never expect such principle to be given.

Details will be published in a near future with related problems.

- 1) M. Fierz, Phys. Rev. **78** (1950), 184.
- 2) O. Hara and H. Shimazu, Prog. Theor. Phys. **5** (1950), 1055.
- 3) H. Yukawa, Phys. Rev. **77** (1950), 219.
- 4) O. Hara and T. Marumori, Prog. Theor. Phys. **9** (1953), 559.
- 5) O. Hara, T. Marumori, Y. Ohnuki and H. Shimodaira, Prog. Theor. Phys. **10** (1953), 114.
- 6) O. Hara, T. Marumori, Y. Ohnuki and H. Shimodaira, Prog. Theor. Phys. in press.
- 7) M. Fierz, Helv. Phys. Acta **12** (1939), 3.
- 8) H. Yukawa, Phys. Rev. **80** (1950), 1047.
- 9) H. Yukawa, Soryushiton Kenkyu (mimeographed circular in Japanese) **5** (1953), 282.

Some Remarks on the Mass Spectrum and Non-local Interaction

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November, 30, 1953

Recently, Yukawa¹⁾ proposed an interesting attempt which introduced the mass spectrum of the elementary particles by non-local field theory.

This attempt seems to be an important step towards the construction of a consistent field theory free from "divergent difficulties". But this theory includes two difficulties in its formalism.

In the first time, the mass spectrum induced by this has the infinite degeneracy, or the finite degeneracy.

In the next time, the non-local interaction obtained by integrating with respect to the internal coordinate destroys the relativistic requirements. As to this point, though Utiyama has pointed out to treat by unitary trick, this method could not obtain the sufficient results.

The author tried to overcome these difficulties by modification of the fundamental equation originated by Yukawa.

As an example of the fundamental equation, let us consider the following one for the scalar field,

$$\left[-\frac{\partial^2}{\partial X_\mu \partial X^\mu} + \gamma^2 - (\beta\lambda)^2 \left\{ -\left(\frac{\partial^2}{\partial X_\mu \partial r^\mu} \right)^2 + \frac{1}{\lambda^4} \left(r_\mu \frac{\partial}{\partial X^\mu} \right)^2 \right\} \right] U(X, r) = 0 \quad (1)$$

where, $X_\lambda = (x_\mu' + x_\mu'')/2$, $r_\mu = x_\mu' - x_\mu''$, and λ is a small constant with the dimension of length, β and γ are the constants to be determined by the practical requirements.

To obtain the particular solution, we put as follows:

$$U(X, r) = e^{ikX} \chi(k, r) \quad (2)$$

and, when, for the internal coordinate, the Lorentz transformation

$$\begin{aligned} r_1 &= 1/\lambda \cdot (k_0 r_1' - i k_3 r_2' + i k_2 r_3' + k_1 r_0') \\ r_2 &= 1/\lambda \cdot (i k_3 r_1' + k_0 r_2' - i k_1 r_3' + k_2 r_0') \\ r_3 &= 1/\lambda \cdot (-i k_2 r_1' + i k_1 r_2' + k_0 r_3' + k_3 r_0') \\ r_0 &= 1/\lambda \cdot (k_1 r_1' + k_2 r_2' + k_3 r_3' + k_0 r_0') \end{aligned} \quad (3)$$

are used under the condition. $k_\mu k^\mu = -x^2$, we can obtain a simple equation of eq. (1). That is

$$[-x^2 + \gamma^2 - (\beta\lambda x^2) \{ \partial^2 / \partial r_0'^2 - r_0'^2 / \lambda^4 \}] \chi(r_0') = 0. \quad (4)$$

This solution is, as is well known, the Hermite polynomial such as

$$\chi(r_0') = H_n(r_0' / \lambda) \exp(-r_0'^2 / 2\lambda^2) \quad (5)$$

and, the mass spectrum in this case will be given by

$$x = r / \sqrt{1 - \beta^2 (2n + 1)}. \quad (6)$$

Also, using the inverse transformation of the above one (2), we can easily represent the particular solution of Eq. (1),

$$U(X, r) = H_n \left(\frac{-k_\mu r^\mu}{x\lambda} \right) \exp \left\{ i k X + \frac{(k_\mu r^\mu)^2}{2\lambda^2 k_\mu k^\mu} \right\}. \quad (7)$$

In our case, the index n of Hermite polynomial will be allowed the odd value by Bloch²⁾'s discussion for the correspondence between the local field and non-local field.

Now, starting from the above representation, let us discuss the interaction Lagrangian by Katayama and Tokuoka³⁾'s method.

The type of interaction Lagrangian is as follows;

$$L = g \int \psi^*(x_1) (x_1 | U | x_2) \psi(x_2) dx_1 dx_2 \quad (8)$$

where ψ^* and ψ are local charged scalar fields, U is non-local neutral scalar field with a certain mass represented by index n , and g is a coupling constant.

The explicit forms of these fields are

$$\psi^*(r) = \frac{1}{(2\pi)^3} \int \psi^*(k) e^{-ikx} dk \quad (9a)$$

and

$$\psi(x) = \frac{1}{(2\pi)^3} \int \psi(q) e^{iqx} dq. \quad (9b)$$

and

$$\begin{aligned} (x_1 | U | x_2) &= U(X, r) \\ &= \frac{1}{(2\pi)^3} \int u(p) e^{ipX} \chi(p, r) dp \end{aligned} \quad (9c)$$

where $A = \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m (n!) 2^{\frac{n}{2}-m}}{(m!) ((n/2)-m)!} \sqrt{\pi}$

using these forms, when we make the integration of (8) with respect to the internal coordinate, we must calculate the integration

$$I = \int d^4r L r_1 \left(\frac{-f_\mu r^\mu}{x\lambda} \right) \exp - \left\{ \frac{(f_\mu r^\mu)^2 + (l_\mu r^\mu) x^2 \lambda^2}{2x^2 \lambda^2} \right\} \quad (10)$$

where $h = k + q$.

In this case, the integrations for r_1, r_2, r_3 , reduce to the product of delta-function

$$\begin{aligned} &\delta(h_1 f_0 + i h_2 f_3 - i h_3 f_2 - h_0 f_1) \\ &\times \delta(i h_1 f_3 - h_2 f_0 - i h_3 f_1 + h_0 f_2) \\ &\times \delta(i h_1 f_2 - i h_2 f_1 + h_3 f_0 - h_0 f_3) \end{aligned} \quad (11)$$

and for r_0 reduces to a finite series.

Making use of the relationship of both the real part and the imaginary part in delta-function, we can easily obtain the following representation as an interaction Lagrangian

$$\begin{aligned} L &= \sum_{m=0}^{\frac{n}{2}} \sum_{s=0}^{\frac{n}{2}-m} \frac{(-1)^{m+s} 2^{\frac{n}{2}-m-s} \pi^{\frac{7}{2}} \lambda^{-2s+1} (n!)}{A(m!) (2s!) ((n/2)-m-s)!} \\ &\times \left(\frac{4x_L^2 - x_n^2}{x_n^4} \right) \int \psi^*(X) \psi(X) \square^{2s} \\ &\times \exp \left(-\frac{\lambda^2 (4x_L^2 - x_n^2)}{8x_n^4} \square^2 \right) U(X) dX \end{aligned} \quad (12)$$

where x_L is the mass of local fields and x_n is the one of non-local field.

The above form is, as is well known, one type of non-local interaction. Therefore, we can induce consistently, satisfying the relativistic requirements, the

mass spectrum with no degeneracy and non-local interaction from non-local field theory.

But this treatment is only a formal success, thus, in future we must discuss the physical meaning of the induction of the fundamental equation, as introduced, using fusion theory of Bosons, by Watanabe⁴⁾ recently.

- 1) H. Yukawa, Lecture at I.C.T.P. (1953).
- 2) C. Bloch, Det. Kgl. Dansk. Vid. Selsk. XXVI, Nr. 1 (1950), 26.
- 3) Z. Tokuoka and Y. Katayama, Prog. Theor. Phys. 6 (1951), 132.
- 4) S. Watanabe, Phys. Rev. 91 (1953), 771.

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New Formulation of One-body Problem in Quantum Electrodynamics

Mikio NAMIKI and Yosio SUZUKI

Prog. Theor. Phys. 9 (1953), 223

The requirements in § 2 are not enough to define the one-electron state. They must be supplemented by the new requirement that, in case of no external field, the space of one-electron state contains the eigenstates of the total energy operator belonging to the eigenvalues satisfying the real energy-momentum condition. Then, since the vectors $\phi_H^* \psi_0$'s cannot fulfil the new requirement, we must replace these vectors with the one satisfying all requirements, for example, the vectors by the adiabatic switch on process from the free stationary states or their combinations. The theory will be formulated along the similar line. The authors wish to express their sincere gratitude to Drs. Suura, Nishijima, and Källén, who have kindly pointed out above situations.

Molecular Structure and Absorption Spectra of Carotenoids

Gentaro ARAKI and Tomokazu MURAI

Prog. Theor. Phys. 8 (1952), 639

The interpretation of the sectional area, A , of the pipe in which π -electrons are enclosed was not appropriate. In the correct interpretation $4A$ should be replaced for A in the previous paper, namely for A in the ninth line of p. 643, eqs. (2.3), (2.4) and (3.1). Consequently we have $A=181.072$ and $A^{1/2}=13.4563$ atomic units. Further for "is expanded", in the second line of p. 643 read "is approximately expanded", and for "the approximate value of k^2 " in the fifth line of the same page read " k ".

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